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A GENERAL APPROACH TO CONFOUNDING PLANS IN MIXED FACTORIAL EXPERIMENTS WHEN THE NUMBER OF LEVELS OF A FACTOR IS ANY POSITIVE INTEGER

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An algebraic technique which maps elements from distinct finite rings into subsets of another finite ring is defined, and a method for combining elements from distinct finite rings is demonstrated. The connection between this mapping and the constructing of confounding plans for the mixed factorial experiment with any number of factors at any number of levels is established, as well as the limitation of the procedure.

1. Summary and introduction. This paper employs a technique which maps elements from distinct finite rings into subsets of another finite ring. A method for combining elements from distinct finite rings is shown and its use demonstrated in constructing confounding plans for any factorial experiment. A paper by White and Hultquist [4] has given methods for the design and analysis of confounding plans for the $p^n \times q^m$ factorial experiment, p and q primes, using a technique for combining elements from distinct finite fields. An equivalent theoretical basis utilizing Ideal Theory was obtained by Raktoe [2], who also provided a generalization which covers all cases in which the number of levels of a factor are all prime numbers. Raktoe [3] has also covered the case where more than one field is based on the same prime and the fields may be prime powered. In a recent paper, Banerjee [1] noted that properties of finite multiplicative groups are sufficient to construct many of the designs given by methods in [2] and [4], and demonstrated the structural identity of this procedure to the others. It was also shown in [1] that it would be possible to provide the confounding plans for mixed factorials of the types $5^n \times 6^m$, where n and m are any two positive integers, which is not covered by the methodology of White and Hultquist [4] or Raktoe [2], since 6 is not a prime number. However, in other cases, a 5×7 factorial experiment, for example, the method of Banerjee [1] fails. The aim of the present paper is to present a methodology which yields confounding plans which cover all possible numbers of factors at all possible levels. The limitation of such plans is also described.

2. The algebraic approach to combining elements from distinct finite rings. Let $n_1, n_2, \cdots, n_s$ be s distinct positive integers and $R(n_1), R(n_2), \cdots, R(n_s)$ the corresponding finite rings. $R(n_j)$ consists of the residue classes of integers modulo the integer n_j. These elements will be denoted as $c(\text{mod}\, n_j)$. Let $n = \prod_{j=1}^{s} n_j$

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and let $c(\text{mod } n)$ be an element of the ring of residue classes of integers modulo n. Also let $\theta_k = \prod_{j=1}^{k} n_j$, $\phi_k = \prod_{j=k+1}^{s} n_j$, and define $\phi_s = 1$. A series of definitions and lemmas are now given to provide the unique decomposition of the integral ring $R(n)$ into subsets whose elements correspond to the elements in $R(n_j)$.

DEFINITION 2.1. The correspondences between the elements of the residue classes of integers (mod n_j) and the subsets of the elements of the residue classes of integers (mod n), are defined as follows

$$c_j(\text{mod } n_j) \rightarrow c_j x_j \phi_j(\text{mod } n) \tag{2.1}$$

for $j = 1, 2, \cdots, s$, where x_j is determined by successive solutions to the equations

$$\textstyle\sum_{i=1}^{j} x_i \theta_j/\theta_i = 1(\text{mod } \theta_j) . \tag{2.2}$$

These solutions may be reduced to

$$x_j = [1 - n_j](\text{mod } \theta_j) . \tag{2.3}$$

The correspondences have been defined so that the unit element in $R(n)$ is the sum of the unit elements in the distinct finite ring.

These correspondences are used in combining elements from distinct finite rings in the following manner.

DEFINITION 2.2. If integers (mod n_j) ($j = 1, 2, \cdots, s$) are to be combined, we define the sum as

$$\textstyle\sum_{j=1}^{s} c_j(\text{mod } n_j) = \sum_{j=1}^{s} c_j x_j \phi_j(\text{mod } n) . \tag{2.4}$$

If there is more than one integer to be combined from any one finite ring, the usual addition can be carried out within the structure of the ring, and the resulting sum combined with elements from other distinct rings in accord with (2.4).

LEMMA 2.1. *Unless all of the terms in the summation* $\sum_{j=1}^{s} c_j(\text{mod } n_j)$ *are equal to zero, then*

$$\textstyle\sum_{j=1}^{s} c_j(\text{mod } n_j) \neq 0(\text{mod } n) . \tag{2.5}$$

PROOF. Assume that $\sum_{j=1}^{s} c_j(\text{mod } n_j)$ is equal to zero for some values of c_j $[j = 1, 2, \cdots, s]$ not all of which are zero. This implies that

$$\textstyle\sum_{j=1}^{s} c_j x_j \phi_j = t'n$$

or equivalently

$$\textstyle\sum_{j=1}^{s} c_j[1 - n_j]\phi_j = t\theta_j . \tag{2.6}$$

Dividing both sides of (2.6) by n_s yields

$$\textstyle\sum_{j=1}^{s-1} c_j[1 - n_j]\phi_j/n_s + c_s[1 - n_s]/n_s = t\theta_{s-1} \tag{2.7}$$

and further simplification gives

$$c_s/n_s = t\theta_{s-1} + c_s - \textstyle\sum_{j=1}^{s-1} c_j[1 - n_j]\phi_j/n_s \tag{2.8}$$

which implies that c_s must be zero, since the left-hand side is a fraction and the right-hand side an integer. Successive application of this technique gives repeated contradictions of c_j being nonzero, and hence the lemma.

LEMMA 2.2. *Each distinct combination* $(c_1(n_1), \cdots, c_s(n_s))$ *gives a distinct value for*

$$\sum_{j=1}^{s} h_j c_j (\text{mod } n_j) , \tag{2.9}$$

where $h_j \neq 0(\text{mod } n_j)$, $j = 1, 2, \cdots, s$.

PROOF. Assume there exist two distinct combinations giving identical values. Then

$$\sum_{j=1}^{s} h_j c_j (\text{mod } n_j) = \sum_{j=1}^{s} h_j d_j (\text{mod } n_j) \tag{2.10}$$

implying that

$$\sum_{j=1}^{s} h_j c_j x_j \phi_j = \sum_{j=1}^{s} h_j d_j x_j \phi_j$$

or equivalently,

$$\sum_{j=1}^{s} (c_j - d_j) h_j x_j \phi_j = 0(\text{mod } n)$$

which contradicts the results of Lemma 2.1. The statement of the lemma then follows.

LEMMA 2.3. *In the residue class ring* (mod n) *every element* c(mod n) *has a unique decomposition*

$$c(\text{mod } n) = \sum_{j=1}^{s} c_j (\text{mod } n_j) . \tag{2.11}$$

PROOF. The proof follows directly from Lemmas 2.2 and 2.3. An example illustrates the use of the preceding correspondences for combining of elements from residue classes of integers (mod 4) and (mod 6). The values of x_1 and x_2 are needed. From (2.3) we have

$$\begin{aligned} x_1 &= (1 - 4)(4) = -3(4) = 1(4) = 1 \\ x_2 &= (1 - 6)(24) = -5(24) = 19(24) = 19 . \end{aligned} \tag{2.12}$$

The correspondences follow from (2.1), and are indicated below.

$$\begin{array}{ll} 0(4) \to 0(24) & 0(6) \to 0(24) \\ 1(4) \to 6(24) & 1(6) \to 19(24) \\ 2(4) \to 12(24) & 2(6) \to 14(24) \\ 3(4) \to 18(24) & 3(6) \to 9(24) \\ & 4(6) \to 4(24) \\ & 5(6) \to 23(24) \end{array} \tag{2.13}$$

Note that the mappings do not always preserve the isomorphism. For example, $3(6) + 4(6) = 1(6)$, but $9(24) + 4(24) = 13(24)$, which does not correspond to any $c(6)$. This is the reason for requiring addition of two elements within the same ring to be defined as in Definition 2.2. To show the combining of elements from residue classes (mod 4) and (mod 6), the following table is constructed;

numbers in the table are understood to be (mod 24).

(2.14)

+	0(6)	1(6)	2(6)	3(6)	4(6)	5(6)
0(4)	0	19	14	9	4	23
1(4)	6	1	20	15	10	5
2(4)	12	7	2	21	16	11
3(4)	18	13	8	3	22	17

To further illustrate the correspondences, consider the combining of elements from residue classes of integers (mod 5), (mod 6), and (mod 8). The x values are calculated from (2.13).

$$x_1 = [1 - n_1](\text{mod } \theta_1) = [1 - 5](5) = 1(5) = 1$$
$$x_2 = [1 - n_2](\text{mod } \theta_2) = [1 - 6](30) = 25(30) = 25$$
$$x_3 = [1 - n_3](\text{mod } \theta_3) = [1 - 8](240) = 233(240) = 233$$

and

$$1(5) \to x_1\phi_1 = 1 \cdot 6 \cdot 8 \to 48(240)$$
$$1(6) \to x_2\phi_2 = 25 \cdot 8 \to 200(240)$$
$$1(8) \to x_3\phi_3 = 233 \cdot 1 \to 233(240)$$

3. Application to the mixed factorial experiment. The technique developed in the previous section for combining elements from distinct finite rings can be used in constructing confounding plans for the mixed factorial experiment, $n_1^{m}1 \times n_2^{m}2 \times \cdots n_s^{m}s$, n_j and m_j being any positive integers. The procedure follows the usual methodology employed in the symmetric factorial experiment and that used by White and Hultquist [4] in the mixed factorial experiment. Consider the mixed factorial experiment $3^2 \times 6$, two factors having three levels each and a third factor having six levels. We denote the three factors by A, B, and C. The standard procedure for confounding $(A^{q(3)}B^{r(3)}C^{s(6)})$ with blocks would be to require that all treatment combinations $(c_1(3), c_2(3), c_3(6))$ with the property $qc_1(3) + rc_2(3) + sc_3(6) = h$ be assigned to a particular block. Using the notation of White and Hultquist the following table indicates confounding plans for the $3^2 \times 6$ factorial experiment.

For example, to confound AB^2C with blocks would require that all treatment combinations with the property $c_1(\text{mod } 3) + 2c_2(\text{mod } 3) + c_3(\text{mod } 6) = h(\text{mod } 18)$ be assigned to block h. In the block denoted 0 would appear the treatment combinations [000, 120, 210], since these are the only combinations which satisfy the above equation with $h = 0$. Note that AB^2 and C would also be confounded and no other effects would be completely confounded with blocks.

4. Connection with other procedures and a limitation. To demonstrate the agreement between correspondences obtained in this paper and those obtained by White and Hultquist [4], consider the mixed factorial experiment $3^2 \times 5$, three factors, A, B, C, the first two having three levels each; the third, five levels.

TABLE 1
Confounding plans for a $3^2 \times 6$ experiment

Y_{ijk}	AB	AB^2	AC	BC	ABC	AB^2C
000	0	0	0	0	0	0
001	0	0	13	13	13	13
002	0	0	8	8	8	8
003	0	0	3	3	3	3
004	0	0	16	16	16	16
005	0	0	11	11	11	11
010	1	2	0	6	6	12
011	1	2	13	1	1	7
012	1	2	8	14	14	2
013	1	2	3	9	9	15
014	1	2	16	4	4	10
015	1	2	11	17	17	5
020	2	1	0	12	12	6
021	2	1	13	7	7	1
022	2	1	8	2	2	14
023	2	1	3	15	15	9
024	2	1	16	10	10	4
025	2	1	11	5	5	17
100	1	1	6	0	6	6
101	1	1	1	13	1	1
102	1	1	14	8	14	14
103	1	1	9	3	9	9
104	1	1	4	16	4	4
105	1	1	17	11	17	17
110	2	0	6	6	12	12
111	2	0	1	1	7	7
112	2	0	14	14	2	2
113	2	0	9	9	15	15
114	2	0	4	4	10	10
115	2	0	17	17	5	5
120	0	2	6	12	0	0
121	0	2	1	7	13	13
122	0	2	14	2	8	8
123	0	2	9	15	3	3
124	0	2	4	10	16	16
125	0	2	17	5	11	11
200	2	2	12	0	12	12
201	2	2	7	13	7	7
202	2	2	2	8	2	2
203	2	2	15	3	15	15
204	2	2	10	16	10	10
205	2	2	5	11	5	5
210	0	1	12	6	0	0
211	0	1	7	1	13	13
212	0	1	2	14	8	8
213	0	1	15	9	3	3
214	0	1	10	4	16	16
215	0	1	5	17	11	11
220	1	0	12	12	6	6
221	1	0	7	7	1	1
222	1	0	2	2	14	14
223	1	0	15	15	9	9
224	1	0	10	10	4	4
225	1	0	5	5	17	17

From (2.3) we have $x_1 = 1$, $x_2 = 11$, and the correspondences are

$$\begin{array}{ll} 0(3) \to 0(15) & 0(5) \to 0(15) \\ 1(3) \to 5(15) & 1(5) \to 11(15) \\ 2(3) \to 10(15) & 2(5) \to 7(15) \\ & 3(5) \to 3(15) \\ & 4(5) \to 14(15) . \end{array} \tag{4.1}$$

The decomposition and the confounding schemes given in Table 1 of White and Hultquist [4] are structurally equivalent to those obtained by the method outlined previously in this paper.

Raktoe [2] has illustrated his procedure with the mixed factorial experiment $2^2 \times 3 \times 5$. The correspondences given by the procedure in Section 2 are

$$\begin{array}{lll} 0(2) \to 0(30) & 0(3) \to 0(30) & 0(5) \to 0(30) \\ 1(2) \to 15(30) & 1(3) \to 20(30) & 1(5) \to 26(30) \\ & 2(3) \to 10(30) & 2(5) \to 22(30) \\ & & 3(5) \to 18(30) \\ & & 4(5) \to 14(30) . \end{array} \tag{4.2}$$

Using Definition 2.2 we see that the ring $R(3) = GF(2) \otimes GF(3) \oplus GF(5)$, i.e.,

$$\begin{array}{ccccccccccccc} GF(2) & + & GF(3) & + & GF(5) & = & C_1(30) & + & C_2(30) & + & C_3(30) & = & R(30) \\ 0 & + & 0 & + & 0 & = & 0 & + & 0 & + & 0 & = & 0 \\ 1 & + & 1 & + & 1 & = & 15 & + & 20 & + & 26 & = & 1 \\ 0 & + & 2 & + & 2 & = & 0 & + & 10 & + & 22 & = & 2 \\ 1 & + & 0 & + & 3 & = & 15 & + & 0 & + & 18 & = & 3 \end{array}$$

etc.

This decomposition is equivalent to that obtained by Raktoe [2]. The procedure described has a limitation which arises when the number of levels of a factor is not a prime or prime-powered. The finite rings worked with are finite fields when the number of levels are prime and all elements have inverses; however, in a finite ring there exist nonzero divisors. Using the procedure described in this paper, confounding plans may be found for any number of levels and any number of factors, but the effects are not mutually orthogonal. Therefore the method of analysis employed by White and Hultquist is not appropriate. The authors have been unable thus far to establish anything like orthogonality for the general case, although one of the referees feels that this can be accomplished. As it exists now, when any of the levels are not prime or prime-powered, some effects that are confounded may be mixed with others not confounded.

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