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A NON-PARAMETRIC TEST OF WHETHER TWO SIMPLE REGRESSION LINES ARE PARALLEL[1]

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This paper provides a means for testing the hypothesis that two simple regression lines are parallel, when the two sets of error terms have two arbitrary unknown continuous distributions. The non-parametric test which is developed here is analogous to the two-sample Wilcoxon test. At the end of the paper, two additional problems in non-parametric regression analysis are briefly referred to.

1. Summary and initial remarks. If one desires to test whether two simple regression lines have the same slope, then one can, of course, use the standard test derived from least-squares theory if one is willing to assume that the distributions of the two sets of error terms are both normal and that they have the same variance. If the two distributions in question are both normal but have different variances, the tests of [1] and [10] are available. In some cases, though, the experimenter may need a test which is valid for two arbitrary distributions of the two sets of error terms. The present paper presents such a test. The test bears a close resemblance to the Wilcoxon two-sample test [13, 7], and may be considered as an extension of the idea developed in [8] concerning the use of the Wilcoxon statistic with respect to a generalized Behrens-Fisher problem.

2. Introduction and statement of results. We suppose that we have M pairs $(Y_1, X_1), (Y_2, X_2), \cdots, (Y_M, X_M)$ such that

$$Y_i = \alpha_Y + \beta_Y X_i + e_i \qquad (i = 1, 2, \cdots, M), \tag{2.1}$$

and N pairs $(Z_1, W_1), (Z_2, W_2), \cdots, (Z_N, W_N)$ such that

$$Z_j = \alpha_Z + \beta_Z W_j + f_j \qquad (j = 1, 2, \cdots, N). \tag{2.2}$$

The α's and β's are unknown regression parameters. The X_i's and W_j's are (known) constants; to avoid complications, we will assume that no two X_i's are equal and that no two W_j's are equal. The error terms $e_1, e_2, \cdots, e_M, f_1, f_2, \cdots, f_N$ are assumed to be mutually independent. We assume that each e_i comes from a distribution with continuous cdf $F_Y(e)$, and that each f_j comes

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from a distribution with continuous cdf $F_Z(f)$, where ordinarily $F_Y(e)$ and $F_Z(f)$ will both be unknown.

Without loss of generality, we may assume that $M \leqq N$.

We wish to test the null hypothesis that the two regression lines are parallel. Specifically, we want to test

$$H_0: \beta_Y = \beta_Z \tag{2.3}$$

against alternatives $\beta_Y \neq \beta_Z$.

Our test of H_0 (2.3) will be based on the statistic

$$w = \binom{M}{2}^{-1}\binom{N}{2}^{-1} \sum_{i<I} \sum_{j<J} u(V_{iIjJ}) , \tag{2.4}$$

where the function $u(V)$ equals 0 if $V \leqq 0$ and 1 if $V > 0$, and where

$$V_{iIjJ} = \frac{Z_J - Z_j}{W_J - W_j} - \frac{Y_I - Y_i}{X_I - X_i} . \tag{2.5}$$

The summation in (2.4) is over all (i, I, j, J) such that $1 \leqq i < I \leqq M$ and $1 \leqq j < J \leqq N$, and thus embraces $\binom{M}{2}\binom{N}{2}$ terms altogether.

In Section 3 we show that, regardless of what $F_Y(e)$ and $F_Z(f)$ are,

$$E(w) = \tfrac{1}{2} \quad \text{if} \quad H_0 \quad \text{is true,} \tag{2.6a}$$

and

$$E(w) \neq \tfrac{1}{2} \quad \text{if} \quad H_0 \quad \text{is not true} \tag{2.6b}$$
$$[E(w) > \tfrac{1}{2} \text{ if } \beta_Z > \beta_Y,\ E(w) < \tfrac{1}{2} \text{ if } \beta_Z < \beta_Y] .$$

In Section 4 we prove that

$$\sup_{F_Y, F_Z} \operatorname{Var}(w) = \frac{2M + 5}{18M(M-1)} \quad \text{if} \quad H_0 \quad \text{is true.} \tag{2.7}$$

Section 5 appeals to a theorem of Hoeffding dealing with U-statistics for independently but non-identically distributed random variables [3] to show that, under certain mild restrictions, $[\operatorname{Var}(w)]^{-\frac{1}{2}}(w - \frac{1}{2})$ is asymptotically $N(0, 1)$ if H_0 (2.3) is true. This result, together with (2.6a) and (2.7), tells us that, if we base a test of H_0 (2.3) on the critical region

$$\left[\frac{2M + 5}{18M(M-1)}\right]^{-\frac{1}{2}} |w - \tfrac{1}{2}| > z_{\frac{1}{2}\alpha} , \tag{2.8}$$

where $z_{\frac{1}{2}\alpha}$ is defined by

$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{z_{\frac{1}{2}\alpha}} e^{-\frac{1}{2}z^2}\, dz = 1 - \tfrac{1}{2}\alpha ,$$

then (disregarding inaccuracies due to the normal approximation) such a test (2.8) will be of size α (where "size" means maximum probability of Type I error). Section 6 uses a slightly refined form of (2.6b) to establish the consistency of the test (2.8) against all alternatives $\beta_Y \neq \beta_Z$.

Our test (2.8) will of course be a conservative test, since Var (w) generally will be smaller than the value on the right-hand side of (2.7). Apparently the test is not unbiased. The question could be raised as to whether one might use in (2.8) a consistent sample estimate of Var (w) instead of the upper bound (2.7), in order to provide increased power. Although the present paper will not attempt to explore this question, such an approach would appear to present a couple of problems. First, whatever estimator one might choose for Var (w) would probably require laborious calculations, perhaps prohibitive calculations (laborious or prohibitive even with a computer, for larger values of M and N). Second, since the estimate of Var (w) will differ from the true value of Var (w) (and can of course be either higher or lower), the true probability of Type I error will differ from its intended value α and one will not know how great the difference is *or whether it is plus or minus*; with the conservative test (2.8), on the other hand, the difference between true and intended probabilities of Type I error can be in the minus direction only (if we ignore inaccuracies caused by the normal approximation to the distribution of w, which are perhaps not of great import). As M and N increase, the second problem diminishes in importance, but the first problem becomes rapidly more serious. Utilizing a consistent sample estimate of Var (w) is reasonable if one is concerned only with "large" M and N and if calculation difficulties are ignored. From a practical standpoint, though, M and N will not always be "large" enough, and calculation difficulties cannot be disregarded.

We may note the following points which are relevant for applications of the test (2.8):

(i) Although the discussion above is in terms of a two-tailed test, the extension to one-tailed tests can of course be made in the usual manner.

(ii) We can obtain confidence bounds on

$$\Delta = \beta_Z - \beta_Y \tag{2.9}$$

which are associated with the test (2.8). The technique for getting the bounds is similar to the one often used with the ordinary Wilcoxon statistic: we find the value (or those two values) of Δ which, when subtracted from every V_{iIjJ} in (2.4), will cause the resulting new value(s) of w to be on the threshold of significance. The reason why this works is that Δ (2.9) is the median of each V_{iIjJ} (2.5), as we shall see in Section 3.

(iii) In practical situations, there may be pair(s) of X_i's or of W_j's whose two members are equal, contrary to one of our assumptions. If such ties do occur, perhaps the best thing to do ordinarily would be simply to count a tally of $\frac{1}{2}$ in the sum (2.4) for each $u(V_{iIjJ})$ whose argument (V_{iIjJ}) would be undefined by virtue of the two X's being alike and/or the two W's being alike.

For a one-sample problem involving one simple regression line, Theil [12] introduced and briefly examined a non-parametric test which resembles our test above for the two-sample problem. Section 7 below will provide a short discussion of the one-sample problem.

Finally, Section 8 poses a two-sample regression problem which is a companion problem to the one covered by the present paper.

3. The expectation of w**.** In this section we prove (2.6). Using (2.5), (2.1)—(2.2), and (2.9), we obtain

$$V_{iIjJ} = z_{jJ} + y_{iI} + \Delta , \tag{3.1}$$

where

$$z_{jJ} = (f_J - f_j)/(W_J - W_j) , \qquad y_{iI} = -(e_I - e_i)/(X_I - X_i) . \tag{3.2}$$

Note that z_{jJ} and y_{iI} (3.2) are independent, and that each is the difference of two independent random variables with a common continuous cdf. It then follows that z_{jJ} and y_{iI} are each distributed symmetrically about 0; that their sum $(z_{jJ} + y_{iI})$ is symmetrically distributed about 0; and that $P\{z_{jJ} + y_{iI} = 0\}$ is 0. Thus, for the case where H_0 (2.3) is true (i.e., $\Delta = 0$), we can write

$$E[u(V_{iIjJ})] = P\{V_{iIjJ} > 0\} = P\{z_{jJ} + y_{iI} > 0\} = \tfrac{1}{2} \qquad \text{if} \quad \beta_Z = \beta_Y , \tag{3.3}$$

and (2.6a) follows at once from (3.3) and (2.4).

For the case $\beta_Z > \beta_Y$ $(\Delta > 0)$, we obtain

$$\begin{aligned} E[u(V_{iIjJ})] &= P\{V_{iIjJ} > \Delta\} + P\{\Delta \geqq V_{iIjJ} > 0\} \\ &= \tfrac{1}{2} + P\{-\Delta < z_{jJ} + y_{iI} \leqq 0\} \\ &\geqq \tfrac{1}{2} + P\{-\tfrac{1}{2}\Delta < z_{jJ} \leqq 0, -\tfrac{1}{2}\Delta < y_{iI} \leqq 0\} \\ &= \tfrac{1}{2} + P\{-\delta_{ZiJ} < f_J - f_j \leqq 0\}P\{-\delta_{YiI} < e_i - e_I \leqq 0\} , \end{aligned} \tag{3.4}$$

where $\delta_{ZjJ} = \frac{1}{2}\Delta|W_J - W_j|$ and $\delta_{YiI} = \frac{1}{2}\Delta|X_I - X_i|$. There exist numbers f^* and $\varepsilon_Z = \varepsilon_Z(\delta_{ZjJ})$ such that $F_Z(f^* + \delta_{ZjJ}) - F_Z(f^*) = \varepsilon_Z > 0$. We have

$$\begin{aligned} &2P\{-\delta_{ZjJ} < f_J - f_j \leqq 0\} \\ &\quad = P\{-\delta_{ZjJ} < f_J - f_j < \delta_{ZjJ}\} \\ &\quad \geqq P\{f^* < f_J < f^* + \delta_{ZjJ}, f^* < f_j < f^* + \delta_{ZjJ}\} \\ &\quad = [\varepsilon_Z(\delta_{ZjJ})]^2 . \end{aligned} \tag{3.5}$$

Looking at (3.4) and (3.5), we can now write

$$E[u(V_{iIjJ})] \geqq \tfrac{1}{2} + \tfrac{1}{4}[\varepsilon_Z(\delta_{ZjJ})]^2[\varepsilon_Y(\delta_{YiI})]^2 \qquad \text{if} \quad \beta_Z > \beta_Y , \tag{3.6}$$

where $\varepsilon_Y(\delta_{YiI})$ is analogous to $\varepsilon_Z(\delta_{ZjJ})$ (and is also > 0). The result (2.6b) for $\beta_Z > \beta_Y$ is established if we apply (3.6) to the expectation of (2.4).

For $\beta_Z < \beta_Y$, the proof of (2.6b) is similar to the above.

4. The variance of w**.** Sections 4 and 5 are concerned with proving certain properties of w under H_0. In this section we prove (2.7). First we will establish that, if the two cdf's are $F_Y(e)$ and $F_Z(f) = F_Z^0(Kf)$, where F_Z^0 is any cdf, then (under H_0) we have

$$\lim_{K\to\infty} \operatorname{Var}(w) = \frac{2M + 5}{18M(M - 1)} . \tag{4.1}$$

Let us start by defining

$$(4.2)\qquad \begin{aligned} C_{iIjJi'I'j'J'} &= \mathrm{Cov}\,[u(V_{iIjJ}),\, u(V_{i'I'j'J'})] \\ &= P\{V_{iIjJ} > 0,\, V_{i'I'j'J'} > 0\} - \tfrac{1}{4}\,; \end{aligned}$$

note that V_{iIjJ} is given by (3.1—3.2) with $\Delta = 0$. By using (2.4), we can write

$$(4.3)\qquad \begin{aligned} \mathrm{Var}\,(w) = \tbinom{M}{2}^{-2}\tbinom{N}{2}^{-2} &\textstyle\sum_{j<J}\sum_{j'<J'}\,[\sum_{i<I} C_{iIjJiIj'J'} \\ &+ 2\textstyle\sum_{i<I<I'} C_{iIjJiI'j'J'} + 2\sum_{i<i'<I} C_{iIjJi'Ij'J'} \\ &+ 2\textstyle\sum_{i<I<I'} C_{iIjJII'j'J'}] + r\,, \end{aligned}$$

where r is the sum of the terms $C_{iIjJi'I'j'J'}$ for which (i, I, i', I') are all different but (j, J, j', J') are not all different. In Sections 4 and 5 we will assume (with no loss of generality) that the X_i's and the W_j's are arranged in ascending order, so that $i < I$ implies $X_i < X_I$ and $j < J$ implies $W_j < W_J$. Now as $K \to \infty$, $F_Z(f) = F_Z{}^0(Kf)$ approaches a distribution function which has all of its probability mass concentrated at a single point (0). We can thus show that, upon taking the limit in (4.3) and using (4.2), (3.1)—(3.2), we obtain

$$(4.4)\qquad \begin{aligned} \lim_{K\to\infty} \mathrm{Var}\,(w) &= \tbinom{M}{2}^{-2}\tbinom{N}{2}^{-2}\tbinom{N}{2}^{2}[\tbinom{M}{2}(\tfrac{1}{2} - \tfrac{1}{4}) + 2\tbinom{M}{3}(\tfrac{1}{3} - \tfrac{1}{4}) \\ &\quad + 2\tbinom{M}{3}(\tfrac{1}{3} - \tfrac{1}{4}) + 2\tbinom{M}{3}(\tfrac{1}{6} - \tfrac{1}{4})] + 0 \\ &= \frac{2M + 5}{18M(M-1)}\,, \end{aligned}$$

which proves (4.1).

The remainder of this section will be devoted to proving that

$$(4.5)\qquad \mathrm{Var}\,(w) \leqq \frac{2M + 5}{18M(M-1)}$$

no matter what F_Y and F_Z are. Clearly, (4.5) taken together with (4.1) will be sufficient to establish (2.7).

Let $\mathbf{R} = (R_1, R_2, \cdots, R_M)$ denote a permutation of M of the numbers $1, 2, \cdots, N$. There exist $N!/(N-M)!$ such permutations altogether. From (2.4) it follows that

$$(4.6)\qquad w = \frac{(N-M)!}{N!} \sum_{\mathbf{R}} \binom{M}{2}^{-1} \sum_{i<I} u(V_{iIR_iR_I})\,,$$

where the left summation (over $\mathbf{R}$) indicates summation over all $N!/(N-M)!$ permutations $\mathbf{R}$. Now if $t_1, t_2, \cdots, t_\nu$ denote any set of random variables, then, regardless of what their joint distribution is, we will have

$$(4.7)\qquad \begin{aligned} &\mathrm{Var}\,[\nu^{-1}(t_1 + t_2 + \cdots + t_\nu)] \\ &\qquad \leqq \nu^{-1}[\mathrm{Var}\,(t_1) + \mathrm{Var}\,(t_2) + \cdots + \mathrm{Var}\,(t_\nu)]\,. \end{aligned}$$

With $\nu = N!/(N-M)!$, we apply this standard inequality (4.7) to (4.6); we obtain

$$\text{(4.8)} \qquad \operatorname{Var}(w) \leqq \frac{(N-M)!}{N!}\binom{M}{2}^{-2} \sum_{\mathbf{R}} \operatorname{Var}\left[\sum_{i<I} u(V_{iIR_iR_I})\right].$$

For each of the $\binom{M}{3}\binom{N}{3}$ sets $(i_1, i_2, i_3; j_1, j_2, j_3)$ which satisfy $1 \leqq i_1 < i_2 < i_3 \leqq M$ and $1 \leqq j_1 < j_2 < j_3 \leqq N$, let us define

$$\text{(4.9)} \qquad S(i_1, i_2, i_3; j_1, j_2, j_3) = \sum_{18} C_{i_{l_1} i_{l_2} j_{L_1} j_{L_2} i_{l_1} i_{l_3} j_{L_1} j_{L_3}},$$

where the summation $\sum_{18}$ in (4.9) is over the 18 sets of values of $(l_1, l_2, l_3; L_1, L_2, L_3)$ which are obtained by taking a double summation in which (L_1, L_2, L_3) runs over all 6 permutations of (1, 2, 3) and (l_1, l_2, l_3) runs over the 3 permutations (1, 2, 3), (2, 1, 3), and (3, 1, 2). Then we can write

$$\text{(4.10)} \qquad \begin{aligned} &\sum_{\mathbf{R}} \operatorname{Var}\left[\sum_{i<I} u(V_{iIR_iR_I})\right] \\ &\qquad = \frac{N!}{(N-M)!}\binom{M}{2}\left(\frac{1}{4}\right) + 2\,\frac{(N-3)!}{(N-M)!} \\ &\qquad\quad \times \sum_{i_1<i_2<i_3} \sum_{j_1<j_2<j_3} S(i_1, i_2, i_3; j_1, j_2, j_3). \end{aligned}$$

Our final task in this section will be to prove that

$$\text{(4.11)} \qquad S(i_1, i_2, i_3; j_1, j_2, j_3) \leqq \tfrac{1}{2}$$

for any values of the indices $(i_1 < i_2 < i_3, j_1 < j_2 < j_3)$. Once (4.11) is established, (4.5) follows easily by combining (4.8), (4.10), and (4.11).

Let

$$\text{(4.12)} \qquad e_{01} < e_{02} < e_{03} \quad \text{and} \quad f_{01} < f_{02} < f_{03},$$

and let $P\{V_{iIjJ} > 0, V_{iI'jJ'} > 0 \mid e_{01}, e_{02}, e_{03}; f_{01}, f_{02}, f_{03}\}$ denote the conditional probability that V_{iIjJ} and $V_{iI'jJ'}$ are both > 0 given that

$$\text{(4.13a)} \qquad e_i, e_I, e_{I'} = e_{01}, e_{02}, e_{03}, \quad \text{but not necessarily in that order}$$

and

$$\text{(4.13b)} \qquad f_j, f_J, f_{J'} = f_{01}, f_{02}, f_{03}, \quad \text{but not necessarily in that order.}$$

Given (4.13), the $3! \times 3! = 36$ possible values of the sextet $(e_i, e_I, e_{I'}; f_j, f_J, f_{J'})$ all have the same probability $(= 1/36)$, inasmuch as (in the unconditional distributions) the three e's are identically distributed, the three f's are identically distributed, and the six variables are mutually independent. Let us define

$$\text{(4.14)} \qquad \begin{aligned} S^* &= S^*(i_1, i_2, i_3; j_1, j_2, j_3 \mid e_{01}, e_{02}, e_{03}; f_{01}, f_{02}, f_{03}) \\ &= \sum_{18} P\{V_{i_{l_1} i_{l_2} j_{L_1} j_{L_2}} > 0, V_{i_{l_1} i_{l_3} j_{L_1} j_{L_3}} > 0 \mid e_{01}, e_{02}, e_{03}; f_{01}, f_{02}, f_{03}\} - \tfrac{18}{4}, \end{aligned}$$

where $\sum_{18}$ has the same meaning as in (4.9). We will prove below that

$$\text{(4.15)} \qquad S^*(i_1, i_2, i_3; j_1, j_2, j_3 \mid e_{01}, e_{02}, e_{03}; f_{01}, f_{02}, f_{03}) \leqq \tfrac{1}{2}$$

for any allowable values of the i's, j's, e_0's, and f_0's. Noting (4.2), we see that S(4.9) is the expectation of S^*(4.14); hence, (4.11) will follow immediately from

(4.15) once the latter is proved. [Incidentally, we note in view of (4.12) that, in (4.14)—(4.15) and the ensuing comments, we are neglecting the point set in which the three e_0's are not all different and/or the three f_0's are not all different. However, this point set is of measure zero (thanks to the continuity of F_Y and F_Z), and so we may legitimately neglect it.]

All that remains, then, is to prove (4.15). If we use (4.14) and recall that the 36 possible values of the sextet of e's and f's are equally probable, we can write

$$36S^* + 162 = \sum_{36} \sum_{18} u\left(\frac{f_{0J_2} - f_{0J_1}}{W_{jL_2} - W_{jL_1}} - \frac{e_{0I_2} - e_{0I_1}}{X_{il_2} - X_{il_1}}\right) \times u\left(\frac{f_{0J_3} - f_{0J_1}}{W_{jL_3} - W_{jL_1}} - \frac{e_{0I_3} - e_{0I_1}}{X_{il_3} - X_{il_1}}\right), \tag{4.16}$$

where the summation $\sum_{36}$ is over the 36 sets of values of $(I_1, I_2, I_3; J_1, J_2, J_3)$ which are obtained by taking a double summation in which (I_1, I_2, I_3) runs over all six permutations of (1, 2, 3) and (J_1, J_2, J_3) likewise runs over all six permutations of (1, 2, 3). The right-hand side of (4.16) thus consists of a summation of $36 \times 18 = 648$ terms; each term is the product of two u-functions, and must therefore be equal to either 0 or 1. We will be able to prove that no more than 180 of these 648 terms can be equal to 1, no matter what the X's, W's, e_0's, and f_0's are. That will mean, in view of (4.16), that $36S^* + 162 \leqq 180$, from which (4.15) (and hence (2.7)) will follow at once.

Thus, it now only remains to show that no more than 180 of the 648 terms can be 1's. Our proof of this proposition is exceedingly lengthy and also rather elementary, and will therefore be omitted here. However, full details are available in [9].

5. Asymptotic normality of w. In this section we show that, under mild restrictions, w (2.4) is asymptotically normal under H_0 (2.3). Our proof will use a theorem of Hoeffding [3, Theorem 8.1] concerning the asymptotic normality of U-statistics for random variables independently but not necessarily identically distributed.

A U-statistic must be of the form [3, equation (5.1)],

$$U = \binom{n}{m}^{-1} \sum{}' \Phi(\mathbf{x}_{\alpha_1}, \mathbf{x}_{\alpha_2}, \cdots, \mathbf{x}_{\alpha_m}), \tag{5.1}$$

where the summation $\sum'$ is over all $\binom{n}{m}$ sets satisfying $1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_m \leqq n$, the function Φ is symmetric in its m (vector) arguments, and the $\mathbf{x}_\alpha$'s ($\alpha = 1, 2, \cdots, n$) are mutually independent (but not necessarily identically distributed) r-dimensional random variables of the form $\mathbf{x}_\alpha = (x_\alpha^{(1)}, x_\alpha^{(2)}, \cdots, x_\alpha^{(r)})$. (In this section we are trying to maintain the same notation which Hoeffding used, except that we are using $\mathbf{x}$ where he used X.)

First we find a U-statistic which is related to w (2.4). In (5.1), let us make the identifications $n = M + N$, $m = 4$, $r = 3$,

$$\begin{aligned} \mathbf{x}_\alpha = (x_\alpha^{(1)}, x_\alpha^{(2)}, x_\alpha^{(3)}) &= (e_\alpha, X_\alpha, 1) \qquad \text{for } 1 \leqq \alpha \leqq M \\ &= (f_{\alpha-M}, W_{\alpha-M}, 2) \qquad \text{for } M + 1 \leqq \alpha \leqq M + N, \end{aligned} \tag{5.2}$$

and

$$(5.3)\qquad \begin{aligned}&\Phi(\mathbf{x}_{\alpha_1}, \mathbf{x}_{\alpha_2}, \mathbf{x}_{\alpha_3}, \mathbf{x}_{\alpha_4})\\ &= \sum_{24} \zeta(\mathbf{x}_h, \mathbf{x}_H, \mathbf{x}_{h'}, \mathbf{x}_{H'})\left[u\left(\frac{x_{H'}^{(1)} - x_{h'}^{(1)}}{x_{H'}^{(2)} - x_{h'}^{(2)}} - \frac{x_H^{(1)} - x_h^{(1)}}{x_H^{(2)} - x_h^{(2)}}\right) - \tfrac{1}{2}\right],\end{aligned}$$

where the summation in (5.3) is over all $4! = 24$ permutations (h, H, h', H') of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with $u(V)$ being considered to be 0 if V is undefined, and where

$$(5.4)\qquad \begin{aligned}&\zeta(\mathbf{x}_h, \mathbf{x}_H, \mathbf{x}_{h'}, \mathbf{x}_{H'})\\ &= 1 \quad \text{if } x_h^{(3)} = x_H^{(3)} = 1,\ x_{h'}^{(3)} = x_{H'}^{(3)} = 2,\\ &\qquad\quad x_H^{(2)} > x_h^{(2)},\ x_{H'}^{(2)} > x_{h'}^{(2)}\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

The $x_\alpha^{(2)}$'s and $x_\alpha^{(3)}$'s of (5.2) are constants, but for our present purposes we regard them as random variables for which all the probability mass is concentrated at a single point. Thus no two $\mathbf{x}_\alpha$'s are identically distributed.

Now note that Φ (5.3) satisfies the specification of being symmetric in its arguments. When H_0 (2.3) is true, the statistic U determined by (5.1)—(5.4) will be related to w (2.4) by the equation

$$(5.5)\qquad U = k_{M,N}(w - \tfrac{1}{2}),$$

where

$$(5.6)\qquad k_{M,N} = \binom{M}{2}\binom{N}{2}\binom{M+N}{4}^{-1}.$$

We will apply Hoeffding's Theorem 8.1 [3] to U (5.5) to prove that $[\text{Var}(U)]^{-\frac{1}{2}}[U - E(U)]$ is asymptotically $N(0, 1)$. At this point we introduce the two mild assumptions which we will use:

ASSUMPTION 5A. We assume that, as $n \to \infty$, N/M approaches some constant c ($c \geqq 1$, since $N \geqq M$).

ASSUMPTION 5B. Let us define

$$(5.7)\qquad K_{iIjJ} = (X_I - X_i)/(W_J - W_j).$$

We assume that there exists a fraction Π ($0 < \Pi < 1$) and a number K_0 ($K_0 > 0$) such that, for all (M, N) and for all integers ν in the interval $1 \leqq \nu \leqq M/2$, the following property holds: in the set of $(M - \nu)\binom{N}{2}$ triples (I, j, J) for which $\nu < I \leqq M$ and $1 \leqq j < J \leqq N$, the number of triples satisfying the condition

$$(5.8)\qquad K_{\nu IjJ} < K_0$$

is $\geqq \Pi(M - \nu)\binom{N}{2}$.

Observing (5.6), we see that Assumption 5A implies that

$$(5.9)\qquad \lim_{n\to\infty} k_{M,N} = 6c^2/(1 + c)^4,$$

which is strictly > 0. We conclude from (5.5) that, once we prove that $[\text{Var}(U)]^{-\frac{1}{2}}[U - E(U)]$ is asymptotically $N(0, 1)$, it will follow immediately that $[\text{Var}(w)]^{-\frac{1}{2}}(w - \frac{1}{2})$ is likewise asymptotically $N(0, 1)$.

To establish the asymptotic normality of U (5.5) by using Hoeffding's Theorem 8.1 [3], we need only show that the three conditions [3, formulas (8.2)—(8.4)] are satisfied. We can see at once that [3, (8.2)] and [3, (8.3)] hold, simply by virtue of the fact that Φ (5.3) is bounded in absolute value. The condition [3, (8.4)] is

$$\lim_{n\to\infty} \frac{\sum_{\nu=1}^{n} E|\bar{\Psi}^3_{1(\nu)}(\mathbf{x}_\nu)|}{[\sum_{\nu=1}^{n} E\{\bar{\Psi}^2_{1(\nu)}(\mathbf{x}_\nu)\}]^{3/2}} = 0\,, \tag{5.10}$$

where $\bar{\Psi}_{1(\nu)}$ is defined by [3, equation (8.1)]. Since $E(\Phi) = 0$ and $|\Phi| \leqq \frac{1}{2}$ [see (5.3—5.4)], it will follow that $|\bar{\Psi}_{1(\nu)}| \leqq \frac{1}{2}$. Hence $|\bar{\Psi}^3_{1(\nu)}| < \bar{\Psi}^2_{1(\nu)}$, and $\sum E|\bar{\Psi}^3_{1(\nu)}| < \sum E\bar{\Psi}^2_{1(\nu)}$. Therefore the fraction on the left side of (5.10) is $< [\sum E\bar{\Psi}^2_{1(\nu)}]^{-\frac{1}{2}}$, which allows us to conclude that (5.10) will be proved if we can show that

$$\sum_{\nu=1}^{n} E\{\bar{\Psi}^2_{1(\nu)}(\mathbf{x}_\nu)\} \to \infty \qquad \text{as } n \to \infty\,. \tag{5.11}$$

Thus all that remains is to verify (5.11). Now using (5.2)—(5.4) above along with [3, equations (8.1), (5.15), and (5.14)], we can write

$$\bar{\Psi}_{1(\nu)}(\mathbf{x}_\nu) = \binom{M+N-1}{3}^{-1}(S_{1\nu} + S_{2\nu}) \qquad (\nu = 1, 2, \cdots, m)\,, \tag{5.12}$$

where we define (for $\nu = 1, 2, \cdots, M$)

$$S_{1\nu} = S_{1\nu}(e_\nu) = \sum_{i=1}^{\nu-1} \sum_{1\leqq j<J\leqq N} \Psi^-_{i\nu jJ}(e_\nu)\,, \tag{5.13}$$

$$S_{2\nu} = S_{2\nu}(e_\nu) = \sum_{I=\nu+1}^{M} \sum_{1\leqq j<J\leqq N} \Psi^+_{\nu IjJ}(e_\nu)\,, \tag{5.14}$$

$$\Psi^-_{i\nu jJ}(e_\nu) = \iiint u(K_{i\nu jJ}[f_J - f_j] + e_i - e_\nu)\, dF_Y(e_i)\, dF_Z(f_j)\, dF_Z(f_J) - \tfrac{1}{2} \qquad (i = 1, 2, \cdots, \nu - 1)\,, \tag{5.15}$$

and

$$\Psi^+_{\nu IjJ}(e_\nu) = \iiint u(K_{\nu IjJ}[f_J - f_j] - e_I + e_\nu)\, dF_Y(e_I)\, dF_Z(f_j)\, dF_Z(f_J) - \tfrac{1}{2} \qquad (I = \nu + 1, \nu + 2, \cdots, M)\,. \tag{5.16}$$

Note that $E(S_{1\nu}) = E(S_{2\nu}) = 0$. Let $\sigma^2_{1\nu}$ and $\sigma^2_{2\nu}$ denote the respective variances of $S_{1\nu}$ and $S_{2\nu}$, and let ρ_ν be the correlation coefficient between $S_{1\nu}$ and $S_{2\nu}$. Then

$$E[(S_{1\nu} + S_{2\nu})^2] = \sigma^2_{1\nu} + \sigma^2_{2\nu} + 2\rho_\nu\sigma_{1\nu}\sigma_{2\nu} \geqq (\sigma_{2\nu} - \sigma_{1\nu})^2\,. \tag{5.17}$$

From (5.15) we see that $|\Psi^-_{i\nu jJ}(e_\nu)| \leqq \frac{1}{2}$. Hence it follows from (5.13) that

$$\sigma^2_{1\nu} \leqq (\nu - 1)^2\binom{N}{2}^2(\tfrac{1}{2})^2\,. \tag{5.18}$$

We will show shortly that there exists a constant $\tau(0 < \tau \leqq 1)$ such that

$$\sigma^2_{2\nu} \geqq (M - \nu)^2\binom{N}{2}^2(\tfrac{1}{2})^2\tau^2 \tag{5.19}$$

for all integers ν in the interval $1 \leqq \nu \leqq M/2$. Now define ν_M to be the largest integer ν such that $(M - \nu)\tau - (\nu - 1) > 0$; thus

$$\frac{(M-1)\tau}{\tau + 1} \leqq \nu_M < \frac{M\tau + 1}{\tau + 1}\,, \tag{5.20}$$

and $\nu_M \leqq M/2$. Using (5.18)—(5.20), we can write

$$\sigma_{2\nu} - \sigma_{1\nu} \geqq \tfrac{1}{2}\binom{N}{2}[(M\tau + 1) - \nu(\tau + 1)] > 0 \qquad (\nu = 1, 2, \cdots, \nu_M) . \tag{5.21}$$

From (5.12), (5.17), and (5.21) we obtain

$$\sum_{\nu=1}^{\nu_M} E\{\bar{\Psi}^2_{1(\nu)}(\mathbf{x}_\nu)\} \geqq \binom{M+N-1}{3}^{-2}\left(\tfrac{1}{2}\right)^2\binom{N}{2}^2\sum_{\nu=1}^{\nu_M}[(M\tau + 1) - \nu(\tau + 1)]^2 . \tag{5.22}$$

But

$$\begin{aligned}\sum_{\nu=1}^{\nu_M}&[(M\tau + 1) - \nu(\tau + 1)]^2\\ &= \tfrac{1}{3}(\tau + 1)^2\nu_M{}^3 - \tfrac{1}{2}(\tau + 1)(2M\tau + 1 - \tau)\nu_M{}^2\\ &\qquad + [(M\tau + 1)(M - 1)\tau + \tfrac{1}{6}(\tau + 1)^2]\nu_M\\ &\geqq \frac{(M - 1)^3\tau^3}{3(\tau + 1)} - \frac{(2M\tau + 1 - \tau)(M\tau + 1)^2}{2(\tau + 1)}\\ &\qquad + \frac{(M\tau + 1)(M - 1)^2\tau^2}{(\tau + 1)} + \frac{(M - 1)\tau(\tau + 1)}{6}\\ &= \frac{(M - 1)^3\tau^3}{3(\tau + 1)} - \tfrac{3}{2}(M - 1)^2\tau^2 - \tfrac{11}{6}(M - 1)\tau(\tau + 1)\\ &\qquad - \tfrac{1}{2}(\tau + 1)^2 ,\end{aligned} \tag{5.23}$$

the inequality in the second line of (5.23) being a consequence of (5.20). Finally, (5.11) can now be established at once if we combine (5.22) and (5.23), let $n \to \infty$, and apply Assumption 5A.

Thus the proof that w is asymptotically normal will be complete once we show that there is a $\tau(> 0)$ for which (5.19) holds.

PROOF OF (5.19). From (5.14) and (5.16) we obtain

$$\sigma^2_{2\nu} = \sum_{I=\nu+1}^{M}\sum_{I'=\nu+1}^{M}\sum_{1\leqq j<J\leqq N}\sum_{1\leqq j'<J'\leqq N} E[\Psi^+_{\nu IjJ}(e_\nu)\Psi^+_{\nu I'j'J'}(e_\nu)] , \tag{5.24}$$

and we also can write

$$\begin{aligned}2E[\Psi^+_{\nu IjJ}&(e_\nu)\Psi^+_{\nu I'j'J'}(e_\nu)]\\ &= \iint [\Psi^+_{\nu IjJ}(v_2) - \Psi^+_{\nu IjJ}(v_1)][\Psi^+_{\nu I'j'J'}(v_2) - \Psi^+_{\nu I'j'J'}(v_1)]\,dF_Y(v_1)\,dF_Y(v_2) .\end{aligned} \tag{5.25}$$

Note that, since w (2.4) depends on the e_i's only through differences between pairs of e_i's, we can assume with no loss of generality that $0 < F_Y(0) < 1$. Since F_Y also is continuous, we can choose numbers D, p, and μ such that

$$\begin{aligned}&D > 0 , \qquad 0 < p < 1 , \qquad \mu > 0 , \qquad F_Y(-D) > \mu ,\\ &1 - F_Y(D) > \mu , \qquad F_Y(pD) - F_Y(-pD) > \mu .\end{aligned} \tag{5.26}$$

Define $q = 1 - p$. If $v_1 \leqq -D$, $v_2 \geqq D$, and $K_{\nu IjJ} < K_0$, then

$$\begin{aligned}\Psi^+_{\nu IjJ}&(v_2) - \Psi^+_{\nu IjJ}(v_1)\\ &= P\{v_1 < K_{\nu IjJ}(f_j - f_J) + e_I \leqq v_2\}\\ &\geqq P\{-D < K_{\nu IjJ}(f_j - f_J) + e_I \leqq D\}\\ &\geqq P\{-qD < K_{\nu IjJ}(f_j - f_J) \leqq qD,\ -pD < e_I \leqq pD\}\\ &= P\{-qD < K_{\nu IjJ}(f_j - f_J) \leqq qD\}P\{-pD < e_I \leqq pD\}\\ &\geqq P\{-qD/K_0 < f_j - f_J \leqq qD/K_0\}[F_Y(pD) - F_Y(-pD)]\\ &> [\varepsilon_Z(qD/K_0)]^2\mu ,\end{aligned} \tag{5.27}$$

the last line being obtained with the aid of (3.5) as well as (5.26). If $v_1 \leqq -D$, $v_2 \geqq D$, and $K_{\nu I'j'J'} < K_0$, then we can obtain for $\Psi^+_{\nu I'j'J'}(v_2) - \Psi^+_{\nu I'j'J'}(v_1)$ a result similar to the result (5.27). From this pair of results it then follows that if $K_{\nu IjJ} < K_0$ and $K_{\nu I'j'J'} < K_0$

$$\begin{aligned} &\textstyle\iint_{v_1 \leqq -D; v_2 \geqq D} [\Psi^+_{\nu IjJ}(v_2) - \Psi^+_{\nu IjJ}(v_1)] \\ (5.28)\qquad &\quad \times [\Psi^+_{\nu I'j'J'}(v_2) - \Psi^+_{\nu I'j'J'}(v_1)]\, dF_Y(v_1)\, dF_Y(v_2) \\ &\quad > [\varepsilon_Z(qD/K_0)]^4 \mu^2 F_Y(-D)[1 - F_Y(D)] > [\varepsilon_Z(qD/K_0)]^4 \mu^4 . \end{aligned}$$

From (5.16) it is clear that both $\Psi^+_{\nu IjJ}(v)$ and $\Psi^+_{\nu I'j'J'}(v)$ are always non-decreasing functions of v. Therefore, the integrand on the right side of (5.25) always is nonnegative everywhere. The latter fact implies that the integral in (5.25) is greater than or equal to the integral in (5.28), which in turn establishes that if $K_{\nu IjJ} < K_0$ and $K_{\nu I'j'J'} < K_0$

$$(5.29)\qquad E[\Psi^+_{\nu IjJ}(e_\nu)\Psi^+_{\nu I'j'J'}(e_\nu)] > \tfrac{1}{2}[\varepsilon_Z(qD/K_0)]^4 \mu^4 ;$$

it also implies that if $K_{\nu IjJ} \geqq K_0$ and/or $K_{\nu I'j'J'} \geqq K_0$

$$(5.30)\qquad E[\Psi^+_{\nu IjJ}(e_\nu)\Psi^+_{\nu I'j'J'}(e_\nu)] \geqq 0 .$$

Finally, we combine (5.24), (5.29), and (5.30), and apply Assumption 5B to obtain, if $1 \leqq \nu \leqq M/2$,

$$(5.31)\qquad \sigma^2_{2\nu} > \tfrac{1}{2}\Pi^2(M - \nu)^2\tbinom{N}{2}^2[\varepsilon_Z(qD/K_0)]^4 \mu^4 .$$

Thus, (5.19) is satisfied with $\tau = 2^{\frac{1}{2}}\Pi[\varepsilon_Z(qD/K_0)]^2\mu^2$.

Remark. It may be noted that our development here in Section 5 suggests a rather interesting way of proving that the Wilcoxon statistic [13, 7] is asymptotically normal, in either the null or the non-null case. The proof is effected by means of Hoeffding's Theorem 8.1 [3], using identification analogous to (5.2)—(5.4) above.

6. Consistency of the test. In this section we show that, against all alternatives $\Delta \neq 0$, the power of the test (2.8) approaches 1 as $M \to \infty$. In order to prove this consistency property, we will impose a mild assumption:

Assumption 6A. We assume that there exists a fraction Π_0 $(0 < \Pi_0 < 1)$ and a number k_0 $(k_0 > 0)$ such that, for all (M, N), the following two properties hold: (i) in the set of $\binom{M}{2}$ pairs (i, I) for which $1 \leqq i < I \leqq M$, the number of pairs satisfying the condition

$$(6.1\text{a})\qquad |X_I - X_i| > k_0$$

is $\geqq \Pi_0 \binom{M}{2}$; and, (ii) in the set of $\binom{N}{2}$ pairs (j, J) for which $1 \leqq j < J \leqq N$, the number of pairs satisfying the condition

$$(6.1\text{b})\qquad |W_J - W_j| > k_0$$

is $\geqq \Pi_0 \binom{N}{2}$.

In proving the consistency of our test (2.8), we will consider explicitly only the case $\Delta > 0$; however, the proof for $\Delta < 0$ is analogous. First we obtain a couple of preliminary results. One of these is related to the material of Section 3. When the conditions (6.1) are satisfied, the expression $[\varepsilon_Z(\delta_{ZjJ})]^2[\varepsilon_Y(\delta_{YiI})]^2$ in (3.6) can be replaced by $[\varepsilon_Z(\frac{1}{2}\Delta k_0)]^2[\varepsilon_Y(\frac{1}{2}\Delta k_0)]^2$ and the inequalities will still hold. Hence it will follow from Assumption 6A that, if $\Delta > 0$, then

$$E(w) \geqq \tfrac{1}{2} + d \qquad \text{for all} \quad (M, N), \tag{6.2}$$

where $d = \frac{1}{4}\Pi_0^2[\varepsilon_Z(\frac{1}{2}\Delta k_0)]^2[\varepsilon_Y(\frac{1}{2}\Delta k_0)]^2 > 0$. The result (6.2) provides a slight refinement to (2.6 b).

We will need an upper bound on Var (w) for the non-null case. First we use (2.4) to write

$$\operatorname{Var}(w) = \binom{M}{2}^{-2}\binom{N}{2}^{-2} \sum_{i<I} \sum_{j<J} \sum_{i'<I'} \sum_{j'<J'} \operatorname{Cov}\,[u(V_{iIjJ}), u(V_{i'I'j'J'})]\,. \tag{6.3}$$

Now $\binom{M}{2}\binom{N}{2}\binom{M-2}{2}\binom{N-2}{2}$ of the $\binom{M}{2}^2\binom{N}{2}^2$ covariance terms on the right side of (6.3) will be 0, and the remaining covariance terms will all be $\leqq 1$. Hence we have

$$\begin{aligned}\operatorname{Var}(w) &\leqq \binom{M}{2}^{-2}\binom{N}{2}^{-2}\left[\binom{M}{2}^2\binom{N}{2}^2 - \binom{M}{2}\binom{N}{2}\binom{M-2}{2}\binom{N-2}{2}\right] \\ &= 1 - \binom{M}{2}^{-1}\binom{M-2}{2}\binom{N}{2}^{-1}\binom{N-2}{2} \leqq 1 - \left(1 - \frac{2}{M-1}\right)^4 < \frac{8}{M-1}\end{aligned} \tag{6.4}$$

for $M > 2$.

Let $P\{R\}$ denote the probability that the test (2.8) will reject H_0. We will show, for $\Delta > 0$, that $P\{R\} \to 1$ as $M \to \infty$. Now

$$\begin{aligned} P\{R\} &= P\left\{\left[\frac{2M+5}{18M(M-1)}\right]^{-\frac{1}{2}}|w - \tfrac{1}{2}| > z_{\frac{1}{2}\alpha}\right\} \\ &\geqq P\{w - \tfrac{1}{2} > M^{-\frac{1}{2}}z_{\frac{1}{2}\alpha}\} \\ &\geqq P\{w - E(w) > M^{-\frac{1}{2}}z_{\frac{1}{2}\alpha} - d\}\,,\end{aligned} \tag{6.5}$$

the last line of (6.5) being a consequence of (6.2). Applying Tchebycheff's inequality to (6.5), we obtain

$$P\{R\} \geqq 1 - \frac{\operatorname{Var}(w)}{(d - M^{-\frac{1}{2}}z_{\frac{1}{2}\alpha})^2} \tag{6.6}$$

for M sufficiently large that $d - M^{-\frac{1}{2}}z_{\frac{1}{2}\alpha} > 0$. Our desired result now follows at once from (6.6) if we utilize (6.4) and let $M \to \infty$.

7. The one-sample problem. Suppose we have just a single sample of size M which conforms to the model (2.1) and to the accompanying assumptions, and suppose we wish to test

$$H_0': \beta_Y = \beta_0\,, \tag{7.1}$$

where β_0 is a specified number, against alternatives $\beta_Y \neq \beta_0$. For this one-sample simple regression problem, a test statistic proposed by Theil [12, page 390] is applicable. This statistic, to which w (2.4) bears a certain similarity, is essentially

$$w' = \binom{M}{2}^{-1} \sum_{i<I} u\left(\frac{Y_I - Y_i}{X_I - X_i} - \beta_0\right). \tag{7.2}$$

As Theil has noted, $(2w' - 1)$ is closely related to the well-known statistic τ which has been considered by M. G. Kendall (see, e.g., [5], page 82, equation (1)) and by others. Although there evidently is not an exact correspondence between $(2w' - 1)$ and τ for the respective non-null cases, it is easily verified that the null distribution of $(2w' - 1)$ is identical with the null distribution of τ. Therefore it follows at once that $E(w') = \frac{1}{2}$ if H_0' is true; that

$$\operatorname{Var}(w') = \frac{2M + 5}{18M(M-1)} \quad \text{if} \quad H_0' \quad \text{is true} \tag{7.3}$$

(incidentally, note how (7.3) compares with (2.7)); and that w' is asymptotically normal under H_0'. Hence a test with critical region

$$\left[\frac{2M + 5}{18M(M-1)}\right]^{-\frac{1}{2}} |w' - \tfrac{1}{2}| > z_{\frac{1}{2}\alpha} \tag{7.4}$$

constitutes a non-parametric test of H_0' (7.1) whose level of significance (disregarding inaccuracies due to the normal approximation) is equal to α. The consistency of the test (7.4) is easily established via techniques similar to those used in Sections 3 and 6 above.

In addition to the test (7.4), another way of testing H_0' (7.1) under the model (2.1) is also available. Hájek [2] has introduced an extensive class of distribution-free tests which are applicable to the one-sample simple regression problem; the Fisher-Yates-Terry-Hoeffding c_1-test is (as he points out) one member of his class. (Explicitly, Hájek just considered the case where β_0 of (7.1) is 0, but this really involves no loss of generality since the formulation (2.1, 7.1) can effectively be put into Hájek's form if we subtract $\beta_0 X_i$ from both sides of (2.1) and subtract β_0 from both sides of (7.1).) In order to pick a specific test from Hájek's class of tests, the user must select a density function which in turn determines a function $\phi(u)$ [2, page 1125, equation (1.6)] upon which the formula for the test statistic is based. The test will have optimal power in a certain sense if the selected density is the (presumably unknown) density of the e_i's. Some users may object to the procedure of arbitrarily selecting a density each time they utilize Hájek's class of tests. However, such users could simply decide in advance that they would always utilize a certain density, such as the normal or the logistic density (densities 3 and 2 respectively in Hájek's Table 1 [2, page 1127]). In fact, it might even be possible, through further theoretical investigation, to discover a specific $\phi(u)$ (i.e., discover a specific choice of the density) which will in some sense possess a minimax property with respect to some criterion related to the power of the test.

Although we will not attempt here to undertake a detailed comparison of

Theil's test (7.4) versus Hájek's tests, we may briefly indicate some of the matters (in addition to the problem of choosing $\phi(u)$ for Hájek's tests) which should perhaps be further examined in such a comparison:

(i) Power is obviously an important consideration. The power of the tests will depend (among other things) on the X_i's, on $F_Y(e)$, on $(\beta_Y - \beta_0)$, and on the choice of $\phi(u)$ for Hájek's tests. Thus power comparisons may be somewhat difficult. For the case where $\phi(u)$ is based on the logistic density, however, the power comparison seems to be relatively clear-cut and dependent mainly on the X_i's (evidently this particular power comparison can never favor the test (7.4); usually favors the Hájek test to a modest degree; and may less commonly (i.e., for certain types of sets of X_i's) favor the Hájek test to a more substantial degree or, contrarily, find the two tests essentially equal in power).

(ii) The relative rapidity with which the null distributions of the competing test statistics approach normality could sometimes be an important matter to the test users. For the particular case where $\phi(u)$ is based on the logistic density and the X_i's consist of the integers from 1 to M, Hájek's test statistic is equivalent to Spearman's rank correlation coefficient ρ. Now the distribution of Kendall's τ evidently converges to normality a bit faster than the distribution of Spearman's ρ [6, page 58]; this situation may be indicative of a small advantage for the test (7.4).

(iii) Ease of computation is a third factor which test users may be expected to take into account. If only a test of H_0' (7.1) is desired and no associated confidence bounds are required, then it appears that the test (7.4) would generally be less simple to compute than one of Hájek's tests. If, on the other hand, a confidence interval for β_Y is needed, then basing it on a Hájek test would seem to necessitate some sort of trial-and-error computations which could be rather lengthy, whereas a confidence interval associated with the test (7.4) can be computed in a straightforward manner.

8. A second two-sample problem. An obvious companion problem to the two-sample problem treated in this paper would be to develop a non-parametric test of whether two parallel simple regression lines are the same. In other words, suppose we have the same model as indicated by (2.1)—(2.2) and the accompanying discussion, except that β_Y and β_Z are each replaced by β (i.e., the two lines are *known* to be parallel), and suppose we wish to test the hypothesis $\alpha_Y = \alpha_Z$ against alternatives $\alpha_Y \neq \alpha_Z$.

It looks as though one interesting possibility for this situation would be to develop a test based on a statistic of the form (2.4), but with V_{iIjJ} in (2.4) replaced by

$$\gamma_{iIjJ}(Z_J - Y_i) + (1 - \gamma_{iIjJ})(Z_j - Y_I), \tag{8.1}$$

where $\gamma_{iIjJ} = (X_I - W_j)/(X_I + W_J - X_i - W_j)$ (for the formulas (8.1) we assume that the X's and W's are in increasing order, so that $X_i < X_I$ and $W_j < W_J$).

Evidently such a test would not be valid under as general conditions on F_Y and F_Z as the test of the present paper: the random variable (8.1) is equal to

$$\gamma_{iIjJ}(f_J - e_i) + (1 - \gamma_{iIjJ})(f_j - e_I) + (\alpha_Z - \alpha_Y) \tag{8.2}$$

and thus has the desired median $(\alpha_Z - \alpha_Y)$ if

(8.3a) F_Y and F_Z are both symmetric (about the origin)

and/or

$$F_Y = F_Z\,, \tag{8.3b}$$

but it may have a different median if (F_Y, F_Z) do not satisfy (8.3).

If the test just proposed were to be fully developed along the lines of the present paper, the outstanding matters to be resolved would include the following:

(i) finding a satisfactory upper bound for the variance of the test statistic under the null hypothesis, given the condition (8.3);

(ii) showing that the null distribution of the test statistic is asymptotically normal under appropriate restrictions, given (8.3); and

(iii) establishing the consistency of the test under appropriate assumptions, given (8.3).

Of course, (8.1, 8.2) would have the proper median even under conditions somewhat broader than (8.3), but such broadening of the conditions might complicate the proofs and at the same time be of limited practical import.

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Final note. After this paper was essentially complete, Hollander [4] proposed a test which is a competitor to the test (2.8) above, and which has certain advantages and disadvantages relative to the test (2.8), as he points out. According to [4], for certain specific situations Monte Carlo samplings show greater power for his test than for the test (2.8). Among other things, though, there is reason to believe that the test (2.8) will be the one with superior power in situations where Var (w) under H_0 is sufficiently close to the upper limit (2.7).

Another recent related article is that of Sen [11].

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