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ON SOME TEST CRITERIA FOR COVARIANCE MATRIX

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Some new test criteria are proposed for testing various hypotheses concerning covariance matrices. Asymptotic expansions of their null distributions are derived in terms of the χ^2 distribution.

1. Introduction and summary. This paper is concerned with problems of testing hypotheses (i) for the equality of covariance matrix to a given matrix (ii) for sphericity (iii) for independence and (iv) for the equality of covariance matrices. For testing these hypotheses when we have a multivariate normal population, the usual test criteria are the likelihood ratio (=LR) tests. When the alternative hypotheses are true, the LR tests of these hypotheses are asymptotically normal. Noting that these limiting distributions are singular at the null hypotheses since the variances vanish, we propose new test statistics which are obtained from these variances by replacing the covariance matrices by their unbiased estimators. Furthermore, we derive asymptotic expansions of the distributions of their test criteria, under the null hypotheses, in terms of χ^2 distributions.

2. Preliminaries. In this section we give some lemmas used in deriving the asymptotic expansions of the distributions under the null hypotheses. First we define the logarithm of a matrix. We denote by $\mathfrak{A}$ the set of all real symmetric $p \times p$ matrices and by $\mathscr{B}$ the set of all real symmetric positive definite $p \times p$ matrices. Now we consider a mapping f from $A\ (\in \mathfrak{A})$ to e^A. This matrix e^A can be defined as follows:

$$e^A = I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots . \tag{2.1}$$

Then we have the following lemma. (See Chevalley [1] page 14.)

LEMMA 2.1. *This function f is one to one mapping and $e^{\mathfrak{A}} = \mathscr{B}$, where $e^{\mathfrak{A}}$ means the set of all matrices e^A for $A \in \mathfrak{A}$.*

By this lemma, we can define the logarithm for a real symmetric positive definite matrix.

Next we consider the distribution of the matrix $Y = (n/2)^{\frac{1}{2}} \log S/n$, where $S(p \times p)$ has the Wishart distribution $W(I, n)$. Since $S = ne^{(2/n)^{\frac{1}{2}}Y}$, expressing the characteristic roots of the matrix Y as $ch_i(Y)$, as in Jack ([6] Lemma 8) the Jacobian is given by

$$\left|\frac{\partial S}{\partial Y}\right| = (2n)^{p(p+1)/4} \operatorname{etr}[(2/n)^{\frac{1}{2}} Y] \prod_{i>j}^{p} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}, \tag{2.2}$$

where $f(\lambda_i) = e^{\lambda_i}$ with $\lambda_i = (2/n)^{\frac{1}{2}} ch_i(Y)$.

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Since we are interested in obtaining asymptotic expansions, we note that the last term $\prod_{i>j}^{p} [f(\lambda_i) - f(\lambda_j)]/[\lambda_i - \lambda_j]$ in the Jacobian (2.2) can be expanded for large n as

$$\prod_{i>j}^{p} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} = 1 + \tfrac{1}{2}(p-1)(2/n)^{\frac{1}{2}} \operatorname{tr} Y + \frac{1}{12n}\{(3p^2 - 6p + 2)(\operatorname{tr} Y)^2 + p \operatorname{tr} Y^2\} + O(n^{-\frac{3}{2}}) . \tag{2.3}$$

Since $|e^A| = \operatorname{etr} A$ for any square matrix A, the "asymptotic" distribution of Y is given by

$$c^* \cdot \operatorname{etr}[\tfrac{1}{2}(n-p+1)(2/n)^{\frac{1}{2}}Y - (n/2)e^{(2/n)^{\frac{1}{2}}Y}] \times \left\{1 + \tfrac{1}{2}(p-1)(2/n)^{\frac{1}{2}} \operatorname{tr} Y + \frac{1}{12n}\{(3p^2 - 6p + 2)(\operatorname{tr} Y)^2 + p \operatorname{tr} Y^2\} + O(n^{-\frac{3}{2}})\right\} , \tag{2.4}$$

where

$$c^* = \{\prod_{\alpha=1}^{p} \Gamma[\tfrac{1}{2}(n+1-\alpha)]\}^{-1}(n/2)^{p(2n-p-1)/4}\pi^{-p(p-1)/4} . \tag{2.5}$$

Finally we state three well-known lemmas.

LEMMA 2.2. *Let a $p(p+1)/2 \times 1$ vector $(y_{11}, y_{22}, \cdots, y_{pp}, y_{12}, y_{13}, \cdots, y_{p-1,p})'$ have a normal distribution with mean zero and covariance matrix $\Sigma^* = (\sigma_{i,j;k,l})$. Put the subscript pairs $(i, j) = a$, $(k, l) = b$, $(m, n) = c$, $(q, r) = d$, $(s, t) = e$ and $(u, v) = f$, then we have*

$$Ey_a y_b y_c y_d = \sigma_{a;b}\sigma_{c;d} + \sigma_{a;c}\sigma_{b;d} + \sigma_{a;d}\sigma_{b;c} , \tag{2.6}$$

and

$$\begin{aligned} Ey_a y_b y_c y_d y_e y_f = {} & \sigma_{a;b}[\sigma_{c;d}\sigma_{e;f} + \sigma_{c;e}\sigma_{d;f} + \sigma_{c;f}\sigma_{d;e}] \\ & + \sigma_{a;c}[\sigma_{b;d}\sigma_{e;f} + \sigma_{b;e}\sigma_{d;f} + \sigma_{b;f}\sigma_{d;e}] \\ & + \sigma_{a;d}[\sigma_{b;c}\sigma_{e;f} + \sigma_{b;e}\sigma_{c;f} + \sigma_{b;f}\sigma_{c;e}] \\ & + \sigma_{a;e}[\sigma_{b;c}\sigma_{d;f} + \sigma_{b;d}\sigma_{c;f} + \sigma_{b;f}\sigma_{c;d}] \\ & + \sigma_{a;f}[\sigma_{b;c}\sigma_{d;e} + \sigma_{b;d}\sigma_{c;e} + \sigma_{b;e}\sigma_{c;d}] . \end{aligned} \tag{2.7}$$

LEMMA 2.3. *Let A and C be $p \times p$ matrices and B and D be $q \times q$ matrices. Then for the Kronecker product of matrices, we have*

$$(A + C) \otimes B = A \otimes B + C \otimes B , \tag{2.8}$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD , \tag{2.9}$$

and

$$|A \otimes B| = |A|^q |B|^p . \tag{2.10}$$

LEMMA 2.4. *Let P_1 and P_2 be symmetric and idempotent matrices such that $P_1 + P_2 = I$ and $P_1 P_2 = 0$. Then for any numbers c_1 and c_2 $(c_1 c_2 \neq 0)$,*

$$(c_1 P_1 + c_2 P_2)^{-1} = c_1^{-1} P_1 + c_2^{-1} P_2 . \tag{2.11}$$

3. Some test statistics for covariance. Let the $p \times 1$ vectors $X_1, X_2, \cdots, X_N$ be a random sample from a normal distribution with mean vector μ and covariance matrix Σ. The modified LR criterion for testing the hypothesis $H_1: \Sigma = \Sigma_0$ against the alternatives $K_1: \Sigma \neq \Sigma_0$ for some given positive definite matrix Σ_0, is given by

$$\lambda_1 = (e/n)^{(np/2)}|S\Sigma_0^{-1}|^{n/2} \operatorname{etr}[-\tfrac{1}{2}\Sigma_0^{-1}S], \tag{3.1}$$

where $S = \sum_{\alpha=1}^{N}(X_\alpha - \bar{X})(X_\alpha - \bar{X})'$, $\bar{X} = N^{-1}\sum_{\alpha=1}^{N} X_\alpha$ and $n = N - 1$. The unbiasedness of this test is shown by Sugiura and Nagao [14] and the monotonicity of the power function with respect to p characteristic roots of $\Sigma\Sigma_0^{-1}$ is established by Nagao [10] and Das Gupta [2]. Also Sugiura [15] gave the asymptotic expansion of the statistic $-2 \log \lambda_1$ under the alternative, the first term of which is a normal distribution. Our concern is the variance of the limiting distribution. Without a constant multiplier, this variance is given by

$$\tau_1^2 = \operatorname{tr}(\Sigma\Sigma_0^{-1} - I)^2. \tag{3.2}$$

Thus τ_1^2 may be regarded as a measure of departure from the null hypothesis. So replacing Σ by the unbiased estimator S/n, we propose a test criterion

$$T_1 = \frac{n}{2}\operatorname{tr}(S\Sigma_0^{-1}/n - I)^2. \tag{3.3}$$

Thus the T_1 test rejects the hypothesis H_1, when the observed value of T_1 is larger than a preassigned constant.

Similarly, using the same notation as above, the modified LR statistic for testing the sphericity hypothesis $H_2: \Sigma = \sigma^2 I$ against the alternatives $K_2: \Sigma \neq \sigma^2 I$, where σ^2 is unspecified, is given by

$$\lambda_2 = |S|^{n/2}(p^{-1}\operatorname{tr} S)^{-np/2}. \tag{3.4}$$

The unbiasedness of this test criterion was proved by Gleser [4] and Sugiura and Nagao [14]. Also Sugiura [15] and Nagao [12] derived the asymptotic expansions under the alternatives and the local alternatives, respectively. The variance of the limiting distribution of $-2 \log \lambda_2$ is proportional to

$$\tau_2^2 = \operatorname{tr}\{\Sigma(\operatorname{tr}\Sigma)^{-1} - p^{-1}I\}^2. \tag{3.5}$$

Thus for the sphericity test, we propose the following test statistic:

$$T_2 = \frac{p^2 n}{2}\operatorname{tr}\{S(\operatorname{tr} S)^{-1} - p^{-1}I\}^2. \tag{3.6}$$

From a different standpoint, the above criterion T_2 was shown to be the locally best invariant test by John [7] and Sugiura [16].

Next we consider the test of independence. Let the $1 \times p$ vector $X' = (X^{(1)\prime}, X^{(2)\prime}, \cdots, X^{(q)\prime})$, having a normal distribution with mean vector $\mu' = (\mu^{(1)\prime}, \cdots, \mu^{(q)\prime})$ and covariance matrix $\Sigma = (\Sigma_{ij})$, $(i, j = 1, 2, \cdots, q)$, be partitioned into q sub-vectors with components $p_1, p_2, \cdots, p_q$, respectively. Given a sample $X_1, X_2, \cdots, X_N$ of N observations on X, we wish to test the hypothesis $H_3: \Sigma_{ij} = 0$

$(i \neq j)$ against the alternatives K_3: $\Sigma_{ij} \neq 0$ for some i, j $(i \neq j)$. Then the LR criterion for this hypothesis is given by

$$\lambda_3 = \frac{|S|^{N/2}}{\prod_{\alpha=1}^{q} |S_{\alpha\alpha}|^{N/2}}, \tag{3.7}$$

where S is partitioned in the same manner as Σ. The corresponding matrix $S_{\alpha\alpha}$ is defined. The unbiasedness of this test is shown by Narain [9]. Also Nagao [13] gave the asymptotic expansion of the distribution under the fixed alternative. The asymptotic variance of $-2 \log \lambda_3$ is proportional to

$$\tau_3^2 = \operatorname{tr}(\Sigma\Sigma_D^{-1} - I)^2, \tag{3.8}$$

where the above Σ_D is a matrix obtained by replacing all submatrices $\Sigma_{ij} = 0$ $(i \neq j)$ in the matrix Σ. Thus for the test of independence, we propose the following test statistic:

$$T_3 = \frac{n}{2} \operatorname{tr}(SS_D^{-1} - I)^2, \tag{3.9}$$

where S_D is a matrix which is obtained by replacing $\Sigma_{\alpha\alpha}$ by $S_{\alpha\alpha}$ in Σ_D. In the case $q = 2$, (3.9) reduces to $T_3 = n \operatorname{tr} S_{12}S_{22}^{-1}S_{21}S_{11}^{-1}$, which is Pillai's statistic.

Finally by the same consideration, for testing for the homogeneity of covariance matrices we propose a test criterion. Let the $p \times 1$ vectors $X_{i1}, X_{i2}, \cdots, X_{iN_i}$ be a random sample from a normal distribution with mean μ_i and covariance matrix Σ_i $(i = 1, 2, \cdots, k)$. The modified LR statistic for testing the hypothesis H_4: $\Sigma_1 = \Sigma_2 = \cdots = \Sigma_k$ against the alternatives K_4: $\Sigma_i \neq \Sigma_j$ for some i, j $(i \neq j)$, is given by

$$\lambda_4 = \{\textstyle\prod_{\alpha=1}^{k} |S_\alpha/n_\alpha|^{n_\alpha/2}\}/|S/n|^{n/2}, \tag{3.10}$$

where $S_\alpha = \sum_{\beta=1}^{N_\alpha} (X_{\alpha\beta} - \bar{X}_\alpha)(X_{\alpha\beta} - \bar{X}_\alpha)'$, $\bar{X}_\alpha = N_\alpha^{-1} \sum_{\beta=1}^{N_\alpha} X_{\alpha\beta}$, $S = \sum_{\alpha=1}^{k} S_\alpha$, $n_\alpha = N_\alpha - 1$ and $n = \sum_{\alpha=1}^{k} n_\alpha$. Under the alternative, the asymptotic expansion of the distribution of the statistic $-2 \log \lambda_4$ was given by the previous paper [11]. Also in the two-sample case this test is shown to be unbiased by Sugiura and Nagao [14]. The variance of the limiting distribution without a constant multiplier is expressed as

$$\tau_4^2 = \textstyle\sum_{\alpha=1}^{k} \rho_\alpha \operatorname{tr}(\Sigma_\alpha \tilde{\Sigma}^{-1} - I)^2, \tag{3.11}$$

where $\rho_\alpha = n_\alpha/n$ and $\tilde{\Sigma} = \sum_{\alpha=1}^{k} \rho_\alpha \Sigma_\alpha$.

Thus for homogeneity of covariances, we propose the following test criterion:

$$T_4 = \tfrac{1}{2} \textstyle\sum_{\alpha=1}^{k} n_\alpha \operatorname{tr}\left\{\frac{S_\alpha}{n_\alpha}\left(\frac{S}{n}\right)^{-1} - I\right\}^2. \tag{3.12}$$

4. Expansion of the distribution of the criterion for $\Sigma = \Sigma_0$. We shall give the asymptotic expansion of the null distribution of the test criterion T_1. Without loss of generality we may assume that S has the Wishart distribution $W(I, n)$. Then the characteristic function of the statistic $T_1 = (n/2) \operatorname{tr}(S/n - I)^2$ is given by

$$C_1(t) = c_{p,n} \int \exp[itT_1]|S|^{\frac{1}{2}(n-p-1)} \operatorname{etr}[-\tfrac{1}{2}S]\, dS\,, \tag{4.1}$$

where

$$c_{p,n}^{-1} = 2^{np/2}\pi^{p(p-1)/4} \prod_{\alpha=1}^{p} \Gamma[\tfrac{1}{2}(n+1-\alpha)]\,. \tag{4.2}$$

By expressing T_1 with $Y = (n/2)^{\frac{1}{2}} \log S/n$, we have

$$T_1 = \operatorname{tr} Y^2 + (2/n)^{\frac{1}{2}} \operatorname{tr} Y^3 + \frac{7}{6n} \operatorname{tr} Y^4 + O(n^{-\frac{3}{2}})\,. \tag{4.3}$$

Therefore using the formula (2.4) we can rewrite the characteristic function $C_1(t)$ as

$$\begin{aligned} C_1(t) = c^* \int \exp\Big[& (it) \operatorname{tr} Y^2 + (2/n)^{\frac{1}{2}}(it) \operatorname{tr} Y^3 \\ & + \frac{7}{6n}(it) \operatorname{tr} Y^4 + \tfrac{1}{2}(n-p+1)(2/n)^{\frac{1}{2}} \operatorname{tr} Y - \frac{n}{2} \operatorname{tr} e^{(2/n)^{\frac{1}{2}}Y}\Big] \\ & \times \Big\{1 + \tfrac{1}{2}(p-1)(2/n)^{\frac{1}{2}} \operatorname{tr} Y \\ & + \frac{1}{12n}\{(3p^2-6p+2)(\operatorname{tr} Y)^2 + p \operatorname{tr} Y^2\} + O(n^{-\frac{3}{2}})\Big\}\, dY\,, \end{aligned} \tag{4.4}$$

where the region of the integral is the set of all real symmetric $p \times p$ matrices. Since we have

$$e^{(2/n)^{\frac{1}{2}}Y} = I + (2/n)^{\frac{1}{2}}Y + \frac{1}{n}Y^2 + \frac{2^{\frac{1}{2}}}{3nn^{\frac{1}{2}}}Y^3 + \frac{1}{6n^2}Y^4 + O(n^{-\frac{5}{2}})\,, \tag{4.5}$$

after some calculation, the function $C_1(t)$ is given by

$$\begin{aligned} C_1(t) = c^* \cdot \exp\Big[-\frac{np}{2}\Big] \int \exp[-\tfrac{1}{2}(1-2it) \operatorname{tr} Y^2]\Big[& 1 + (2/n)^{\frac{1}{2}}(\mathrm{it} - \tfrac{1}{6}) \operatorname{tr} Y^3 \\ & + \frac{1}{n}\Big\{-\tfrac{1}{12}(\operatorname{tr} Y)^2 + \frac{p}{12} \operatorname{tr} Y^2 + \tfrac{1}{12}(14it-1) \operatorname{tr} Y^4 \\ & + (it - \tfrac{1}{6})^2(\operatorname{tr} Y^3)^2\Big\} + O(n^{-\frac{3}{2}})\Big] dY\,. \end{aligned} \tag{4.6}$$

Thus we may regard the variable Y as having a $p(p+1)/2$ dimensional normal distribution, with mean zero and covariance matrix $(\sigma_{i,j;k,l})$ with

$$\sigma_{i,j;k,l} = (1-2it)^{-1}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2\,.$$

Since $|(\sigma_{i,j;k,l})| = (1-2it)^{-f}2^{-p(p-1)/2}$, we have

$$\begin{aligned} C_1(t) = c \cdot (1-2it)^{-(f/2)}E\Big[& 1 + (2/n)^{\frac{1}{2}}(it - \tfrac{1}{6}) \operatorname{tr} Y^3 \\ & + \frac{1}{n}\Big\{-\tfrac{1}{12}(\operatorname{tr} Y)^2 + \frac{p}{12} \operatorname{tr} Y^2 + \tfrac{1}{12}(14it-1) \operatorname{tr} Y^4 \\ & + (it - \tfrac{1}{6})^2(\operatorname{tr} Y^3)^2\Big\} + O(n^{-\frac{3}{2}})\Big]\,, \end{aligned} \tag{4.7}$$

where $f = \frac{1}{2}p(p+1)$ and

$$c = c^*(2\pi)^{\frac{1}{4}p(p+1)}2^{-p(p-1)/4}\exp[-\tfrac{1}{2}pn]\,. \tag{4.8}$$

Using Lemma 2.2, we can obtain the following moments:

$$E(\operatorname{tr} Y)^2 = p(t)_1\,, \qquad E\operatorname{tr} Y^2 = \tfrac{1}{2}p(p+1)(t)_1\,, \tag{4.9}$$

$$E\operatorname{tr} Y^4 = \frac{p}{4}(2p^2+5p+5)(t)_2\,, \qquad E(\operatorname{tr} Y^3)^2 = \tfrac{3}{4}p(4p^2+9p+7)(t)_3\,,$$

where $(t)_\alpha = (1-2it)^{-\alpha}$.
Since all product moments of odd degree from a normal population with mean zero are zero, the expectations in (4.7) are given by

$$1+\frac{1}{n}\left\{\frac{p}{2}(p+1)^2(t)_1 - \frac{p}{8}(6p^2+13p+9)(t)_2 + \frac{p}{12}(4p^2+9p+7)(t)_3\right\} + O(n^{-2})\,. \tag{4.10}$$

Applying Stirling's formula to the constant factor in (4.7), we have

$$(1-2it)^{-(f/2)}\left\{1-\frac{p}{24n}(2p^2+3p-1)\right\} + O(n^{-2})\,. \tag{4.11}$$

Thus we obtain the following asymptotic formula:

$$\begin{aligned} C_1(t) = (1-2it)^{-(f/2)}\Big[1+\frac{1}{n}\Big\{&\frac{p}{12}(4p^2+9p+7)(t)_3 \\ &-\frac{p}{8}(6p^2+13p+9)(t)_2 + \frac{p}{2}(p+1)^2(t)_1 \\ &-\frac{p}{24}(2p^2+3p-1)\Big\} + O(n^{-2})\Big]\,. \end{aligned} \tag{4.12}$$

Inversion of this characteristic function yields the following theorem:

THEOREM 4.1. *The null distribution of the T_1 given by* (3.3), *expanded asymptotically in terms of the χ^2-distribution for large n, is*

$$\begin{aligned} \Pr(T_1 \leqq x) = P_f + \frac{1}{n}\Big\{&\frac{p}{12}(4p^2+9p+7)P_{f+6} - \frac{p}{8}(6p^2+13p+9)P_{f+4} \\ &+\frac{p}{2}(p+1)^2P_{f+2} - \frac{p}{24}(2p^2+3p-1)P_f\Big\} + O(n^{-2})\,, \end{aligned} \tag{4.13}$$

where $f = \frac{1}{2}p(p+1)$ and $P_f = P(\chi_f^2 \leqq x)$.

5. Expansion of the distribution of the criterion for sphericity. By the same consideration as above, the characteristic function of the criterion T_2 is given by

$$\begin{aligned} C_2(t) = c\cdot(1-2it)^{-(f/2)}E\Big[1+\frac{1}{n}\Big\{&\frac{p}{12}\operatorname{tr} Y^2 - \tfrac{1}{12}(\operatorname{tr} Y)^2 \\ &+\tfrac{1}{12}(14it-1)\operatorname{tr} Y^4 - \frac{14}{3p}(it)\operatorname{tr} Y\operatorname{tr} Y^3 - \frac{5}{2p}(it)(\operatorname{tr} Y^2)^2 \end{aligned} \tag{5.1}$$

$$+\frac{12}{p^2}(it)(\mathrm{tr}\,Y)^2\,\mathrm{tr}\,Y^2-\frac{6}{p^3}(it)(\mathrm{tr}\,Y)^4$$
$$+\left[(it-\tfrac{1}{6})\,\mathrm{tr}\,Y^3-\frac{3}{p}(it)\,\mathrm{tr}\,Y\,\mathrm{tr}\,Y^2+\frac{2}{p^2}(it)(\mathrm{tr}\,Y)^3\right]^2\Big\}$$
$$+O(n^{-2})\Big],$$

where $f=\frac{1}{2}p(p+1)-1$ and the expectation in (5.1) is taken under a normal distribution with mean zero and covariance matrix $(\sigma_{i,j;k,l})$ with $\sigma_{i,j;k,l}=(t)_1(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})/2+p^{-1}\{1-(t)_1\}\delta_{ij}\delta_{kl}$. After using Lemma 2.2 to calculate the expectations in (5.1) and simplifying the resulting expression, we obtain

$$\begin{aligned}C_2(t)=(1-2it)^{-(f/2)}[1&+n^{-1}\{\tfrac{1}{12}(p^3+3p^2-8p-12-200p^{-1})(t)_3\\&+\tfrac{1}{8}(-2p^3-5p^2+7p+12+420p^{-1})(t)_2\\&+\tfrac{1}{4}(p^3+2p^2-p-2-216p^{-1})(t)_1\\&+\tfrac{1}{24}(-2p^3-3p^2+p+436p^{-1})\}]+O(n^{-2}),\end{aligned}\tag{5.2}$$

where $f=\frac{1}{2}p(p+1)-1$.

By inverting this characteristic function, we have the following theorem:

THEOREM 5.1. *The null distribution of the test* T_2 *given by* (3.6), *expanded asymptotically in terms of the* χ^2*-distribution for large* n, *is*

$$\begin{aligned}\Pr(T_2\leqq x)=P_f&+n^{-1}\{\tfrac{1}{12}(p^3+3p^2-8p-12-200p^{-1})P_{f+6}\\&+\tfrac{1}{8}(-2p^3-5p^2+7p+12+420p^{-1})P_{f+4}\\&+\tfrac{1}{4}(p^3+2p^2-p-2-216p^{-1})P_{f+2}\\&+\tfrac{1}{24}(-2p^3-3p^2+p+436p^{-1})P_f\}+O(n^{-2}),\end{aligned}\tag{5.3}$$

where $P_f=P(\chi_f{}^2\leqq x)$ *with* $f=\frac{1}{2}p(p+1)-1$.

6. Expansion of the distribution of the criterion for independence. The characteristic function of the criterion T_3 is given by

$$\begin{aligned}C_3(t)=c\cdot(1-2it)^{-(f/2)}E\Big[1&+\frac{1}{n}\Big\{\frac{p}{12}\,\mathrm{tr}\,Y^2-\tfrac{1}{12}(\mathrm{tr}\,Y)^2\\&+\tfrac{1}{12}(14it-1)\,\mathrm{tr}\,Y^4-\tfrac{14}{3}(it)\,\mathrm{tr}\,Y_DY^3\\&-\tfrac{5}{2}(it)\,\mathrm{tr}\textstyle\sum_{\alpha=1}^q(\sum_{\beta=1}^q Y_{\alpha\beta}Y_{\beta\alpha})^2+10(it)\,\mathrm{tr}\,Y_D{}^2Y^2\\&+2(it)\,\mathrm{tr}\,(Y_DY)^2-6(it)\,\mathrm{tr}\,Y_D{}^4\\&+[(it-\tfrac{1}{6})\,\mathrm{tr}\,Y^3-3(it)\,\mathrm{tr}\,Y_DY^2+2(it)\,\mathrm{tr}\,Y_D{}^3]^2\Big\}+O(n^{-2})\Big],\end{aligned}\tag{6.1}$$

where $f=\frac{1}{2}(p^2-\sum_{\alpha=1}^q p_\alpha{}^2)$ and Y_D is a matrix corresponding to Σ_D. The expectation of (6.1) is taken under the normal distribution with mean zero and covariance matrix $(\sigma_{i,j;k,l})$ with $\sigma_{i,j;k,l}=\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\{(t)_1+\{1-(t)_1\}\delta_{\alpha\beta}\delta_{\alpha\gamma}\delta_{\alpha\delta}\}$ for $p_1+\cdots+p_{\alpha-1}+1\leqq i\leqq p_1+\cdots+p_\alpha$, $p_1+\cdots+p_{\beta-1}+1\leqq j\leqq p_1+\cdots+p_\beta$, $p_1+\cdots+p_{\gamma-1}+1\leqq k\leqq p_1+\cdots+p_\gamma$ and $p_1+\cdots+p_{\delta-1}+1\leqq l\leqq p_1+\cdots+p_\delta$ with $p_0=0$ $(\alpha,\beta,\gamma,\delta=1,2,\cdots,q)$. Thus by Lemma

2.2 the characteristic function $C_3(t)$ can be expanded asymptotically as

$$
\begin{aligned}
C_3(t) = (1-2it)^{-(f/2)}[1 &+ n^{-1}\{\tfrac{1}{12}(p^3 - 3p\bar{p}_2 + 2\bar{p}_3)(t)_3 \\
&+ \tfrac{1}{8}(-2p^3 + 4p\bar{p}_2 - 2\bar{p}_3 - p^2 + \bar{p}_2)(t)_2 \\
&+ \tfrac{1}{4}(p^3 - p\bar{p}_2 + p^2 - \bar{p}_2)(t)_1 + \tfrac{1}{24}(-2p^3 + 2\bar{p}_3 - 3p^2 + 3\bar{p}_2)\}] \\
&+ O(n^{-2}),
\end{aligned} \tag{6.2}
$$

where $\bar{p}_\beta = \sum_{\alpha=1}^{q} p_\alpha{}^\beta$.

Inverting this characteristic function, we have the following theorem:

THEOREM 6.1. *The null distribution of the statistic T_3 given by* (3.9) *is expanded asymptotically for large n as*

$$
\begin{aligned}
\Pr(T_3 \leqq x) = P_f + n^{-1}\{&\tfrac{1}{12}(p^3 - 3p\bar{p}_2 + 2\bar{p}_3)P_{f+6} \\
&+ \tfrac{1}{8}(-2p^3 + 4p\bar{p}_2 - 2\bar{p}_3 - p^2 + \bar{p}_2)P_{f+4} \\
&+ \tfrac{1}{4}(p^3 - p\bar{p}_2 + p^2 - \bar{p}_2)P_{f+2} \\
&+ \tfrac{1}{24}(-2p^3 + 2\bar{p}_3 - 3p^2 + 3\bar{p}_2)P_f\} + O(n^{-2}),
\end{aligned} \tag{6.3}
$$

where $\bar{p}_\beta = \sum_{\alpha=1}^{q} p_\alpha{}^\beta$ and $f = \frac{1}{2}(p^2 - \bar{p}_2)$.

We remark that this formula (6.3) agrees, in the case $q = 2$, with expansions obtained previously by Lee [8] and Fujikoshi [3].

7. Expansion of the distribution of the criterion for homogeneity of covariance matrices. By putting $Y_\alpha = (y_{ij}^{(\alpha)}) = (n_\alpha/2)^{\frac{1}{2}} \log S_\alpha/n_\alpha$ and $\rho_\alpha = n_\alpha/n$ $(\alpha = 1, 2, \cdots, k)$ with $n = \sum_{\alpha=1}^{k} n_\alpha$, the characteristic function of the statistic T_4 is given by $C_4(t) = C_4{}^*(t)C_4{}^{**}(t)$, where

$$
\begin{aligned}
C_4{}^*(t) = \textstyle\prod_{\alpha=1}^{k} [\{\prod_{j=1}^{p} \Gamma[\tfrac{1}{2}(n_\alpha + 1 - j)]\}^{-1} n_\alpha{}^{\frac{1}{4}p(2n_\alpha - p - 1)}] \exp[-\tfrac{1}{2}np] \\
\times \pi^{(k/2)p} 2^{-kp(p-1)/4} (1-2it)^{-(f/2)},
\end{aligned} \tag{7.1}
$$

and

$$
\begin{aligned}
C_4{}^{**}(t) = E\Big[1 + \frac{1}{n}\Big\{&\frac{p}{12} \textstyle\sum_{\alpha=1}^{k} \operatorname{tr} Y_\alpha{}^2/\rho_\alpha - \tfrac{1}{12} \sum_{\alpha=1}^{k} (\operatorname{tr} Y_\alpha)^2/\rho_\alpha \\
&- 6(it) \operatorname{tr} (\textstyle\sum_{\alpha=1}^{k} \rho_\alpha{}^{\frac{1}{2}} Y_\alpha)^4 - \tfrac{5}{2}(it) \operatorname{tr} (\sum_{\alpha=1}^{k} Y_\alpha{}^2)^2 \\
&+ 2(it) \operatorname{tr} \textstyle\sum_{\beta=1}^{k} [Y_\beta(\sum_{\alpha=1}^{k} \rho_\alpha{}^{\frac{1}{2}} Y_\alpha)]^2 + \tfrac{1}{12}(14it - 1) \operatorname{tr} \sum_{\alpha=1}^{k} Y_\alpha{}^4/\rho_\alpha \\
&+ 10(it) \operatorname{tr} (\textstyle\sum_{\alpha=1}^{k} \rho_\alpha{}^{\frac{1}{2}} Y_\alpha)^2 (\sum_{\alpha=1}^{k} Y_\alpha{}^2) \\
&- \tfrac{14}{3}(it) \operatorname{tr} (\textstyle\sum_{\alpha=1}^{k} Y_\alpha{}^3/\rho_\alpha{}^{\frac{1}{2}})(\sum_{\alpha=1}^{k} \rho_\alpha{}^{\frac{1}{2}} Y_\alpha) \\
&+ [(it - \tfrac{1}{6}) \operatorname{tr} \textstyle\sum_{\alpha=1}^{k} Y_\alpha{}^3/\rho_\alpha{}^{\frac{1}{2}} + 2(it) \operatorname{tr} (\sum_{\alpha=1}^{k} \rho_\alpha{}^{\frac{1}{2}} Y_\alpha)^3 \\
&- 3(it) \operatorname{tr} \textstyle\sum_{\alpha=1}^{k} Y_\alpha{}^2 (\sum_{\alpha=1}^{k} \rho_\alpha{}^{\frac{1}{2}} Y_\alpha)]^2\Big\} + O(n^{-2})\Big],
\end{aligned} \tag{7.2}
$$

with $f = (k-1)p(p+1)/2$. The expectation in (7.2) is taken under a $kp(p+1)/2$ dimensional normal distribution with mean zero and covariance matrix $(\sigma_{i,j;k,l}^{(\alpha,\beta)})$. This covariance matrix can be calculated as follows: Let $y = (y_{11}^{(1)}, \cdots, y_{pp}^{(1)}, y_{12}^{(1)}, \cdots, y_{p-1,p}^{(1)}, \cdots, y_{11}^{(k)}, \cdots, y_{pp}^{(k)}, y_{12}^{(k)}, \cdots, y_{p-1,p}^{(k)})'$, $\gamma = (\rho_1{}^{\frac{1}{2}}, \rho_2{}^{\frac{1}{2}}, \cdots, \rho_k{}^{\frac{1}{2}})'$ and $\tilde{I} = \operatorname{diag}(1, \cdots, 1, 2, \cdots, 2)$ where the multiplicities of the diagonal elements 1 and 2 are p and $p(p-1)/2$, respectively. Then by Lemma 2.3 and 2.4 we have

$$(7.3)\quad \begin{aligned}&(1-2it)\textstyle\sum_{\alpha=1}^{k}\operatorname{tr}Y_\alpha{}^2+2(it)\operatorname{tr}(\sum_{\alpha=1}^{k}\rho_\alpha{}^{\frac{1}{2}}Y_\alpha)^2\\ &\qquad=y'\{(1-2it)I_k\otimes\tilde I+2(it)\gamma\gamma'\otimes\tilde I\}y\\ &\qquad=y'[\{(1-2it)(I_k-\gamma\gamma')+\gamma\gamma'\}\otimes\tilde I\}y\\ &\qquad=y'[\{(1-2it)^{-1}(I_k-\gamma\gamma')+\gamma\gamma'\}\otimes\tilde I^{-1}]^{-1}y\,.\end{aligned}$$

Thus the covariance between $y_{ij}^{(\alpha)}$ and $y_{kl}^{(\beta)}$ is given by

$$\sigma_{i,j;k,l}^{(\alpha;\beta)}=\{(t)_1(\delta_{\alpha\beta}-(\rho_\alpha\rho_\beta)^{\frac{1}{2}})+(\rho_\alpha\rho_\beta)^{\frac{1}{2}}\}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})/2\,.$$

Since we have

$$(7.4)\qquad C_4{}^*(t)=(1-2it)^{-(f/2)}\left[1-\frac{1}{24n}\tilde\rho p(2p^2+3p-1)+O(n^{-2})\right],$$

by using Lemma 2.2 we can express $C_4(t)$ in the form

$$(7.5)\quad \begin{aligned}C_4(t)=(1-2it)^{-(f/2)}[1&+n^{-1}[\tfrac{1}{12}\{\tilde\rho p(4p^2+9p+7)-3k^2p(p+1)^2\\ &-(3k-2)p(p^2+3p+4)\}(t)_3+\tfrac{1}{8}\{-\tilde\rho p(6p^2+13p+9)\\ &+4k^2p(p+1)^2+(2k-1)p(2p^2+5p+5)\}(t)_2\\ &+\tfrac{1}{4}(2\tilde\rho-k^2-k)p(p+1)^2(t)_1+\tfrac{1}{24}(1-\tilde\rho)p(2p^2+3p-1)]\\ &+O(n^{-2})]\,,\end{aligned}$$

where $\tilde\rho=\sum_{\alpha=1}^{k}\rho_\alpha{}^{-1}$.
Inverting (7.5) gives

THEOREM 7.1. *The null distribution of the test T_4 given by* (3.12), *for large $n=\sum_{\alpha=1}^{k}n_\alpha$, is*

$$(7.6)\quad \begin{aligned}\Pr(T_4\leqq x)=P_f&+n^{-1}[\tfrac{1}{12}\{\tilde\rho p(4p^2+9p+7)-3k^2p(p+1)^2\\ &-(3k-2)p(p^2+3p+4)\}P_{f+6}+\tfrac{1}{8}\{-\tilde\rho p(6p^2+13p+9)\\ &+4k^2p(p+1)^2+(2k-1)p(2p^2+5p+5)\}P_{f+4}\\ &+\tfrac{1}{4}(2\tilde\rho-k^2-k)p(p+1)^2P_{f+2}\\ &+\tfrac{1}{24}(1-\tilde\rho)p(2p^2+3p-1)P_f]+O(n^{-2})\,,\end{aligned}$$

where $\tilde\rho=\sum_{\alpha=1}^{k}\rho_\alpha{}^{-1}$ and $P_f=P(\chi_f{}^2\leqq x)$ with $f=\frac{1}{2}(k-1)p(p+1)$.

8. Numerical example. The null distribution of each test statistic T_i has the following asymptotic expansion:

$$(8.1)\qquad \Pr(T_i\leqq x)=P_f+n^{-1}\{a_6P_{f+6}+a_4P_{f+4}+a_2P_{f+2}+a_0P_f\}+O(n^{-2})\,,$$

where $\sum_{i=0}^{3}a_{2i}=0$ and the different coefficients a's and f degrees of freedom depend on T_i. Applying the general inverse expansion formula due to Hill and Davis [5], we obtain the following asymptotic formula for the 100α % point of T_i:

$$(8.2)\quad u+\frac{1}{n}\left[\frac{2a_6u}{f(f+2)(f+4)}\{u^2+(f+4)u+(f+2)(f+4)\}\right.\\ \left.+\frac{2a_4u}{f(f+2)}(u+f+2)+\frac{2a_2u}{f}\right]+O(n^{-2})\,,$$

where $\Pr(\chi_f{}^2\geqq u)=\alpha$.

We give the approximate 5 % point of each test T_i ($i = 1, 2, 3, 4$).

test	T_1	T_2	T_3	T_4
case	$p = 2$, $n = 100$	$p = 2$, $n = 100$	$p_1 = p_2 = p_3 = 1$, $n = 100$	$p = 2$, $n_1 = n_2 = 100$
first term	7.815	5.991	7.815	7.815
second term	0.033	0.150	−0.076	−0.149
approx. value	7.848	6.141	7.739	7.666

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