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EXPECTATION OF ELEMENTARY SYMMETRIC FUNCTIONS OF A WISHART MATRIX

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Some conjectures made by De Waal [*Ann. Math. Statist.* **43** (1972) 344–347] on the expectation of elementary symmetric functions of the roots of a noncentral Wishart matrix are proved true. The method of proof is through a simple, though perhaps obscure, property of these elementary symmetric functions and simple properties of the Wishart distribution.

1. Introduction. If $X(p \times n)$ has independently, normally distributed columns with covariance V and $\mathscr{E}(X) = M$, then $A = XX'$ has the noncentral Wishart distribution. Write $\text{tr}_j(A)$ for the jth elementary symmetric function of the latent roots of A and set $\text{tr}_0(A) = 1$ for convenience. Put $\Omega = V^{-1}MM'$ and $(a)^{(i)} = a(a-1)(a-2)\cdots(a-i+1)$.

De Waal (1972) conjectures that

$$\mathscr{E}\,\text{tr}_p(A) = \mathscr{E}|A| = |V| \textstyle\sum_{i=0}^{p} (n-i)^{(p-i)}\, \text{tr}_i(\Omega)\,, \quad p \leqq n \tag{1.1}$$

and, further, that when $V = I$,

$$\mathscr{E}\,\text{tr}_j(A) = \textstyle\sum_{i=0}^{j} (n-i)^{(j-i)} \binom{p-i}{j-i}\, \text{tr}_i(\Omega)\,, \quad i \leqq j \leqq p \leqq n\,. \tag{1.2}$$

We will show

$$\mathscr{E}\,\text{tr}_j(V^{-1}A) = \textstyle\sum_{i=0}^{j} (n-i)^{(j-i)} \binom{p-i}{j-i}\, \text{tr}_i(\Omega)\,, \quad i \leqq j \leqq p \leqq n\,. \tag{1.3}$$

Labelling the claimed identity in (1.3) as $C_{j,p}(V)$ it is clear that $C_{p,p}(V)$ is equivalent to (1.1) and that $C_{j,p}(I)$ is identical to (1.2). However, choosing K so that $KVK' = I$ and transforming $X \to KX$, we see that $C_{j,p}(V)$ is true if and only if $C_{j,p}(I)$ is true $:1 \leqq j \leqq p$. We show first that $C_{p,p}(I)$ implies $C_{j,p}(I)$ and then prove that $C_{p,p}(I)$ is true thus validating the entire set of De Waal's conjectures, more generally expressed in (1.3).

2. $C_{p,p}(I)$ implies $C_{j,p}(I)$. Let $J = \{i_1, i_2, \cdots, i_j\}$ with $i_1 < i_2, \cdots, < i_j$ be an ordered subset of the integers $1, 2, \cdots, p$. For any matrix $B(p \times p)$, define B_J as the $j \times j$ matrix formed from B by preserving only those rows and columns corresponding to elements of J. If $\lambda_1, \lambda_2, \cdots, \lambda_p$ are the latent roots of B then $|B - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_p - \lambda)$. Differentiate this equation $(p-j)$ times with respect to λ and set $\lambda = 0$ to obtain

$$\text{tr}_j(B) = \textstyle\sum_J |B_J| \tag{2.1}$$

the summation extending over all possible J. From (2.1)

$$\textstyle\sum_J \text{tr}_k(B_J) = \sum_{K(J)} |(B_J)_{K(J)}| \tag{2.2}$$

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where $K(J)$ is an ordered, size k subset of the elements of J. Clearly $(B_J)_{K(J)} \equiv B_K$ for some K, an ordered, size k subset of $\{1, 2, \cdots, p\}$. Further all possible such B_K will be included on the right-hand side of (2.2) an equal number of times. We have therefore

$$\sum_J \sum_{K(J)} |(B_J)_{K(J)}| = \binom{p-k}{j-k} \sum_K |B_K| \tag{2.3}$$

the combinational multiplier having been determined by comparing the number of determinants on the left and right sides of the equation. Use (2.1) on the right-hand side of (2.3) and then (2.2) to obtain

$$\sum_J \operatorname{tr}_k(B_J) = \binom{p-k}{j-k} \operatorname{tr}_k(B) \qquad k \leqq j\,. \tag{2.4}$$

Assume $C_{pp}(I)$ to be true, $1 \leqq p \leqq n$. Since

$$\mathscr{E} \operatorname{tr}_j(A) = \sum_J \mathscr{E}|A_J| \tag{2.5}$$

then

$$\mathscr{E} \operatorname{tr}_j(A) = \sum_J \sum_{i=0}^{j} (n-i)^{(j-i)} \operatorname{tr}_i(\Omega_J) \tag{2.6}$$

and so, $C_{j,p}(I)$ is true $:1 \leqq j \leqq p \leqq n$ after using (2.4) on the right side of (2.6).

3. Validation of $C_{p,p}(I)$. Since $\mathscr{E}|A|$ depends on Ω only through the latent roots of the latter we may assume that $\Omega = \operatorname{diag}(\omega_1, \omega_2, \cdots, \omega_p)$ and correspondingly that $X = Y + M$ wherein Y is a matrix of pn mutually independent standard normal deviates and $M = (m_{ij})$ is the matrix $m_{ii} = \omega_i^{\frac{1}{2}}$, $1 \leqq i \leqq p$; $m_{ij} = 0$ otherwise. Clearly $\mathscr{E}|A|$ is a symmetric function of $\{\omega_1^{\frac{1}{2}}, \omega_2^{\frac{1}{2}}, \cdots, \omega_p^{\frac{1}{2}}\}$ of order at most two in each element. Since the elements of Y have a distribution symmetric about zero, $\mathscr{E}|A|$ is invariant under the transformation $\omega_i^{\frac{1}{2}} \to -\omega_i^{\frac{1}{2}}$. Evidently $\mathscr{E}|A|$ must be a symmetric function of $\{\omega_1, \omega_2, \cdots, \omega_p\}$ of order at most one in each element therefore, for some $d_0, d_1, \cdots, d_p$, we may write

$$\mathscr{E}|A| = d_0 + d_1 \operatorname{tr}_1(\Omega) + \cdots + d_p \operatorname{tr}_p(\Omega)\,. \tag{3.1}$$

We first show

$$\mathscr{E} \frac{\partial}{\partial \omega_1} \frac{\partial}{\partial \omega_2}, \cdots, \frac{\partial}{\partial \omega_p} |A| = 1\,. \tag{3.2}$$

The differential of $|A|$ with respect to ω_1 is the sum of p determinants the ith of which is $|A|$ with its ith row differentiated with respect to ω_1, $i = 1, 2, \cdots, p$. The differential in (3.2) therefore is the sum of p^p determinants. Of these, those which have any row of A differentiated at least twice will be zero due to the presence of a row of zeros. Of the remaining $p!$ determinants, all but one will have zero expectation since their expansion will be seen to contain an element of Y raised to an odd power. The surviving determinant is that in which the ith row of A has been differentiated with respect to ω_i, $i = 1, 2, \cdots, p$. The product of the diagonal elements of this determinant is $(1 + \omega_{11}^{\frac{1}{2}} y_{11})(1 + \omega_{22}^{\frac{1}{2}} y_{22}) \cdots (1 + \omega_{pp}^{\frac{1}{2}} y_{pp})$ which has unit expectation. Every other product occurring in the expansion of this determinant has an element of Y raised to an odd power and has zero expectation.

Now

$$d_i = \mathscr{E} \frac{\partial}{\partial \omega_1} \frac{\partial}{\partial \omega_2} \cdots \frac{\partial}{\partial \omega_i} |A| \quad \text{when } \omega_{i+1} = \omega_{i+2} = \cdots = \omega_p = 0 . \tag{3.3}$$

Let A_{11} be the first i rows of columns of A; A_{22} be the last $p - i$ rows and columns of A and $A_{12} = A'_{21}$ be formed from the first i rows and last $p - i$ columns of A. When $\omega_{i+1} = \omega_{i+2} = \cdots = \omega_p = 0$, it is well known that A_{11} and $A_{22} - A_{21}A_{11}^{-1}A_{12}$ are independent and that the latter, which has a central Wishart distribution, has expectation $(n - i)^{(p-i)}$. Since $|A| = |A_{11}||A_{22} - A_{21}A_{11}^{-1}A_{12}|$ and applying (3.1) to $|A_{11}|$ we see that $d_i = (n - i)^{(p-i)}$ which proves $C_{pp}(I)$.

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