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CONSISTENT AUTOREGRESSIVE SPECTRAL ESTIMATES

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We consider an autoregressive linear process $\{x_t\}$, a one-sided moving average, with summable coefficients, of independent identically distributed variables $\{e_t\}$ with zero mean and fourth moment, such that $\{e_t\}$ is expressible in terms of past values of $\{x_t\}$. The spectral density of $\{x_t\}$ is assumed bounded and bounded away from zero. Using data $x_1, \cdots, x_n$ from the process, we fit an autoregression of order k, where $k^3/n \to 0$ as $n \to \infty$.

Assuming the order k is asymptotically sufficient to overcome bias, the autoregression yields a consistent estimator of the spectral density of $\{x_t\}$. Furthermore, assuming k goes to infinity so that the bias from using a finite autoregression vanishes at a sufficient rate, the autoregressive spectral estimates are asymptotically normal, uncorrelated at different fixed frequencies. The asymptotic variance is the same as for spectral estimates based on a truncated periodogram.

1. Introduction. There has been interest recently in estimating the spectral density function of a time series by a fitted autoregression. This approach has the advantage of estimating simultaneously a predictor and a frequency analysis for the time series. Parzen (1969) lists other advantages. His student Kromer (1970) studies the asymptotic distribution of the estimated spectral density as first the number n of observations and then the order k of autoregression go to infinity. Akaike (1969) has applied autoregressive spectral estimation with considerable success, although his theory is based on the assumption that the time series be a true finite autoregression. Consistency and asymptotic normality for autoregression estimates were proved by Mann and Wald (1943) under the assumption that the data come from an autoregression of known order, but it is rare that such an assumption can be justified. A more reasonable assumption would be that the data belong to a stationary process, with some regularity assumptions. In this case consistent estimation requires that n and k increase simultaneously.

Here, with assumptions on the regularity of the stationary process and with assumptions on the relative asymptotic rates of k and n, we show in Theorem 1 that the autoregressive spectral estimates are consistent. With further assumptions we also show in Theorem 6 that the estimates are asymptotically normal, uncorrelated at different frequencies. The limiting distribution is stated by Parzen (1969) and proved as an iterated limit by Kromer (1970). Although the limiting distribution is the same as that of periodogram estimates based on autocorrelations truncated at k (Rosenblatt (1959), Anderson (1970) page 534), the implications

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for practical spectral estimation are not clear. It is relevant that, although an autoregression gives an asymptotically efficient estimate in case the process is a finite autoregression, the periodogram gives an inefficient estimate in case the process is a finite moving average (Parzen (1971)).

2. Probability limits of autoregression estimates. Suppose that $\{\cdots x_{-1}, x_0, x_1, \cdots\}$ is a linear process:

$$x_t = e_t + b_1 e_{t-1} + b_2 e_{t-2} + \cdots, \tag{2.1}$$

where $b_1, b_2, \cdots$ are real numbers and $\{\cdots, e_{-1}, e_0, e_1, \cdots\}$ is a sequence of independent identically distributed random variables with mean $Ee_t = 0$, variance $Ee_t^2 = \sigma^2$. Assume also that $B(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is bounded and bounded away from zero for $|z| \leqq 1$. As Akutowicz (1957) has shown, this is equivalent to assuming that

$$x_t + a_1 x_{t-1} + a_2 x_{t-2} + \cdots = e_t, \tag{2.2}$$

where

$$A(z) = 1 + a_1 z + a_2 z^2 + \cdots = 1/B(z) \tag{2.3}$$

is bounded and bounded away from zero, $|z| \leqq 1$. A special case is the autoregressive-moving-average process

$$x_t + \phi_1 x_{t-1} + \cdots + \phi_m x_{t-m} = e_t + \psi_1 e_{t-1} + \cdots + \psi_j e_{t-j},$$

where $\phi(z) = 1 + \phi_1 z + \cdots + \phi_m z^m$ and $\psi(z) = 1 + \psi_1 z + \cdots + \psi_j z^j$ are nonzero for $|z| \leqq 1$, in which case $A(z) = \phi(z)/\psi(z)$ is a rational function.

Define the autocovariance $r_t = E(x_t x_{t+k})$. Then

$$r_k = (\sigma^2/2\pi) \int_{-\pi}^{\pi} e^{ik\lambda} |B(e^{i\lambda})|^2 \, d\lambda = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) \, d\lambda, \tag{2.4}$$

where $f(\lambda) = \sigma^2 |B(e^{i\lambda})|^2 / 2\pi$ is the spectral density.

Having observed $x_1, \cdots, x_n$, a least squares predictor of x_t of order k can be estimated by minimizing

$$(n-k)^{-1} \sum_{j=0}^{n-1-k} (x_{k+j+1} + c_1 x_{k+j} + \cdots + c_k x_{1+j})^2. \tag{2.5}$$

This yields the autoregression coefficients $c_1 = \hat{a}_{1k}, \cdots, c_k = \hat{a}_{kk}$, where $\hat{a}(k)' = (\hat{a}_{1k}, \cdots, \hat{a}_{kk})$ satisfies $\hat{R}(k)\hat{a}(k) = -\hat{r}(k)$, with

$$\hat{R}(k) = \sum_{j=k}^{n-1} X_j(k) X_j(k)'/(n-k), \qquad \hat{r}(k) = \sum_{j=k}^{n-1} X_j(k) x_{j+1}/(n-k),$$
$$X_j(k)' = (x_j, \cdots, x_{j-k+1}).$$

Call the minimum of (2.5) $\hat{\sigma}_k^2$. Also let

$$\hat{A}_k(z) = 1 + \hat{a}_{1k} z + \cdots + \hat{a}_{kk} z^k$$
$$\hat{f}_n(\lambda) = \hat{\sigma}_k^2 / (2\pi |\hat{A}_k(e^{i\lambda})|^2).$$

The corresponding theoretical parameters will also be needed. Let $a(k)' = (a_1, \cdots, a_k)$ and let $c_1 = a_{1k}, \cdots, c_k = a_{kk}$ be the values that minimize

$E(x_t + c_1 x_{t-1} + \cdots + c_k x_{t-k})^2$, with minimum σ_k^2. Let $r(k)' = (r_1, \cdots, r_k)$ and $\sum(k) = \mathrm{Diag}(\sigma_0^2, \sigma_1^2, \cdots, \sigma_{k-1}^2)$,

$$R(k) = \begin{pmatrix} r_0 & r_1 & \cdots & r_{k-1} \\ r_1 & r_0 & \cdots & r_{k-2} \\ \cdot & \cdot & \cdot & \cdot \\ r_{k-1} & r_{k-2} & \cdots & r_0 \end{pmatrix}, \qquad L(k) = \begin{pmatrix} 1 & 0 & & \cdots & 0 \\ a_{11} & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k-1,k-1} & \cdots & a_{1,k-1} & & 1 \end{pmatrix}.$$

Then a useful identity is

$$R(k)^{-1} = L(k)' \textstyle\sum (k)^{-1} L(k) . \tag{2.6}$$

(See Akaike (1969) Kromer (1970, page 98).)

A result which will be needed is that, if $0 < F_1 < f(\lambda) < F_2$ and $\lambda_1 < \cdots < \lambda_p$ are the eigenvalues of $R(k)$, then (Grenander and Szegö (1958, page 64))

$$2\pi F_1 \leqq \lambda_1 < \cdots < \lambda_p \leqq 2\pi F_2 . \tag{2.7}$$

We consider the difference between $a(k)$ and $\hat{a}(k)$. If $\hat{R}(k)^{-1}$ exists,

$$\begin{aligned} a(k) - \hat{a}(k) &= a(k) + \hat{R}(k)^{-1}\hat{r}(k) = \hat{R}(k)^{-1}(\hat{R}(k)a(k) + \hat{r}(k)) \\ &= \hat{R}(k)^{-1} \textstyle\sum_{j=k}^{n-1} X_j(k)(x_{j+1} + a_1 x_j + \cdots + a_k x_{j-k+1})/(n-k) \\ &= \hat{R}(k)^{-1} \textstyle\sum_{j=k}^{n-1} X_j(k) e_{j+1,k}/(n-k) , \end{aligned} \tag{2.8}$$

where $e_{j+1,k} = x_{j+1} + a_1 x_j + \cdots + a_k x_{j-k+1}$.

The difference $\hat{\sigma}_k^2 - \sigma^2$ can be expressed in terms of $a(k) - \hat{a}(k)$, letting $\hat{r}_0 = (x_{k+1}^2 + \cdots + x_n^2)/(n-k)$:

$$\begin{aligned} \hat{\sigma}_k^2 - \sigma^2 &= (n-k)^{-1} \textstyle\sum_{j=k+1}^{n} (x_j + \hat{a}_{1k} x_{j-1} + \cdots + \hat{a}_{nk} x_{j-k})^2 - \sigma^2 \\ &= \hat{r}_0 + \hat{a}(k)'\hat{r}(k) - \sigma^2 \\ &= (\hat{r}_0 - r_0) + (\hat{a}(k) - a(k))' r(k) + a(k)'(\hat{r}(k) - r(k)) \\ &\quad + (\hat{a}(k) - a(k))'(\hat{r}(k) - r(k)) - \textstyle\sum_{j=k+1}^{\infty} a_j r_j . \end{aligned} \tag{2.9}$$

Furthermore, Hannan (1960, page 39) has shown that

$$\begin{aligned} (n-t)&E((n-t)^{-1} \textstyle\sum_{j=t}^{n-1} x_j x_{j-t} - r_t)^2 \\ &= \textstyle\sum_{s=-(n-t)+1}^{n-t-1} (1 - |s|/(n-t)) \\ &\quad \times (r_s^2 + r_{s-t} r_{s+t} + K_4 \textstyle\sum_{j=0}^{\infty} b_j b_{j+s} b_{j+t} b_{j+s+t}) , \end{aligned} \tag{2.10}$$

where K_4 is the fourth cumulant of e_t. Thus (2.10) is bounded by

$$\begin{aligned} &\textstyle\sum_{s=-\infty}^{\infty} (r_s^2 + |r_{s-t} r_{s+t}| + |K_4| \sum_{j=0}^{\infty} |b_j b_{j+t}| \sum_{s=-\infty}^{\infty} |b_{j+s} b_{j+s+t}|) \\ &\leqq \textstyle\sum_{s=-\infty}^{\infty} r_s^2 + (\sum_{s=-\infty}^{\infty} r_s^2)^{\frac{1}{2}} (\sum_{s=-\infty}^{\infty} r_s^2)^{\frac{1}{2}} \\ &\quad + |K_4| \textstyle\sum_{j=0}^{\infty} |b_j b_{j+t}| (\sum_{s=0}^{\infty} b_s^2)^{\frac{1}{2}} (\sum_{s=0}^{\infty} b_s^2)^{\frac{1}{2}} \\ &\leqq 2 \textstyle\sum_{s=-\infty}^{\infty} r_s^2 + |K_4| (\sum_{s=0}^{\infty} b_s^2)^2 . \end{aligned} \tag{2.11}$$

The sum $\sum r_s^2$ is bounded if $f(\lambda) = 1/2\pi \sum r_s e^{is\lambda}$ is bounded.

The following three lemmas will be needed.

LEMMA 1. *If $\{u_t\}$ and $\{v_t\}$ are linear processes,*

$$u_t = c_0 e_t + c_1 e_{t-1} + \cdots, \qquad v_t = d_0 e_t + d_1 e_{t-1} + \cdots, \qquad \text{with}$$
$$C^2 = c_0^2 + c_1^2 + \cdots, \qquad D^2 = d_0^2 + d_1^2 + \cdots,$$

where $\{e_t\}$ is an independent identically distributed sequence with $E(e_t^4) < \infty$, then

$$E(\textstyle\sum_{t=1}^m u_t v_t)^2 \leqq 3Ee_t^4 m^2 C^2 D^2 .$$

PROOF. By expanding the product $u_0 v_0 u_j v_j$ in terms of $e_j, e_{j-1}, \cdots$, cancelling terms having odd powers of e_t, and using $(E(e_t^2))^2 \leqq Ee_t^4$, we obtain

$$\begin{aligned} &E(u_0 v_0 u_j v_j) \\ &\quad \leqq Ee_t^4[(\textstyle\sum_{t=0}^\infty |c_t d_t|)^2 + \sum_{t=0}^\infty |c_t c_{t+j}| \sum_{t=0}^\infty |d_t d_{t+j}| + \sum_{t=0}^\infty |c_t d_{t+j}| \sum_{t=0}^\infty |c_{t+j} d_t|] \\ &\quad \leqq Ee_t^4[C^2D^2 + C^2D^2 + C^2D^2] , \end{aligned}$$

applying the Schwarz inequality. Therefore,

$$E(\textstyle\sum_{t=1}^m u_t v_t)^2 = \sum_{t=1}^m \sum_{j=1}^m E(u_t v_t u_j v_j) \leqq 3Ee_t^4 m^2 C^2 D^2 .$$

LEMMA 2. *If $u_t = c_0 e_t + c_1 e_{t-1} + \cdots$, as in Lemma 1,*

$$e_{tk} = x_t + a_1 x_{t-1} + \cdots + a_k x_{t-k}, \qquad e_t = x_t + a_1 x_{t-1} + \cdots ,$$

and the spectral density of $\{x_t\}$ is bounded, $f(\lambda) < F_2$, then

$$E(\textstyle\sum_{t=k+1}^n u_t(e_{tk} - e_t))^2 \leqq 6\pi E(e_t/\sigma)^4 F_2 E(u_t^2)(n-k)^2(a_{k+1}^2 + a_{k+2}^2 + \cdots) . \tag{2.12}$$

PROOF. We can write

$$\begin{aligned} e_{tk} - e_t &= a_{k+1} x_{t-k-1} + a_{k+2} x_{t-k-2} + \cdots \\ &= d_{k+1,k} e_{t-k-1} + d_{k+2,k} e_{t-k-2} + \cdots . \end{aligned}$$

Therefore, letting $D_k^2 = d_{k+1,k}^2 + d_{k+2,k}^2 + \cdots$, by Lemma 1,

$$E(\textstyle\sum_{t=k+1}^n u_t(e_{tk} - e_t))^2 \leqq 3Ee_t^4 C^2 D_k^2 (n-k)^2 .$$

Now

$$\begin{aligned} \sigma^2 D_k^2 &= E(a_{k+1} x_{t-k+1} + a_{k+2} x_{t-k+2} + \cdots)^2 \\ &= \textstyle\int_{-\pi}^{\pi} |a_{k+1} e^{i(k+1)\lambda} + a_{k+2} e^{i(k+2)\lambda} + \cdots|^2 f(\lambda)\, d\lambda \\ &\leqq F_2 \textstyle\int_{-\pi}^{\pi} |a_{k+1} e^{i(k+1)\lambda} + a_{k+2} e^{i(k+2)\lambda} + \cdots|^2\, d\lambda \\ &= 2\pi F_2 (a_{k+1}^2 + a_{k+2}^2 + \cdots) \end{aligned}$$

and this completes the proof.

Applying Lemma 2 to the sum in (2.8), we find

$$E\|\textstyle\sum_{j=k}^{n-1} X_j(k)(e_{j+1,k} - e_{j+1})/(n-k)\|^2 \leqq \text{const}\; k(a_{k+1}^2 + a_{k+2}^2 + \cdots) . \tag{2.13}$$

In the following lemma, a matrix norm will be used: if $C = (c_{ij})$ is a matrix, then

$$\|C\| = \sup \|Cx\| , \qquad \|x\| \leqq 1 ,$$

using the Euclidean norm for the vector x, $\|x\|^2 = x'x$. Thus $\|C\|^2 \leqq \sum_{i,j} c_{ij}^2$, and $\|C\|$ also is dominated by the largest modulus of the eigenvalues of C.

Therefore, (2.7) implies

$$(2.14) \qquad ||R(k)|| \leqq 2\pi F_2\,, \qquad ||R(k)^{-1}|| \leqq 1/(2\pi F_1)\,.$$

LEMMA 3. *Let* $\{x_t\}$ *satisfy* (2.2) *with* $Ee_t^4 < \infty$. *Assume* $0 < F_1 < f(\lambda) < F_2$, *and let* k *be chosen as a function of* n *so* $k^3/n \to 0$. *Then* $k^{\frac{1}{2}}||\hat{R}(k)^{-1} - R(k)^{-1}||$ *converges to zero in probability.*

PROOF. We have, denoting $||R(k)^{-1}|| = p$, $||\hat{R}(k)^{-1} - R(k)^{-1}|| = q$, $||\hat{R}(k) - R(k)|| = Q$,

$$\begin{aligned} q &= ||\hat{R}(k)^{-1} - R(k)^{-1}|| = ||\hat{R}(k)^{-1}(R(k) - \hat{R}(k))R(k)^{-1}|| \\ &\leqq ||\hat{R}(k)^{-1}||\,||R(k) - \hat{R}(k)||\,||R(k)^{-1}|| \\ &\leqq (p + q)Qp \end{aligned}$$

so, if $pQ < 1$,

$$(2.15) \qquad q \leqq p^2Q/(1 - pQ)\,.$$

By (2.10) and (2.11), $(n - t)E((n - t)^{-1} \sum_{j=t}^{n-1} x_j x_{j-t} - r_t)^2$ is bounded by a constant P and thus

$$(2.16) \qquad E(Q^2) \leqq pk^2/(n - k)\,.$$

Then

$$E(k^{\frac{1}{2}}Q)^2 \leqq k^3p/(n - k) \to 0$$

because $k^3/n \to 0$ by assumption. Furthermore, (2.14) implies that p is bounded. Thus, by (2.15) and (2.16)

$$k^{\frac{1}{2}}q \leqq p^2k^{\frac{1}{2}}Q/(1 - pQ) \to 0\,,$$

in probability, and this completes the proof.

Note that by (2.8), (2.13), (2.14), Lemma 3, and $E||\sum_{j=k}^{n-1} X_j(k)e_{j+1}||^2 = k(n - k)\sigma^2Ex_t^2$,

$$\begin{aligned} (2.17) \qquad ||\hat{a}(k) - a(k)|| \leqq\; & ||\hat{R}(k)^{-1} - R(k)^{-1}||\,||\textstyle\sum_{j=k}^{n-1} X_j(k)e_{j+1,k}/(n - k)|| \\ &+ ||R(k)^{-1}||\,||\textstyle\sum_{j=k}^{n-1} X_j(k)(e_{j+1,k} - e_{j+1})/(n - k)|| \\ &+ ||R(k)^{-1}||\,||\textstyle\sum_{j=k}^{n-1} X_j(k)e_{j+1}/(n - k)|| \end{aligned}$$

converges to zero in probability under the assumptions of Lemma 3, if also $k(a_{k+1}^2 + a_{k+2}^2 + \cdots)$ goes to zero.

In Theorem 1 and Theorem 2, a regularity condition on the spectral density $f(\lambda)$ is implied by assumptions (i) and (iv). By Wiener's Theorem (Zygmund (1959, page 245)) and (2.3), $\sum |a_i| < \infty$ and $A(e^{i\lambda}) \neq 0$ imply that $\sum |b_i| < \infty$. Thus $B(e^{i\lambda})$ is continuous and nonzero and it follows that $f(\lambda)$ is continuous, and there are constants F_1 and F_2 such that

$$(2.18) \qquad 0 < F_1 < f(\lambda) < F_2\,.$$

THEOREM 1. *Let* $x_t = e_t + b_1e_{t-1} + \cdots$ *where* $\{e_t\}$ *is a sequence of independent* *identically distributed random variables, mean* 0, *variance* σ^2. *Assume* $B(z) = 1 + b_1 z + \cdots = 1/(1 + a_1 z + \cdots) = 1/A(z)$ *is nonzero for* $|z| < 1$. *Assume*

(i) $A(e^{i\lambda})$ *is nonzero*, $-\pi < \lambda \leqq \pi$.

(ii) $E(e_t^4) < \infty$.

(iii) *The choice of* k *in terms of* n *is such that* $k^3/n \to 0$.

(iv) *The choice of* k *in terms of* n *is such that* $k^{\frac{1}{2}}(|a_{k+1}| + |a_{k+2}| + \cdots) \to 0$.

Then $\hat{f}_n(\lambda)$ *converges to* $f(\lambda)$ *in probability.*

PROOF. It will be sufficient to show that $\hat{\sigma}_k^2 \to \sigma^2$ and $\hat{A}_k(e^{i\lambda}) \to A(e^{i\lambda})$ in probability. Let $q(k) = (q_{1k}, \cdots, q_{kk})'$ be such that $||q(k)||^2 = (q_{1k}^2 + \cdots + q_{kk}^2) \leqq k$. Then by (2.8),

$$
\begin{aligned}
&|q(k)'(\hat{a}(k) - a(k))| \\
&\quad \leqq ||q(k)||\,||\hat{R}(k)^{-1} - R(k)^{-1}||\,||\textstyle\sum X_j(k)e_{j+1,k}/(n-k)|| \\
&\qquad + |\textstyle\sum_{j=k}^{n-1} q(k)R(k)^{-1}X_j(k)e_{j+1}/(n-k)| \\
&\qquad + ||\textstyle\sum_{j=k}^{n-1} q(k)R(k)^{-1}X_j(k)(e_{j+1,k} - e_{j+1})/(n-k)||
\end{aligned} \tag{2.19}
$$

where the first term on the right converges to zero in probability by (2.13), Lemma 3, and condition (iv). The second term has mean square

$$q(k)'R(k)^{-1}q(k)\sigma^2/(n-k)$$

and therefore converges to zero in probability by (2.14) and condition (iii).

To see that the third term converges to zero in probability, we use Lemma 2 with $u_t = q(k)'R(k)^{-1}X_t(k)$, where $\operatorname{Var} u_t = q(k)'R(k)^{-1}q(k) \leqq k/(2\pi F_1)$ by (2.14). Then

$$E(\textstyle\sum_{j=k}^{n-1} u_j(e_{j+1,k} - e_{j+1})/(n-k))^2 \leqq \text{const } k(a_{k+1}^2 + a_{k+2}^2 + \cdots)$$

which converges to zero by condition (iv).

For the difference,

$$\hat{A}_k(e^{i\lambda}) - A(e^{i\lambda}) = \textstyle\sum_{j=1}^{k} (\hat{a}_{jk} - a_j)e^{ij\lambda} - \sum_{j=k+1}^{\infty} a_j e^{ij\lambda},$$

the real and imaginary parts of the first sum are of the form (2.19), and therefore converge to zero in probability, while the second sum converges to zero by (iv).

To show that $\hat{\sigma}_k^2 \to \sigma^2$ in probability, we apply (2.10), (2.11), (2.18) to the terms on the right-hand side of (2.9). The first term on the right-hand side of (2.9) converges to zero in probability, by (2.10), (2.11), and (2.18). The second term converges to zero in probability by (2.17). By (2.10), (2.11) and (2.18),

$$E||\hat{r}(k) - r(k)||^2 \leqq \text{const } k/(n-k)\,. \tag{2.20}$$

Thus the third term, $a(k)'(\hat{r}(k) - r(k))$, converges to zero in probability. The fourth term converges in probability to zero by (2.17) and (2.20). The fifth term converges to zero by (iv). The proof of Theorem 1 is complete.

THEOREM 2. *Assume that* $\{x_t\}$ *satisfies* $x_t = e_t + b_1 e_{t-1} + \cdots$, *where* $\{e_t\}$ *is a sequence of independent identically distributed random variables with mean* 0 *and* *variance* σ^2. *Let* $B(z) = 1 + b_1 z + \cdots = 1/(1 + a_1 z + \cdots) = 1/A(z)$ *be nonzero in the unit circle. Assume*

(i) $A(e^{i\lambda})$ *is nonzero*, $-\pi < \lambda \leqq \pi$.
(ii) $E(e_t^4) < \infty$.
(iii) *The choice of* k *in terms of* n *is such that* $k^3/n \to 0$.
(iv) *The choice of* k *in terms of* n *is such that* $n^{\frac{1}{2}}(|a_{k+1}| + |a_{k+2}| + \cdots) \to 0$.
(v) $\gamma(k) = (\gamma_{1k}, \cdots, \gamma_{kk})'$ *is such that* $\|\gamma(k)\|^2 = \gamma_{1k}^2 + \cdots + \gamma_{kk}^2$ *is bounded.*

Then the difference

$$(n-k)^{\frac{1}{2}}\gamma(k)'(\hat{a}(k) - a(k)) - \gamma(k)'R(k)^{-1}\sum_{j=k}^{n-1} X_j(k)e_{j+1}/(n-k)^{\frac{1}{2}},$$

converges in probability to zero.

Proof. By (2.8),

$$\begin{aligned}(n-k)^{\frac{1}{2}}\gamma(k)'(a(k) - \hat{a}(k)) &- \gamma(k)'R(k)^{-1}\textstyle\sum_{j=k}^{n-1} X_j(k)e_{j+1}(n-k)^{-\frac{1}{2}} \\ &= \gamma(k)(\hat{R}(k)^{-1} - R(k)^{-1})\textstyle\sum_{j=k}^{n-1} X_j(k)e_{j+1,k}(n-k)^{-\frac{1}{2}} \\ &\quad + \textstyle\sum_{j=k}^{n-1}\gamma(k)R(k)^{-1}X_j(k)(e_{j+1,k} - e_{j+1})(n-k)^{-\frac{1}{2}},\end{aligned}$$

where the first term on the right converges to zero in probability by (2.13), Lemma 3, and condition (iv). For the second term we use Lemma 2 with $u_j = \gamma(k)R(k)^{-1}X_j(k)$, where $\operatorname{Var} u_j = \gamma(k)'R(k)^{-1}\gamma(k) \leqq 1/(2\pi F_1)$ by (2.14). Then

$$E\left(\textstyle\sum_{j=k}^{n-1} u_j(e_{j+1,k} - e_{j+1})(n-k)^{-\frac{1}{2}}\right)^2 \leqq \text{const}\,(n-k)(a_{k+1}^2 + a_{k+2}^2 + \cdots),$$

which converges to zero by condition (iv).

3. Asymptotic normality. Having shown in Theorem 2 that the distribution of the autoregression coefficients could be studied using a quadratic expression with the same asymptotic distribution, we now investigate the distribution of this expression. It will be convenient here to restrict the γ's of Theorem 2 to be trigonometric; otherwise, the assumptions here are weaker. Theorem 3, in which we compute the asymptotic variance of the equivalent expressions, is preceded by a lemma.

Lemma 4. *Let* $\{x_t\}$ *satisfy* (2.2) *with* $\sum |a_t| < \infty$. *Define* $A_k(z) = 1 + a_{1k}a + \cdots + a_{kk}z^k$. *Then* $\lim_{k\to\infty} A_k(z) = A(z)$, $|z| \leqq 1$.

Proof. By Theorem 2.2 of Baxter (1962), taking $\lambda = 0$ in his theorem and letting $a_{0k} = 1$, $a_0 = 1$,

$$\textstyle\sum_{m=0}^{k} |a_{mk}/\sigma_k^2 - a_m/\sigma^2| \leqq \text{const}\sum_{m=k+1}^{\infty} |a_m|/\sigma^2.$$

The proof follows from this and the result (Grenander and Szegö (1958, pages 44, 183))

$$\lim_{k\to\infty}\sigma_k^2 = \sigma^2. \tag{3.1}$$

Theorem 3. *Let* $f(\lambda) = \sigma^2/(2\pi|1 + a_1e^{i\lambda} + \cdots|^2)$ *satisfy* $0 < F_1 < f(\lambda) < F_2$. *Assume* $|a_1| + |a\,| + \cdots < \infty$. *Let* $c_1 = \bar{d}_1, \cdots, c_p = \bar{d}_p$ *be complex numbers,* c_0

and d_0 *real,*

$$q_j = c_0 + c_1 e^{ij\lambda_1} + \cdots + c_p e^{ij\lambda_p} + d_0 e^{ij\pi} + d_1 e^{-ij\lambda_1} + \cdots + d_p e^{-ij\lambda_p}\,, \quad j = 1, 2, \cdots, 0 < \lambda_1 < \cdots < \lambda_p < \pi\,, \tag{3.2}$$

and let $\gamma(k) = (q_1, \cdots, q_k)'/k^{\frac{1}{2}}$.

Then

$$\lim_{k\to\infty} \gamma(k)'R(k)^{-1}\gamma(k) = v\,, \tag{3.3}$$

where

$$v = c_0^2/2\pi f(0) + c_1 d_1/\pi f(\lambda_1) + \cdots + c_p d_p/\pi f(\lambda_p) + d_0^2/2\pi f(\pi)\,. \tag{3.4}$$

PROOF. In order to prove (3.3), it will be sufficient to evaluate the limit of

$$k^{-1}\alpha(k)'R(k)^{-1}\beta(k)\,, \tag{3.5}$$

where

$$\alpha(k)' = (1, e^{i\lambda}, \cdots, e^{(k-1)i\lambda})\,, \qquad \beta(k)' = (1, e^{i\mu}, \cdots, e^{i(k-1)\mu})\,, \quad -\pi < \lambda, \mu \leqq \pi\,. \tag{3.6}$$

By (2.6), (3.5) can be written

$$k^{-1} \sum_{j=0}^{k-1} A_j(e^{-i\lambda})A_j(e^{-i\mu})e^{j(\lambda+\mu)i}/\sigma_j^2\,. \tag{3.7}$$

Thus, in the special cases $\lambda = -\mu$ or $\lambda = \mu = \pi$, using Lemma 4 and (3.1),

$$\begin{aligned} \lim_{k\to\infty} k^{-1}\alpha(k)'R(k)^{-1}\beta(k) &= \lim_{j\to\infty} |A_j(e^{i\lambda})|^2/\sigma_j^2 \\ &= |A(e^{i\lambda})|^2/\sigma^2 \\ &= (2\pi f(\lambda))^{-1}\,. \end{aligned} \tag{3.8}$$

On the other hand, if $e^{-i\lambda} \neq e^{i\mu}$, denote

$$A_j(e^{-i\lambda})A_j(e^{-i\mu})/\sigma_j^2 = u_j\,, \qquad e^{j(\lambda+\mu)i} = v_j\,, \qquad v_0 + \cdots + v_j = V_j\,. \tag{3.9}$$

Then the limit of (3.7) becomes, using Lemma 4 and (3.1).

$$\begin{aligned} \lim_{k\to\infty} k^{-1} \sum_{j=0}^{k-1} u_j v_j &= \lim_{k\to\infty} k^{-1}(\sum_{j=0}^{k-2} (u_j - u_{j+1})V_j + u_{k-1}V_{k-1}) \\ &= \lim_{k\to\infty} k^{-1} \sum_{j=0}^{k-2} (u_j - u_{j+1})V_j \\ &= \lim_{j\to\infty} (u_j - u_{j+1})V_j \\ &= 0\,. \end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10), we get (3.3), and this proves the theorem.

In Theorem 4, we show that the equivalent quadratic expression found in Theorem 2 is asymptotically normal. This is similar to the asymptotic behavior of the smoothed periodogram, as given by Anderson (1970, page 534). Note that nothing is assumed here about the zeros of $B(z)$ within the unit circle, and the assumptions $k/n \to 0$ and $\sum |b_i| < \infty$ represent a considerable weakening of assumptions (iii) and (iv) in Theorem 2.

THEOREM 4. *Let*

$$s_n = \gamma(k)'R(k)^{-1} \sum_{j=k}^{n-1} X_j(k)e_{j+1}(n-k)^{-\frac{1}{2}}\,,$$

where $\gamma(k)' = q(k)'/k^{\frac{1}{2}} = (q_1, \cdots, q_k)/k^{\frac{1}{2}}$ *and* $\{q_j\}$, *as given by* (3.2), *is trigonometric, not identically zero. Assume that* $x_t = e_t + b_1 e_{t-1} + \cdots$, *where* $\{e_t\}$ *is a sequence of independent identically distributed random variables with* $Ee_t = 0$ *and*

(i) $|b_1| + |b_2| + \cdots < \infty$.
(ii) $E(e_t^4) < \infty$.
(iii) *The choice of* k *in terms of* n *is such that* $k \to \infty$ *and* $k/n \to 0$.
(iv) $B(e^{i\lambda}) \neq 0$, $-\pi < \lambda \leqq \pi$.

Then s_n *is asymptotically normal with mean* 0 *and variance* $v\sigma^2$, *where* v *is given by Theorem* 3.

PROOF. Assume first that $\{x_t\}$ is a finite moving average of order m. Letting

$$u_{jn} = q(k)'R(k)^{-1}X_j(k) , \qquad j = k, \cdots, n-1 ,$$

u_{jn} is a linear combination of $e_{j-k-m+1}, \cdots, e_j$. To see that the coefficients of this linear combination are bounded, we use the representation (2.6). Because $\{x_t\}$ is assumed to be a finite moving average with nonzero spectral density, Theorem 3.2 of Baxter (1962) shows that there are constants a, b, c, $0 < a < 1$, $0 < b < 1$, $0 < c$, such that $|a_{jk}| < c(a^j + b^k)$. This gives a bound for the sum of the absolute values of the entries in any row or column of $L(k)$. It follows then, since the sequence $\{q_j\}$ is bounded, that u_{jn} is a linear combination of $e_{j-k-m+1}, \cdots, e_j$ with bounded coefficients.

Calling this bound B, letting $w_{jn} = u_{j+k-1,n} e_{j+k} k^{-\frac{1}{2}}$, $j = 1, \cdots, n-k$, and letting $K = k + m$, w_{jn} can be written

$$w_{jn} = L(e_{j-m}, \cdots, e_{j+k-1}) e_{j+k} k^{-\frac{1}{2}} \tag{3.11}$$

where $L(e_{j-m}, \cdots, e_{j+k-1})$ is a linear combination of $e_{j-m}, \cdots, e_{j+k-1}$ with bounded coefficients. It follows that

$$Ew_{jn}^4 \leqq B^4(K + 3K(K-1))(Ee_t^4)^2/k^2 , \tag{3.12}$$

so Ew_{jn}^4 is bounded.

Let N be an integer such that $N/(n-K) \to 0$, $K/N \to 0$ (the largest integer less than $((n-K)K)^{\frac{1}{2}}$ will do), and let M be the greatest integer less than or equal to $(n-K)/N$. Define the random variables

$$\begin{aligned} z_{1n} &= (w_{1n} + \cdots + w_{N-K,n})/N^{\frac{1}{2}} \\ z_{2n} &= (w_{N+1,n} + \cdots + w_{2N-K,n})/N^{\frac{1}{2}} \\ &\cdots\cdots \\ z_{Mn} &= (w_{(M-1)N+1,n} + \cdots + w_{MN-K,n})/N^{\frac{1}{2}} , \end{aligned}$$

which are independent, by (3.11). Since $w_{1n}, \cdots, w_{n-k,n}$ are uncorrelated,

$$\lim_{n\to\infty} \operatorname{Var} z_{jn} = \lim_{n\to\infty} \frac{N-K}{N} \operatorname{Var} w_{1n} = v\sigma^2 ,$$

using Theorem 3.

In order to apply the Lyapounov central limit theorem (Anderson (1970, page

426)) to the z's we wish to show that Ez_{jn}^4 is bounded,

$$Ez_{jn}^4 = N^{-2} \sum_{q,r,s,t=1}^{N-K} E(w_{qn} w_{rn} w_{sn} w_{tn}) . \tag{3.13}$$

In this sum the nonzero terms are at most

(a) $(N - K)$ terms of the form Ew_{tn}^4,
(b) $\frac{1}{2}(N - K)(N - K - 1)6$ terms of the form $Ew_{tn}^2 w_{sn}^2$, $s \neq t$,
(c) $\frac{1}{2}(N - K)(N - K - 1)4$ terms of the form $Ew_{tn} w_{sn}^3$, $s > t$,
(d) $(N - K)(K)(K)12$ terms of the form $Ew_{tn} w_{sn} w_{qn}^2$, $q > s > t$.

In (d) we have allowed for the independence of w_{tn}, w_{sn}, w_{qn} for $q - s > K$, $s - t > K$. By (3.11) the nonzero terms in (d) contain $E(e_{t+k}^2 e_{s+k}^2 e_{s+i}^2 e_{q+k}^2)$ where $k - K \leqq i < k$, and

$$E(w_{tn} w_{sn} w_{qn}^2) \leqq K(Ee_j^4)^2 B^4/k^2 . \tag{3.14}$$

Furthermore, $K/k \to 1$ and $K/N \to 0$, so the sum of the terms in (d) is bounded by a constant times N^2. Also, by (3.12), the sums of the terms in (a), (b), and (c) are bounded by a constant times N^2. It follows that (3.13) is bounded. Now

$$\begin{aligned} \operatorname{Var} z_{jn} &= \frac{N-k}{N} \operatorname{Var} w_{1n} \\ &= ((N - k)/N)(\sigma^2/k) \operatorname{Var} u_{kn} \\ &= ((N - k)/N)(\sigma^2/k) q(k)' R(k)^{-1} R(k) R(k)^{-1} q(k) \\ &\to v\sigma^2 \end{aligned} \tag{3.15}$$

by Theorem 3. Thus, by the Lyapounov central limit theorem, $(1/M^{\frac{1}{2}}) \sum_{j=1}^{M} z_{jn}$ is asymptotically normal, mean 0, variance $v\sigma^2$. Furthermore,

$$(1/M^{\frac{1}{2}}) \sum_{j=1}^{M} z_{jn} - (1/(n - k)^{\frac{1}{2}}) \sum_{j=1}^{n-k} w_{jn}$$

has limiting variance zero, and therefore

$$s_n = (1/(n - k)^{\frac{1}{2}}) \sum_{j=1}^{n-k} w_{jn}$$

is asymptotically normal, mean 0, variance $v\sigma^2$.

We now remove the assumption that $\{x_t\}$ be a finite moving average. Writing x_t in the form

$$x_t = e_t + b_1 e_{t-1} + \cdots ,$$

let $\{x_{tm}\}$ be the finite moving average

$$x_{tm} = e_t + b_1 e_{t-1} + \cdots + b_m e_{t-m} .$$

By (i), $\sum |b_i| < \infty$, the spectral density of $\{x_{tm}\}$,

$$f_m(\lambda) = (\sigma^2/2\pi)|1 + b_1 e^{i\lambda} + \cdots + b_m e^{mi\lambda}|^2$$

converges uniformly to the spectral density $f(\lambda)$ of $\{x_t\}$, and is therefore bounded away from zero for sufficiently large m. Then, by Theorem 3, the limit

$$\lim_{k\to\infty} \gamma(k)' R_m(k)^{-1} \gamma(k) = v_m ,$$

where $R_m(k)$ is the covariance matrix of $x_{1m}, \cdots, x_{km}$, exists.

Define $X_{jm}(k) = (x_{jm}, \cdots, x_{j-k+1,m})'$ and

$$(3.16) \qquad \begin{aligned} s_n &= (n-k)^{-\frac{1}{2}}\gamma(k)'R(k)^{-1} \textstyle\sum_{j=k}^{n-1} X_j(k)e_{j+1} \\ s_{nm} &= (n-k)^{-\frac{1}{2}}\gamma(k)'R_m(k)^{-1} \textstyle\sum_{j=k}^{n-1} X_{jm}(k)e_{j+1} . \end{aligned}$$

So far, it has been proved that s_{nm} is asymptotically normal, mean 0, variance $v_m\sigma^2$. According to Anderson's (1970) Corollary 7.7.1, it will be sufficient to prove the theorem to show that $v_m \to v$ and Var $(s_n - s_{nm})$ converges to zero, uniformly in n, as $m \to \infty$.

Now

$$(3.17) \qquad \begin{aligned} &\mathrm{Var}\,(s_n - s_{nm}) \\ &\quad = \mathrm{Var}\ \textstyle\sum_{j=k}^{n-1} \gamma(k)'(R(k)^{-1}X_j(k) - R_m(k)^{-1}X_{jm}(k))e_{j+1}(n-k)^{-\frac{1}{2}} \\ &\quad = \mathrm{Var}\,[\gamma(k)'R(k)^{-1}X_j(k) - \gamma(k)'R_m(k)^{-1}X_{jm}(k)\}\sigma^2 \\ &\quad = \sigma^2\,\mathrm{Var}\,\{[\gamma(k)'R(k)^{-1}(X_j(k) - X_{jm}(k)] \\ &\qquad + [\gamma(k)'(R(k)^{-1} - R_m(k)^{-1})X_{jm}(k)]\} . \end{aligned}$$

Thus it will be sufficient to show that the variances of the two square-bracketed terms go to zero. For the first of these,

$$(3.18) \qquad \mathrm{Var}\,[\gamma(k)'R(k)^{-1}(X_j(k) - X_{jn}(k))] = \gamma(k)'R(k)^{-1}R_d(k)R(k)^{-1}\gamma(k) ,$$

where $R_d(k)$ is the covariance matrix of $x_j - x_{jm}, \cdots, x_{j-k+1} - x_{j-k+1,m}$. By (2.14), $||R_d(k)||$ is bounded by 2π times the maximum of the spectral density $f_d(\lambda) = (1/2\pi)|b_{m+1}e^{\iota\lambda} + b_{m+2}e^{2\iota\lambda} + \cdots|^2\sigma^2$ of the process $\{x_t - x_{tm}\}$, which converges to zero uniformly by (i). By (2.14), $||R(k)^{-1}||$ is bounded by 2π times the reciprocal of the minimum of the spectral density $f(\lambda)$, which is positive by (iv). Thus (3.18) is bounded by

$$||\gamma(k)||^2||R(k)^{-1}||^2||R_d(k)||$$

and converges to zero, uniformly in n, as $m \to \infty$.

For the second expression in square brackets in (3.17),

$$(3.19) \qquad \begin{aligned} &\mathrm{Var}\,[\gamma(k)'(R(k)^{-1} - R_m(k)^{-1})X_{jm}] \\ &\quad = \gamma(k)'(R(k)^{-1} - R_m(k)^{-1})R_m(k)(R(k)^{-1} - R_m(k)^{-1})\gamma(k) \\ &\quad \leqq ||\gamma(k)||^2||R(k)^{-1} - R_m(k)^{-1}||^2||R_m(k)||^2 . \end{aligned}$$

Now

$$\begin{aligned} ||R(k)^{-1} - R_m(k)^{-1}|| &= ||R(k)^{-1}(R_m(k) - R(k)R_m(k)^{-1})|| \\ &\leqq ||R(k)^{-1}||\,||R_m(k) - R(k)||\,||R_m(k)^{-1}|| . \end{aligned}$$

By (2.14), $||R_m(k)||$ is bounded because $f_m(\lambda)$ is bounded, since $f_m(\lambda)$ converges to $f(\lambda)$ uniformly. Also, by (2.14), $||R(k)^{-1}||$ and $||R_m(k)^{-1}||$ are bounded, since $f_m(\lambda) > \frac{1}{2}\min_\lambda f(\lambda) > 0$ for m sufficiently large. Furthermore, if $\alpha' = (\alpha_1, \cdots, \alpha_k)$,

$$\begin{aligned} ||R_m(k) - R(k)|| &= \sup_{||\alpha||=1} |\alpha'(R_m(k) - R(k))\alpha| \\ &= \sup_{||\alpha||=1} |\textstyle\int_{-\pi}^{\pi} |\alpha_1 e^{\iota\lambda} + \cdots + \alpha_k e^{\iota k\lambda}|^2(f_m(\lambda) - f(\lambda))\, d\lambda| \\ &\leqq 2\max_\lambda |f_m(\lambda) - f(\lambda)| \to 0 \qquad \text{as } m \to \infty . \end{aligned}$$

Thus (3.19) goes to zero uniformly in n, as $m \to \infty$.

This completes the proof that Var $(s_n - s_{nm})$ converges to zero uniformly in n as $m \to \infty$. Furthermore, denoting standard deviation by SD,

$$|v^{\frac{1}{2}} - (v_m)^{\frac{1}{2}}|\sigma = \lim_{n\to\infty} |\mathrm{SD}s_n - \mathrm{SD}s_{nm}| \leqq \sup_n \mathrm{SD}(s_n - s_{nm}) \to 0 \quad \text{as } m \to \infty ,$$

so $v_m \to v$ as $m \to \infty$. We conclude from Anderson's (1970) Corollary 7.7.1 that s_n is asymptotically normal, mean 0, variance $v\sigma^2$, and Theorem 4 is proved.

4. Asymptotically normal spectral estimates. Here we apply the results of the previous theorems to the autoregressive estimate of the spectral density. Let

$$(4.1) \qquad \begin{aligned} \hat{C}_k(\lambda) &= 1 + \textstyle\sum_{j=1}^{k} \hat{a}_{jk} \cos j\lambda , & C(\lambda) &= 1 + \textstyle\sum_{j=1}^{\infty} a_j \cos j\lambda , \\ \hat{S}_k(\lambda) &= \textstyle\sum_{j=1}^{k} \hat{a}_{jk} \sin j\lambda , & S(\lambda) &= \textstyle\sum_{j=1}^{\infty} a_j \sin j\lambda . \end{aligned}$$

A similar result to Theorem 5 is stated by Parzen (1969), page 403.

THEOREM 5. *Assume that* $\{x_t\}$ *satisfies* $x_t = e_t + b_1 e_{t-1} + \cdots$, *where* $\{e_t\}$ *is a sequence of independent, identically distributed random variables with mean* 0 *and variance* σ^2. *Let* $B(z) = 1 + b_1 z + \cdots = 1/(1 + a_1 z + \cdots) = 1/A(z)$ *be nonzero in the unit circle and let* $f(\lambda) = \sigma^2|B(e^{i\lambda})|^2/2\pi$. *Assume* (i)—(iv) *of Theorem* 2, *with* $k \to \infty$.

If $0 < \lambda_1 < \cdots < \lambda_p < \pi$, *then the joint distribution of*

$$(4.2) \qquad \begin{aligned} &(n/k)^{\frac{1}{2}}(\hat{C}_k(0) - C(0)), (n/k)^{\frac{1}{2}}(\hat{C}_k(\lambda_1) - C(\lambda_1)), \cdots, (n/k)^{\frac{1}{2}}(\hat{C}_k(\lambda_p) - C(\lambda_p)) \\ &(n/k)^{\frac{1}{2}}(\hat{C}_k(\pi) - C(\pi)), (n/k)^{\frac{1}{2}}(\hat{S}_k(\lambda_1) - S(\lambda_1)), \cdots, (n/k)^{\frac{1}{2}}(\hat{S}_k(\lambda_p) - S(\lambda_p)) \end{aligned}$$

is asymptotically independent normal, with zero means and variances

$$(4.3) \qquad \begin{aligned} &\frac{\sigma^2}{2\pi f(0)}, \frac{\sigma^2}{4\pi f(\lambda_1)}, \cdots, \frac{\sigma^2}{4\pi f(\lambda_p)} \\ &\frac{\sigma^2}{2\pi f(\pi)}, \frac{\sigma^2}{4\pi f(\lambda_1)}, \cdots, \frac{\sigma^2}{4\pi f(\lambda_p)} . \end{aligned}$$

PROOF. Let $C_k(\lambda) = 1 + \sum_{j=1}^{k} a_j \cos j\lambda$, $S_k(\lambda) = \sum_{j=1}^{k} a_j \sin j\lambda$. Then the joint asymptotic distribution of (4.2) is unchanged if, for each λ, $C(\lambda)$ is replaced by $C_k(\lambda)$ and $S(\lambda)$ is replaced by $S_k(\lambda)$, because

$$\lim_{n\to\infty} (n/k)^{\frac{1}{2}}(a_{k+1}e^{i(k+1)\lambda} + a_{k+2}e^{i(k+2)\lambda} + \cdots) = 0$$

by assumption (iv).

Then from Theorem 4 and Theorem 2 it is apparent that any linear combination of (4.2) is asymptotically normal, so the joint distribution of (4.2) is asymptotically multivariate normal. The variances (4.3) are obtained by appropriate choices of the c's and d's in Theorem 3. To obtain the asymptotic variance of $(n/k)^{\frac{1}{2}}(\hat{C}_k(0) - C(0))$, let $c_0 = 1$ with others zero. To obtain the asymptotic variance of $(n/k)^{\frac{1}{2}}(\hat{C}_k(\lambda_j) - C(\lambda_j))$, use $c_j = \frac{1}{2} = d_j$ with the others zero. To obtain the asymptotic variance of $(n/k)^{\frac{1}{2}}(\hat{S}_k(\lambda_j) - S(\lambda_j))$, use $c_j = -i/2$, $d_j = i/2$, with the others zero. To obtain the asymptotic variance of $(n/k)^{\frac{1}{2}}(\hat{C}_k(\pi) - C(\pi))$, use $d_0 = 1$ with the others zero. A computation using the above method shows that the variance of the asymptotic distribution of the sum of any pair of terms in

(4.2) is the sum of the asymptotic variances of the two terms. Thus the terms in (4.2) have an uncorrelated multivariate normal asymptotic distribution, and this proves the theorem.

The asymptotic distribution of the spectral estimates is similar to the asymptotic distribution of truncated periodogram estimates (Anderson (1970), page 534), as stated by Parzen (1969), page 406.

THEOREM 6. *Suppose that* $\{x_t\}$ *is a linear process,* $x_t = e_t + b_1 e_{t-1} + \cdots$, *where* $\{e_t\}$ *is a sequence of independent identically distributed random variables,* $Ee_t = 0$, *such that* $B(z) = 1 + b_1 z + \cdots$ *is nonzero in the unit circle and*

(i) *The spectral density* $f(\lambda) = (\sigma^2/2\pi)|B(e^{i\lambda})|^2$ *is positive.*

(ii) $E(e_t^4) < \infty$.

(iii) k *is chosen so* $k \to \infty$ *and* $k^3/n \to 0$.

(iv) k *is chosen so* $n^{\frac{1}{2}}(|a_{k+1}| + |a_{k+2}| + \cdots) \to 0$, *where* $A(z) = 1 + a_1 z + \cdots = 1/B(z)$.

The spectral density estimate is $\hat{f}_n(\lambda) = \hat{\sigma}_k^2/[2\pi(\hat{C}_k(\lambda)^2 + \hat{S}_k(\lambda)^2)]$.

Then the joint asymptotic distribution of

$$(n/k)^{\frac{1}{2}}(\hat{f}_n(0) - f(0)),\ (n/k)^{\frac{1}{2}}(\hat{f}_n(\lambda_1) - f(\lambda_1)),\ \cdots,$$
$$(n/k)^{\frac{1}{2}}(\hat{f}_n(\lambda_p) - f(\lambda_p)),\ (n/k)^{\frac{1}{2}}(\hat{f}_n(\pi) - f(\pi)), \qquad 0 < \lambda_1 < \cdots < \lambda_p < \pi,$$

is independent, normal, mean 0, *with variances*

$$4f^2(0),\ 2f^2(\lambda_1),\ \cdots,\ 2f^2(\lambda_p),\ 4f^2(\pi). \tag{4.4}$$

The first part of the proof of Theorem 6 is devoted to proving that $(n/k)^{\frac{1}{2}}(\hat{\sigma}_k^2 - \sigma^2)$ converges to zero in probability.

We consider the five terms on the right-hand side of (2.9).

(a) $\mathrm{Var}\,[(n/k)^{\frac{1}{2}}(\hat{r}_0 - r_0)] \leqq \mathrm{const}/k \to 0$, by (2.10) and (2.11).

(b) Assumptions (iii) and (iv) imply that $f(\lambda)$ is continuous, as discussed before Theorem 1. Therefore $r_1^2 + r_2^2 + \cdots < \infty$, so Theorem 2 can be used with $\gamma(k) = (r_1, \cdots, r_k)$, yielding

$$(n-k)^{\frac{1}{2}}(\hat{a}(k) - a(k))'r(k) \sim r(k)'R(k)^{-1} \textstyle\sum_{j=k}^{n-1} X_j(k)e_{j+1}/(n-k)^{\frac{1}{2}}.$$

The variance of the right-hand side is

$$r(k)'R(k)^{-1}r(k)\sigma^2,$$

so the second term converges to zero because $\|R(k)^{-1}\|$ is bounded, by (i) and (2.14).

(c) $$\{\mathrm{Var}\,[(n/k)^{\frac{1}{2}}a(k)'(\hat{r}(k) - r(k))]\}^{\frac{1}{2}} \leqq (n/k)^{\frac{1}{2}} \textstyle\sum_{j=1}^{k} |a_j|c/(n-k)^{\frac{1}{2}} \to 0,$$

using (2.10), (2.11) and (iv).

(d) $$[(n/k)^{\frac{1}{2}}(\hat{a}(k) - a(k))'(\hat{r}(k) - r(k))]^2 \leqq (n/k)\|\hat{a}(k) - a(k)\|^2\|\hat{r}(k) - r(k)\|^2,$$

which converges to zero in probability, by (2.10), (2.11), and (2.17).

(e) $$((n/k)^{\frac{1}{2}} \textstyle\sum_{j=k+1}^{\infty} a_j r_j)^2 \leqq (n/k) \sum_{j=k+1}^{\infty} a_j^2 \sum_{j=k+1}^{\infty} r_j^2 \to 0, \qquad \text{by (iv).}$$

Thus $(n/k)^{\frac{1}{2}}(\hat{\sigma}_k^2 - \sigma^2)$ converges to zero in probability.

Now let $\Delta\hat{f}_n(\lambda) = (n/k)^{\frac{1}{2}}(\hat{f}_n(\lambda) - f(\lambda))$, $\Delta\hat{S}_k(\lambda) = (n/k)^{\frac{1}{2}}(\hat{S}_k(\lambda) - S(\lambda))$, $\Delta\hat{C}_k(\lambda) = (n/k)^{\frac{1}{2}}(\hat{C}_k(\lambda) - C(\lambda))$, $\Delta\hat{\sigma}_k^2 = (n/k)^{\frac{1}{2}}(\hat{\sigma}_k^2 - \sigma^2)$ and write

$$\Delta\hat{f}_n(\lambda) = \frac{\partial\hat{f}_n(\lambda)}{\partial\hat{\sigma}_k^2}\Delta\hat{\sigma}_k^2 + \frac{\partial\hat{f}_n(\lambda)}{\partial\hat{C}_k(\lambda)}\Delta\hat{C}_k(\lambda) + \frac{\partial\hat{f}_n(\lambda)}{\partial\hat{S}_k(\lambda)}\Delta\hat{S}_k(\lambda) + o([\Delta\hat{\sigma}_k^2)^2 + (\Delta\hat{C}_k(\lambda))^2 + (\Delta\hat{S}_k(\lambda))^2]^{\frac{1}{2}}),$$

where the last term converges to zero in probability. The result (4.4) follows then by a direct application of Theorem 5. This completes the proof of Theorem 6.

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