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ADMISSIBILITY IMPLICATIONS FOR DIFFERENT CRITERIA IN CONFIDENCE ESTIMATION

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It will be shown that if a confidence procedure is admissible when the criterion is probability of not covering the true value and expected length (or something more general than length), then the same confidence procedure is at least almost admissible when the criterion is probability of not covering the true value and probability of covering false values. The result is true (under mild conditions) for virtually all confidence region estimation problems.

It will be shown that if a confidence procedure is admissible when the criterion is probability of not covering the true value and expected length (or something more general than length), then the same confidence procedure is at least almost admissible when the criterion is probability of not covering the true value and probability of covering false values. The result is true (under mild conditions) for virtually all confidence region estimation problems. (See Pratt [3] page 550 for a variety of problems to which the present result is applicable.)

A stronger conclusion is obtainable (admissibility instead of almost admissibility) in special cases by making use of the duality of confidence sets and tests of hypotheses.

We need some notation and definitions. Let X be a random variable taking values in $\mathscr{X}$. The distribution of X is of the form $P_{\theta'}(\cdot)$, for the parameter θ'. The parameter space is $\{\theta'\} = \Omega$, and $(\Omega, \mathscr{A}, \mu)$ represents a measure space. We distinguish between θ', the true value of the parameter, and θ which can be any element in Ω. Randomized confidence sets are determined by bimeasurable functions $\varphi(x, \theta)$ on $\mathscr{X} \times \Omega$, where for each x, $\varphi(x, \theta)$ is the probability that the point θ is included in the confidence set when x is the observed value. (See Joshi [2] page 1044 for a more thorough explanation and for remarks on admissibility of confidence sets.) Consider the function

$$\beta_\varphi(\theta, \theta') = E_{\theta'} \varphi(x, \theta) , \tag{1}$$

for every $\theta \neq \theta'$. Thus $\beta_\varphi(\theta, \theta')$ represents the probability of covering the false value θ when θ' is true. A procedure φ is then said to be inadmissible if there exists another procedure φ^* such that

$$\text{(2)}\qquad \begin{aligned} &\text{(i)} \quad \beta_{\varphi^*}(\theta, \theta') \leqq \beta_\varphi(\theta, \theta') \qquad \text{for every} \quad \theta \neq \theta' , \\ &\text{(ii)} \quad E_{\theta'} \varphi^*(x, \theta') \geqq E_{\theta'} \varphi(x, \theta') \qquad \text{for every} \quad \theta' , \end{aligned}$$

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with strict inequality in (i) for some pair (θ, θ') or strict inequality in (ii) for some θ'. A procedure φ is said to be almost admissible if there exists no φ^* satisfying (i) and (ii) above with strict inequality in (i) on a set of positive two dimensional $(\mu \times \mu)$ measure in (θ, θ') space.

Now we introduce a different criterion. Let A denote a measurable set in $\mathscr{A}$, let $M(A, \theta')$ be such that for each θ', $M(\cdot, \theta')$ is a nonnegative σ-finite measure on Ω, with μ absolutely continuous with respect to M, and for each A, $M(A, \cdot)$ is a measurable function. Also let

$$m_\varphi(x, \theta') = \int \varphi(x, t) M(dt, \theta') . \tag{3}$$

(Note if t is one dimensional and $M(dt, \theta') = dt$, then $m_\varphi(x, \theta') = m_\varphi(x)$, just the Lebesgue measure (generalized length) of the confidence set.) Finally let the new criterion be

$$\text{(4)} \qquad \begin{aligned} &\text{(i)} \quad E_{\theta'} \varphi(x, \theta') \\ &\text{(ii)} \quad E_{\theta'} m_\varphi(x, \theta') . \end{aligned}$$

A procedure φ is said to be almost admissible by criterion (4), if there exists no φ^* such that

(i) $E_{\theta'} \varphi^*(x, \theta') \geqq E_{\theta'} \varphi(x, \theta)$ and
(ii) $E_{\theta'} m_{\varphi^*}(x, \theta') \leqq E_{\theta'} m_\varphi(x, \theta)$,

with strict inequality in (ii) on a set of positive (μ) measure.

Now we can prove the

THEOREM. *Suppose $\varphi(x, \theta)$ is an almost admissible confidence procedure when the criterion is* (4). *Assume*

$$E_{\theta'} m_\varphi(x, \theta') < \infty , \qquad \text{a.e. } (\mu) . \tag{5}$$

Then $\varphi(x, \theta)$ is almost admissible when the criterion is (2).

PROOF. The idea of the proof is suggested in an article by Pratt [3]. Note that

$$E_{\theta'} m_\varphi(x, \theta') = \int \int \varphi(x, t) M(dt, \theta') \, dP_{\theta'}(x) . \tag{6}$$

Interchange the order of integration in (6) and get

$$\begin{aligned} E_{\theta'} m_\varphi(x, \theta') &= \int [\int \varphi(x, t) \, dP_{\theta'}(x)] M(dt, \theta') \\ &= \int E_{\theta'} \varphi(x, t) M(dt, \theta') \\ &= \int \beta(t, \theta') M(dt, \theta') . \end{aligned} \tag{7}$$

Now suppose φ is not almost admissible so that there exists a φ^* which is better than φ. Then from (2) it follows that

$$\beta_{\varphi^*}(\theta, \theta') \leqq \beta_\varphi(\theta, \theta') \qquad \text{for every } \theta \neq \theta' , \tag{8}$$

with strict inequality on a (θ, θ') set of positive $\mu \times \mu$ measure and

$$E_{\theta'} \varphi^*(x, \theta') \geqq E_{\theta'} \varphi(x, \theta') . \tag{9}$$

Using the strict inequality in (8) along with (5), (7) and the fact that μ is absolutely continuous with respect to M, we have

$$\int \beta_{\varphi^*}(t, \theta')M(dt, \theta') < \int \beta_{\varphi}(t, \theta')M(dt, \theta') , \tag{10}$$

for some θ' set of positive μ measure. Hence by (7), (9) and (10) we have a contradiction to the hypothesis that $\varphi(x, \theta)$ is almost admissible when the criterion is (4). This completes the proof of the theorem.

REMARK 1. Consider the risk function

$$\rho(\theta', \varphi) = (1 - \beta_\varphi(\theta', \theta')) + CE_{\theta'}(m_\varphi(x, \theta')) , \tag{11}$$

where C is any positive constant. Then if φ is admissible with respect to the risk (11), and (5) holds, then φ is almost admissible by criterion (4). Obviously if φ is admissible by criterion (4), it is almost admissible by criterion (4). Finally note, if φ is admissible by criterion (11), and (5) holds for all θ', then φ is admissible by criterion (4).

REMARK 2. The theorem is in general false without condition (5). For example, let $P_{\theta'}(dx) = e^{-(x-\theta')}\,dx$ for $x > \theta'$, $M(dt, \theta') = dt$, $\varphi \equiv 1$. Here φ is admissible for criterion (4) but not in the sense of (2).

By way of examples and counter examples, note that if X is a single observation from a normal distribution with mean γ and variance 1 then $X \pm k$ is an admissible confidence interval for γ if the criterion is probability of not covering the true γ and expected length. The theorem implies that the procedure is almost admissible when the criterion is (2). In fact, in this case the duality between tests of hypotheses and confidence intervals yields admissibility instead of almost admissibility.

The reverse implication is obviously false. Just consider the above example. The confidence interval $(X + C_1, X + C_2)$, $C_1 \neq -C_2$, is an admissible confidence interval for γ when the criterion is (2). However, if the criterion is probability of not covering the true value and expected length, then the interval $X \pm k$, $k = (C_2 - C_1)/2$ is better. Another interesting counter example is a confidence region $\|Y - \gamma\| \leqq C$, where Y is an observation on a multivariate normal distribution with mean vector γ and covariance I and the dimension of Y is three or more. (See Brown [1] Theorem 3.3.2.)

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