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LIMITING BEHAVIOR OF THE EXTREMUM OF CERTAIN SAMPLE FUNCTIONS[1]

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For a sequence of random variables forming an m-dependent stochastic process (not necessarily stationary), asymptotic distribution and other convergence properties of the extremum of certain functions of the empirical distribution are studied. In this context, it is shown that the asymptotic probability of the classical Kolmogorov-Smirnov statistic exceeding any positive real number provides an upper bound for the corresponding probability when the underlying random variables are not necessarily identically distributed. The theory is specifically applied to the study of the limiting distribution, strong convergence and convergence of the first moment of the strength of a bundle of parallel filaments (which is shown to be the extremum of a function of the empirical distribution).

1. Introduction. Let $X_{n,1} \leqq \cdots \leqq X_{n,n}$ be the n ordered values of $X_1, \cdots, X_n$, representing the strengths (nonnegative random variables) of individual filaments in a bundle of n parallel filaments of equal length. If we assume that the force of a free load on the bundle is distributed equally on each filament and the strength of an individual filament is independent of the number of filaments in a bundle, then the minimum load B_n beyond which all the filaments of the bundle give way is defined to be the strength of the bundle.

Now, if a bundle breaks under a load L, then the inequalities $nX_{n,1} \leqq L$, $(n-1)X_{n,2} \leqq L, \cdots, X_{n,n} \leqq L$ are simultaneously satisfied. Consequently, the bundle strength can be represented as

$$B_n = \max\{nX_{n,1}, (n-1)X_{n,2}, \cdots, X_{n,n}\}. \tag{1.1}$$

When the X_i are i.i.d. rv (independent and identically distributed random variables), Daniels [4] investigated the probability distribution of B_n and established the asymptotic normality of the standardized form of B_n by very elaborate and complicated analysis.

We observe that if $S_n(x)$ be the empirical distribution function for $X_1, \cdots, X_n$, then $n^{-1}B_n$ can be written as $\sup_{x \geqq 0} x[1 - S_n(x)]$ (see Section 7). This leads us to consider a general class of statistics of the form $\sup_x \psi(x, S_n(x))$ to which B_n belongs, and by probabilistic arguments on the fluctuations of S_n, we are able to study the distribution theory even in a more complicated situation when the $\{X_i\}$ forms an m-dependent process, not necessarily stationary.

Section 2 is devoted to the detailed statement of the problem. The main theorem along with the needed regularity conditions are stated in Section 3. In Section 4

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some preliminary results including the derivation of the Kolmogorov–Smirnov bound in the case of non-identically distributed random variables are obtained. These results are then used in the proof of the main theorem in Section 5. In Section 6, under a convexity assumption, moment convergence of the statistic is proved. In Section 7, it is shown that if the X_i are i.i.d. rv with a distribution function $F(x)$ and if $x[1 - F(x)]$ besides having a unique maximum h_0 is monotonically decreasing for $x \geqq x_1$, where x_1 is finite, then $n^{\frac{1}{2}}|E[\sup_{x\geqq 0}\{x[1 - S_n(x)]\} - h_0| \to 0$ as $n \to \infty$. Some possible generalizations are briefly sketched in Section 8.

2. Statement of the problem. Let $\{X_1, X_2, \cdots\}$ be a sequence of rv's forming an m-dependent stochastic process, not necessarily stationary. The marginal and the joint df of X_i and (X_i, X_{i+h}) are denoted by $F_i(x)$ and $F_{i,h}(x, y)$, respectively, for $h = 1, \cdots, m$, $i = 1, 2, \cdots$, and let $F_i(x)$ admit of a continuous density function $f_i(x)$. The empirical df (distribution function) when the sample is $(X_1, \cdots, X_n)$ is defined by

$$S_n(x) = n^{-1} \sum_{i=1}^{n} c(x - X_i)\,, \qquad -\infty < x < \infty\,, \tag{2.1}$$

where $c(u)$ is equal to 1 or 0 according as $u \geqq 0$ or not. Also, the average df $\bar{F}_{(n)}(x)$ is defined by

$$\bar{F}_{(n)}(x) = n^{-1} \sum_{i=1}^{n} F_i(x)\,, \qquad \text{so that} \quad ES_n(x) = \bar{F}_{(n)}(x)\,, \qquad -\infty < x < \infty\,. \tag{2.2}$$

Consider a sequence of nonnegative real valued functions $h_n(x) = \psi(x, \bar{F}_{(n)}(x))$ $(-\infty < x < \infty)$ where $h_n(x)$ assumes a finite global maximum h_n^0 at a unique (unknown) point $x = x_n^0$. Our primary concern is to provide a suitable estimator of h_n^0 based on the sample $(X_1, \cdots, X_n)$. Since $S_n(x)$ unbiasedly estimates $\bar{F}_{(n)}(x)$ for all x, we are intuitively led to the following estimator of h_n^0. Let

$$Z_{ni} = \psi(X_i, S_n(X_i))\,, \qquad i = 1, 2, \cdots, n\,, \tag{2.3}$$

$$Z_n^* = \max_{1 \leqq i \leqq n} Z_{ni}\,. \tag{2.4}$$

Our central problem is to derive the asymptotic normality of $n^{\frac{1}{2}}(Z_n^* - h_n^0)$. Since $Z_{n1}, \cdots, Z_{nn}$ are not independent (even when $m = 0$) nor necessarily identically distributed, the usual techniques of deriving the distribution of the sample maximum fail to provide our desired result. The task is accomplished here by first showing that under certain regularity conditions, to be stated in Section 3, as $n \to \infty$, $n^{\frac{1}{2}}(Z_n^* - h_n^0)$ is, in probability, proportional to $n^{\frac{1}{2}}[S_n(x_n^0) - \bar{F}_{(n)}(x_n^0)]$, and then applying the central limit theorem on the latter variable.

3. Basic regularity conditions and the main theorem. Let

$$A = \{(x, y): -\infty < x < \infty, 0 \leqq y \leqq 1\}\,. \tag{3.1}$$

We assume that $\psi(x, y)$ is a nonnegative, continuous function defined for all $(x, y) \in A$. We impose the following regularity conditions on h_n^0, x_n^0 and $\bar{F}_{(n)}$:

(I)

$$(3.2)\qquad 0 < \inf_n h_n^0 \leqq \sup_n h_n^0 < \infty \quad \text{and} \quad \sup_n |x_n^0| < \infty .$$

(II) The sequence of the marginal densities $\{f_i(x)\}$ is assumed to be equal to $\{F_i(x)\}$ and is equicontinuous and uniformly bounded in the interval $[\inf_n x_n^0 - \rho, \sup_n x_n^0 + \rho]$, where ρ (> 0) is arbitrarily small. This implies that

$$(3.3)\qquad \sup_n \bar{f}_{(n)}(x_n^0) \leqq f^0 < \infty \quad \text{where} \quad \bar{f}_{(n)}(x) = (d/dx)\bar{F}_{(n)}(x) .$$

In addition, it is assumed that

$$(3.4)\qquad \inf_n \bar{f}_{(n)}(x_n^0) \geqq f_0 > 0 .$$

Also, let $p_n = \bar{F}_{(n)}(x_n^0)$. Then, Assumption II implies that

$$(3.5)\qquad 0 < p_0 \leqq \inf_n p_n \leqq \sup_n p_n \leqq p^0 < 1 .$$

(III) For every $\eta > 0$ and $\rho > 0$, let

$$(3.6)\qquad B(\rho, \eta) = \bigcup_n \{(x, y) : x_n^0 - \rho \leqq x \leqq x_n^0 + \rho, p_n - \eta \leqq y \leqq p_n + \eta\} .$$

Let $\psi_{10}(x, y)$ and $\psi_{01}(x, y)$ denote the partial derivatives of $\psi(x, y)$ with respect to x and y respectively whenever they exist. We assume that $\psi_{10}(x, y)$ exists and is jointly continuous in (x, y) for all $(x, y) \in B(\rho, \eta)$, where both ρ and η are small. Also $\psi_{01}(x, y)$ exists and is jointly continuous in (x, y) for all $(x, y) \in A$. It is assumed that

(i) if $\xi_n = \psi_{01}(x_n^0, p_n)$, then

$$(3.7)\qquad 0 < \inf_n |\xi_n| \leqq \sup_n |\xi_n| < \infty ,$$

(ii) for every $y : 0 < y < 1$, $|\psi_{01}(x, y)| \leqq g(x)$ where $g(x)$ is continuous in $x(-\infty < x < \infty)$, and

(iii) $g^2(x)$ is uniformly integrable with respect to the $\{F_i\}$, which ensures that

$$(3.8)\qquad \sup_i \int_{g(x)>t} g^2(x)\, dF_i(x) = \alpha^*(t) \to 0 \quad \text{as} \quad t \to \infty ,$$

(IV) To avoid the possibility of having another local maximum of $h_n(x)$ to be arbitrary close (in abscissa or value) to $h_n^0 = h(x_n^0)$, we impose the following separability and monotonicity conditions. Let $h_n^{(1)}$ be a second largest local maximum of $h_n(x)$ (if any), then

$$(3.9)\qquad \inf_n [h_n(x_n^0) - h_n^{(1)}] \geqq h^* > 0 .$$

Also, the continuity and differentiability of $h_n(x)$ in the neighborhood of x_n^0, implied by (III) is strengthened to the following: there exist positive finite constants C_i, $k_i(> 1)$, $i = 1, 2$, independent of n, such that for $x : |x - x_n^0| \leqq \rho(> 0)$,

$$(3.10)\qquad h_n(x_n^0) - C_1|x - x_n^0|^{k_1} \leqq h_n(x) \leqq h_n(x_n^0) - C_2|x - x_n^0|^{k_2} ,$$

where ρ may be taken sufficiently small. In fact, if $h_n(x)$ has a continuous second derivative which is negative and uniformly (in n) bounded away from 0 and bounded below in the interval $[x_n^0 - \rho, x_n^0 + \rho]$, then (3.10) holds with $k_1 = k_2 = 2$.

We now introduce the following notations. Let

$$\bar{f}_{(n)j}(x_n^0) = n_j^{-1} \sum_{k=0}^{n_j-1} f_{j+k(m+1)}(x_n^0); \quad n_j = [(n+1+m-j)/(m+1)],\ j = 1, \cdots, m+1, \tag{3.11}$$

where $[q]$ is the largest integer contained in q,

$$\bar{f}_{(n)}(x_n^0) = n^{-1} \sum_{j=1}^{m+1} n_j \bar{f}_{(n)j}(x_n^0), \tag{3.12}$$

where we note that $n/n_j \to (m+1)$ as $n \to \infty$, for $j = 1, \cdots, m+1$;

$$p_{n,i} = F_i(x_n^0), \qquad i = 1, \cdots, n; \tag{3.13}$$

$$\bar{F}_{(n)h}(x, y) = (n-h)^{-1} \sum_{i=1}^{n-h} F_{i,h}(x, y);$$

$$\alpha_{n,h} = \bar{F}_{(n)h}(x_n^0, x_n^0) - p_n^2, \qquad \beta_{n,h} = (n-h)^{-1} \sum_{i=1}^{n-h} \{p_{n,i} p_{n,i+h} - p_n^2\} \tag{3.14}$$

for $h = 1, \cdots, m$, and $\alpha_{n,0} = p_n(1-p_n)$, $\beta_{n,0} = n^{-1} \sum_{i=1}^{n} (p_{n,i} - p_n)^2$. Let then

$$\nu_{n,m}^2 = (\alpha_{n,0} - \beta_{n,0}) + 2 \sum_{h=1}^{m} \{(n-h)/n\}\{\alpha_{n,h} - \beta_{n,h}\}; \tag{3.15}$$

$$\gamma_{n,m}^2 = \nu_{n,m}^2 \xi_n^2; \qquad \xi_n = \psi_{01}(x_n^0, p_n). \tag{3.16}$$

Then, the main theorem of the paper is the following:

THEOREM 3.1. *Under the conditions stated above, if* $\inf_n \gamma_{n,m} > 0$, *then for all real* x,

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}}[Z_n^* - h_n^0]/\gamma_{n,m} \leqq x\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}t^2)\, dt. \tag{3.17}$$

The proof of the theorem is postponed to Section 5.

It may be remarked that if the $\{X_i\}$ forms a stationary $m(\geqq 0)$-dependent process where the marginal df of X_i is $F(x)$, then all the Conditions, (I) to (IV), simplify considerably. Here $h_n(x) = \psi(x, F(x))$ does not depend on n, and hence, $h_n^0 = h^0$ and $x_n^0 = x^0$ do not depend on n. Thus, (3.2) is not needed. Also, instead of (3.3) and (3.4), we need to assume only that in the neighborhood of x^0, $f(x) = F'(x)$ is continuous, positive and finite. $B(\rho, \eta)$ reduces to a neighborhood of (x^0, p) (where $p_n = p = F(x^0)$ does not depend on n), and $\xi_n = \xi = \psi_{01}(x^0, p)$ has to be only nonzero. (3.8) is automatically granted by the square integrability of $g(x)$ (with respect to F), while, in (3.9) and (3.10), the uniformity conditions with respect to n are not needed.

4. Some preliminary results. Let $Y_{n,1} \leqq \cdots \leqq Y_{n,n}$ be the order statistics corresponding to the random variables

$$Y_{ni} = \psi(X_i, \bar{F}_{(n)}(X_i)), \qquad i = 1, 2, \cdots, n. \tag{4.1}$$

Now $Y_{n1}, \cdots, Y_{nn}$ form an m-dependent stochastic process with the marginal df's $G_1^{(n)}(x), \cdots, G_n^{(n)}(x)$, respectively. Note that by Assumption II,

$$\sup_x \psi(x, \bar{F}_{(n)}(x)) = h_n^0 \Rightarrow G_i^{(n)}(h_n^0) = 1 \qquad \text{for all } 1 \leqq i \leqq n. \tag{4.2}$$

LEMMA 4.1. *Under the regularity conditions of Section* 3, *for every* $\varepsilon > 0$,

$$\lim_{n\to\infty} P\{Y_{n,n} \geqq h_n^0 - \varepsilon/n\} = 1, \tag{4.3}$$

PROOF. By virtue of (3.9) and (3.10), it follows that for every $\varepsilon > 0$, there exists an n, say $n_0(\varepsilon)$, such that $\varepsilon/n < h^*$ for all $n \geqq n_0(\varepsilon)$, and there exists $\{\varepsilon(n)\}$ such that for all $n \geqq n_0(\varepsilon)$,

$$\{x : h_n(x) \geqq h_n^0 - \varepsilon/n\} \supset \{x : |x - x_n^0| \leqq \tfrac{1}{2}\varepsilon(n)\} , \tag{4.4}$$

where [by (3.10)]

$$\lim_{n\to\infty} \varepsilon(n) = 0 \quad \text{but} \quad \lim_{n\to\infty} \{n\varepsilon(n)\} = \infty . \tag{4.5}$$

Hence, for every $\varepsilon > 0$, $n \geqq n_0(\varepsilon)$,

$$\begin{aligned} 1 - G_i^{(n)}(h_n^0 - \varepsilon/n) &\geqq P\{|X_i - x_n^0| \leqq \tfrac{1}{2}\varepsilon(n)\} \\ &= F_i(x_n^0 + \tfrac{1}{2}\varepsilon(n)) - F_i(x_n^0 - \tfrac{1}{2}\varepsilon(n)) \cong \varepsilon(n) f_i(x_n^0) , \end{aligned} \tag{4.6}$$

by Assumption II. Let then

$$Y_{(n)}^{(j)} = \max [Y_{nj}, Y_{nj+m+1}, \cdots, Y_{nj+(n_j-1)(m+1)}] , \qquad j = 1, \cdots, m+1 .$$

Then, $Y_{n,n} \geqq Y_{(n)}^{(j)}$ for all $j = 1, \cdots, m+1$. Thus,

$$P\{Y_{n,n} \leqq h_n^0 - \varepsilon/n\} \leqq \min_j P\{Y_{(n)}^{(j)} \leqq h_n^0 - \varepsilon/n\} . \tag{4.7}$$

Now, $Y_{(n)}^{(j)}$ is the maximum over n_j independent random variables. Hence,

$$\begin{aligned} P\{Y_{(n)}^{(j)} \leqq h_n^0 - \varepsilon/n\} &= \textstyle\prod_{k=0}^{n_j-1} G_{j+k(m+1)}^{(n)}(h_n^0 - \varepsilon/n) \\ &\leqq [n_j^{-1} \textstyle\sum_{k=0}^{n_j-1} G_{j+k(m+1)}^{(n)}(h_n^0 - \varepsilon/n)]^{n_j} \\ &\leqq\!\!\!\cong [1 - \varepsilon(n)\bar{f}_{(n)j}(x_n^0)]^{n_j} , \qquad j = 1, 2, \cdots, m+1 . \end{aligned} \tag{4.8}$$

Since $\sup_j \bar{f}_{(n)j}(x_n^0) \geqq \bar{f}_{(n)}(x_n^0) > 0$ by (3.4) and $n\varepsilon(n) \to \infty$ as $n \to \infty$, we have

$$\min_j P\{Y_{(n)}^{(j)} \leqq h_n^0 - \varepsilon/n\} \leqq\!\!\!\cong [1 - \varepsilon(n)\bar{f}_{(n)}(x_n^0)^{n_j} \to 0 \qquad \text{as } n \to \infty . \tag{4.9}$$

Hence the lemma follows from (4.7) and (4.9). □

LEMMA 4.2. *Let $\{X_i\}$ be a sequence of m dependent random variables with continuous* df *$\{F_i\}$ and let $S_n(x)$ and $\bar{F}_{(n)}(x)$ be defined as in* (2.1) *and* (2.2), *then for every $\varepsilon > 0$, there exists a finite $c(\varepsilon) > 0$ such that*

$$\limsup_n P\{\sup_x n^{\frac{1}{2}}|S_n(x) - \bar{F}_{(n)}(x)| > c(\varepsilon)\} < \varepsilon . \tag{4.10}$$

PROOF. Let us define

$$\begin{aligned} S_{n,j}(x) &= n_j^{-1} \textstyle\sum_{k=0}^{n_j-1} c(x - X_{j+k(m+1)}) \qquad \text{and} \\ \bar{F}_{(n)j}(x) &= n_j^{-1} \textstyle\sum_{k=0}^{n_j-1} F_{j+k(m+1)}(x) , \end{aligned} \tag{4.11}$$

for $j = 1, \cdots, m+1$. Then by definition,

$$\sup_x |n^{\frac{1}{2}}\{S_n(x) - \bar{F}_{(n)}(x)\}| \leqq \textstyle\sum_{j=1}^{m+1} (n/n_j)^{-\frac{1}{2}}\{\sup_x |n_j^{\frac{1}{2}}\{S_{n,j}(x) - \bar{F}_{(n)j}(x)\}|\} . \tag{4.12}$$

Since $(m+1)n_j \sim n, j = 1, \cdots, m+1$, it suffices to prove the following theorem.

THEOREM 4.3. *Let $\{X_i\}$ be a sequence of independent* rv'*s with continuous* df'*s $\{F_i\}$, and let $S_n(x)$ and $\bar{F}_{(n)}(x)$ be defined as in* (2.1) *and* (2.2). *Then, for every $\lambda > 0$,*

$$\limsup_n P\{\sup_x n^{\frac{1}{2}}|S_n(x) - \bar{F}_{(n)}(x)| > \lambda\} \leqq 2 \textstyle\sum_{k=1}^{\infty} (-1)^{k+1} e^{-2k^2\lambda^2} , \tag{4.13}$$

where the equality sign holds when all the F_i are identical.

PROOF. Define $Q_{ni} = \bar{F}_{(n)}(X_i)$, $t_{ni} = P\{Q_{ni} \leqq t\}$, $0 \leqq t \leqq 1$, $i = 1, \cdots, n$. Then $n^{-1} \sum_{i=1}^n t_{ni} = t: 0 \leqq t \leqq 1$. Also, let

$$U_n(t) = n^{\frac{1}{2}}[G_n^*(t) - t]\,; \qquad G_n^*(t) = n^{-1} \sum_{i=1}^n c(t - Q_{ni})\,, \qquad 0 \leqq t \leqq 1\,. \tag{4.14}$$

Note that $U_n(0) = U_n(1) = 0$. Thus, is suffices to show that for all $\{F_i\}$,

$$\limsup_n P\{\sup_{0<t<1} |U_n(t)| > \lambda\} \leqq 2 \sum_{k=1}^\infty (-1)^{k+1} e^{-2k^2\lambda^2}\,. \tag{4.15}$$

Since $U_n(t)$ has n discontinuities (jumps of $n^{-\frac{1}{2}}$), it does not belong to the space $C[0, 1]$ of all continuous real functions on $[0, 1]$. We consider the space $D[0, 1]$ of real functions $y(t)$ defined on $[0, 1]$ with the properties that (i) $y(t - 0)$ and $y(t + 0)$ exist for all $0 < t < 1$, (ii) $y(t) = y(t + 0)$, $0 \leqq t < 1$, and (iii) $y(t)$ is continuous at $t = 0$ and $t = 1$. With $D[0, 1]$, we associate the Skorokhod J_1-topology

$$\rho_D(x, y) = \inf_{\lambda \in \Lambda} [\sup_t |x(t) - y(\lambda(t))| + \sup_t |t - \lambda(t)|]\,, \tag{4.16}$$

where Λ is the class of strictly increasing, continuous mapping of $[0, 1]$ onto itself. Then, for every $n \geqq 1$, $U_n = \{U_n(t): 0 \leqq t \leqq 1\}$ belongs to $D[0, 1]$.

Note that for every $(0 \leqq)\ t_1 < t < t_2\ (\leqq 1)$ and $n \geqq 1$,

$$\begin{aligned} E\{[U_n(t) &- U_n(t_1)]^2[U_n(t_2) - U_n(t)]^2\} \\ &= n^{-2}\{\textstyle\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{q=1}^n E(d_{ni}^{(1)} d_{nj}^{(1)} d_{nk}^{(2)} d_{nq}^{(2)})\} \\ &= n^{-2}\{\textstyle\sum_{i=1}^n E(d_{ni}^{(1)} d_{ni}^{(2)})^2 + \sum_{i\neq j=1}^n E(d_{ni}^{(1)})^2 E(d_{nj}^{(2)})^2 \\ &\qquad + 2 \textstyle\sum_{i\neq j=1}^n E(d_{ni}^{(1)} d_{ni}^{(2)}) E(d_{nj}^{(1)} d_{nj}^{(2)})\} \\ &\leqq n^{-2}\{\textstyle\sum_{i=1}^n E(d_{ni}^{(1)} d_{ni}^{(2)})^2 + [\sum_{i=1}^n E(d_{ni}^{(1)})^2][\sum_{i=1}^n E(d_{ni}^{(2)})^2] \\ &\qquad + 2[\textstyle\sum_{i=1}^n E(d_{ni}^{(1)} d_{ni}^{(2)})]^2\}\,, \end{aligned} \tag{4.17}$$

where

$$d_{ni}^{(1)} = c(t - Q_{ni}) - c(t_1 - Q_{ni}) - p_{ni}^{(1)}\,; \qquad p_{ni}^{(1)} = P\{t_1 \leqq Q_{ni} \leqq t\}\,, \tag{4.18}$$

$$d_{ni}^{(2)} = c(t_2 - Q_{ni}) - c(t - Q_{ni}) - p_{ni}^{(2)}\,; \qquad p_{ni}^{(2)} = P\{t \leqq Q_{ni} \leqq t_2\}\,, \tag{4.19}$$

for $i = 1, \cdots, n$. On denoting by

$$p_1 = t - t_1\,, \qquad p_2 = t_2 - t\,, \qquad p = p_1 + p_2 = t_2 - t_1\,, \tag{4.20}$$

and noting that

(i) $n^{-1} \sum_{i=1}^n p_{ni}^{(s)} = p_s$, $s = 1, 2$,

(ii) $n^{-2} \sum_{i=1}^n p_{ni}^{(1)} [p_{ni}^{(2)}]^2 \leqq (n^{-1} \sum_{i=1}^n p_{ni}^{(1)})(n^{-1} \sum_{i=1}^n [p_{ni}^{(2)}]^2) \leqq (n^{-1} \sum_{i=1}^n p_{ni}^{(1)}) \times (n^{-1} \sum_{i=1}^n p_{ni}^{(2)}) = p_1 p_2$, similarly, $n^{-2} \sum_{i=1}^n [p_{ni}^{(1)}]^2 p_{ni}^{(2)} \leqq p_1 p_2$, and

(iii) $(n^{-1} \sum_{i=1}^n p_{ni}^{(1)} p_{ni}^{(2)})^2 \leqq [n^{-1} \sum_{i=1}^n (p_{ni}^{(1)})^2][n^{-1} \sum_{i=1}^n (p_{ni}^{(2)})^2] \leqq p_1 p_2$, we obtain from (4.18), (4.19) and (4.20) that the right-hand side of (4.17) can be written as

$$\begin{aligned} n^{-2}\{&\textstyle\sum_{i=1}^n p_{ni}^{(1)} p_{ni}^{(2)} [p_{ni}^{(1)} + p_{ni}^{(2)} - 3 p_{ni}^{(1)} p_{ni}^{(2)}] \\ &+ [\textstyle\sum_{i=1}^n p_{ni}^{(1)}(1 - p_{ni}^{(1)})][\sum_{i=1}^n p_{ni}^{(2)}(1 - p_{ni}^{(2)})] + 2[\sum_{i=1}^n p_{ni}^{(1)} p_{ni}^{(2)}]^2\} \\ &\leqq n^{-2}\{\textstyle\sum_{i=1}^n [p_{ni}^{(1)}]^2 p_{ni}^{(2)} + \sum_{i=1}^n p_{ni}^{(1)} [p_{ni}^{(2)}]^2 + n^2 p_1 p_2 + 2n^2 p_1 p_2\} \\ &\leqq 5 p_1 p_2 \leqq \tfrac{5}{4}(p_1 + p_2)^2 = 1.25 p^2 = 1.25 (t_2 - t_1)^2\,. \end{aligned} \tag{4.21}$$

Now, for every $n \geqq 1$, let $Z_n = \{Z_n(t): 0 \leqq t \leqq 1\}$ be a Gaussian process with $EZ_n(t) = 0$ and $E[Z_n(t)Z_n(s)] = n^{-1} \sum_{i=1}^n s_{ni}(1 - t_{ni})$, $0 \leqq s \leqq t \leqq 1$, where $s_{ni} = P\{Q_{ni} \leqq s\}$, $i = 1, \cdots, n$. This Gaussian process can be conceived as an average of n independent Gaussian processes. Since, by definition, uniformly in $n \geqq 1$,

$$
\begin{aligned}
E[Z_n(t) - Z_n(s)]^4 &= 3\{E[Z_n(t) - Z_n(s)]^2\}^2 \\
&= 3\{n^{-1} \textstyle\sum_{i=1}^n [s_{ni}(1 - s_{ni}) - 2s_{ni}(1 - t_{ni}) + t_{ni}(1 - t_{ni})]\}^2 \qquad (4.22)\\
&= 3\{(t - s) - n^{-1} \textstyle\sum_{i=1}^n (t_{ni} - s_{ni})^2\}^2 \\
&\leqq 3(t - s)^2, \qquad 0 \leqq s \leqq t \leqq 1,
\end{aligned}
$$

according to the Kolmogorov existence theorem [cf. Hájek and Šidák ((1967) page 177)], such a process exists in the space $C[0, 1]$ of all continuous real functions on $[0, 1]$.

For every finite $m(\geqq 1)$, $0 \leqq t^{(1)} < \cdots < t^{(m)} \leqq 1$, and $\boldsymbol{\lambda}_m = (\lambda_1, \cdots, \lambda_m)$ with real and finite λ's, the random variable $\tilde{Z}_n = \sum_{j=1}^m \lambda_j Z_n(t^{(j)})$ is distributed normally with 0 mean and variance $\phi_n^2 = E(\tilde{Z}_n^2) \leqq \frac{1}{4}(\sum_{j=1}^m |\lambda_j|)^2 < \infty$, where, of course, ϕ_n may be arbitrarily close to 0, depending on the $G_i^{(n)}(t^{(j)})$, $i = 1, \cdots, n$, $j = 1, \cdots, m$. Let $\tilde{U}_n = \sum_{j=1}^m \lambda_j U_n(t^{(j)})$, which has also mean 0 and variance ϕ_n^2. Since $\tilde{U}_n$ involves a summation over n independent (bounded valued) random variables, by the classical central limit theorem [viz., Loève (1963) page 277], $\tilde{U}_n$ is asymptotically normal with 0 mean and variance ϕ_n^2, whenever $n\phi_n^2 \to \infty$ as $n \to \infty$. On the other hand, if $n\phi_n^2$ does not go to ∞ as $n \to \infty$, $\phi_n^2 \to 0$ as $n \to \infty$, and hence, both $\tilde{Z}_n$ and $\tilde{U}_n$ have asymptotically a degenerate distribution which attaches a unit mass at the point 0. Hence, for every finite $m(\geqq 1)$, and $0 \leqq t^{(1)} < \cdots < t^{(m)} \leqq 1$, $[U_n(t^{(1)}), \cdots, U_n(t^{(m)})]$ and $[Z_n(t^{(1)}), \cdots, Z_n(t^{(m)})]$ have convergent equivalent finite dimensional distributions. Consequently, by (4.21) and Theorem 15.6 of Billingsley ((1967) page 128), it follows that U_n and Z_n are convergent equivalent in law in the Skorokhod J_1-topology on $D[0, 1]$. Hence, to prove the theorem, it suffices to show that for every $\lambda > 0$,

$$\limsup_{n\to\infty} P\{\sup_{0<t<1} |Z_n(t)| > \lambda\} \leqq 2 \sum_{k=1}^\infty (-1)^{k+1} e^{-2k^2\lambda^2}. \tag{4.23}$$

Consider now a Brownian bridge $[Z(t): 0 \leqq t \leqq 1]$ with $EZ(t) = 0$ and $E[Z(s)Z(t)] = s(1 - t)$ for $0 \leqq s \leqq t \leqq 1$. For finitely many $0 < t^{(1)} < \cdots < t^{(m)} < 1$, let $\mathbf{D}_n^{(m)}$ and $\mathbf{D}^{(m)}$ be the covariance matrices of $[Z_n(t^{(1)}), \cdots, Z_n(t^{(m)})]$ and $[Z(t^{(1)}), \cdots, Z(t^{(m)})]$ respectively. It is then easy to verify that

$$\mathbf{D}^{(m)} - \mathbf{D}_n^{(m)} = ((n^{-1} \textstyle\sum_{i=1}^n (t_{ni}^{(j)} - t^{(j)})(t_{ni}^{(l)} - t^{(l)}))) = \mathbf{D}^{*(m)}, \tag{4.24}$$

where $\mathbf{D}^{*(m)}$ is positive semi-definite (p.s.d.) and $\mathbf{D}^{(m)}$ is p.d. Also, by the well-known Kolmogorov–Smirnov theorem

$$\lim_{m\to\infty} P\{\max_{1\leqq j\leqq m} |Z(t^{(j)})| > \lambda\} \leqq 2 \sum_{k=1}^\infty (-1)^{k+1} e^{-2k^2\lambda^2}. \tag{4.25}$$

So the desired result will follow if we can show that for every m and $0 \leqq t^{(1)} < \cdots < t^{(m)} \leqq 1$,

$$P\{\max_{1\leqq j\leqq m} |Z_n(t^{(j)})| > \lambda\} \leqq P\{\max_{1\leqq j\leqq m} |Z(t^{(j)})| > \lambda\}. \tag{4.26}$$

which really follows from (4.24), (4.25) and the following lemma due to Anderson [1].

LEMMA 4.4. *Let C_p be a convex set (in p-dimensions) symmetric about the origin. Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be independent normally distributed random variables (p-vectors) with null means and dispersion matrices* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *respectively, where* $\mathbf{B}_1$ *is* p.d. *and* $\mathbf{B}_2$ *and* $\mathbf{B}_3 = \mathbf{B}_1 - \mathbf{B}_2$ *are at least* p.s.d. *Then*

$$P\{\mathbf{X} \in C_p\} \leqq P\{\mathbf{Y} \in C_p\} . \tag{4.27}$$

where the equality holds only when $\mathbf{B}_3$ *is a null matrix.*

Now, by definition, $Y_{n,n}$, defined just before (4.1), is equal to $\max_{1 \leqq i \leqq n} \psi(X_i, \bar{F}_{(n)}(X_i)) = \psi(X_{n,r}, \bar{F}_{(n)}(X_{n,r}))$, where r is a random variable which can assume integral values between 1 and n, and $X_{n,1} \leqq \cdots \leqq X_{n,n}$ are the ordered variables corresponding to $X_1, \cdots, X_n$.

LEMMA 4.5. $n^{-1}r - p_n \to 0$, *in probability, as* $n \to \infty$.

PROOF. By Lemma 4.1, (3.9) and (3.10), it follows that $|X_{n,r} - x_n^0| \to_p 0$, and hence, by Assumption II of Section 3, $\bar{F}_{(n)}(X_{n,r}) - \bar{F}_{(n)}(x_n^0) \to_p 0$ as $n \to \infty$. Now, upon noting that $S_n(X_{n,r}) = r/n$ and $\bar{F}_{(n)}(x_n^0) = p_n$, we have

$$\bar{F}_{(n)}(X_{n,r}) - \bar{F}_{(n)}(x_n^0) = [\bar{F}_{(n)}(X_{n,r}) - S_n(X_{n,r})] + (r/n - p_n) , \tag{4.28}$$

where by Lemma 4.2, the first term on the right-hand side of (4.28) is $O_p(n^{-\frac{1}{2}})$, and hence, the convergence of the left-hand side to zero, in probability, as $n \to \infty$, implies that $r/n - p_n \to_p 0$ as $n \to \infty$. □

Consider now a sequence of intervals

$$I_n(\delta) = \{x : x_n^0 - \delta \leqq x \leqq x_n^0 + \delta\} , \qquad n \geqq 1, \delta > 0 . \tag{4.29}$$

Let then

$$J_n(\delta) = \sup_{x \in I_n(n)} \{|n^{\frac{1}{2}}[S_n(x) - \bar{F}_{(n)}(x)] - n^{\frac{1}{2}}[S_n(x_n^0) - p_n]|\} . \tag{4.30}$$

LEMMA 4.6. *For every positive* ε *and* η, *there exists a* $\delta(> 0)$, *such that*

$$\limsup_n P\{J_n(\delta) > \varepsilon\} < \eta . \tag{4.31}$$

The proof readily follows from (4.12) and the tightness of the $m + 1$ component processes as shown in the proof of Theorem 4.3. A direct consequence of the preceding two lemmas is the following.

LEMMA 4.7. $|n^{\frac{1}{2}}[S_n(X_{n,r}) - \bar{F}_{(n)}(X_{n,r})] - n^{\frac{1}{2}}[S_n(x_n^0) - p_n]| \to_p 0$, *as* $n \to \infty$.

Also, writing

$$W_n^* = n^{\frac{1}{2}}[S_n(x_n^0) - p_n] , \tag{4.32}$$

we obtain from Lemma 2.3 of Sen [9] the following.

LEMMA 4.8. *If* $\inf_n \nu_{n,m} > 0$, *then* $\mathscr{L}(W_n^*/\nu_{n,m}) \to \mathscr{N}(0, 1)$.

We define $g(x)$ as in after (3.7) and denote the df of $g(X_i)$ by $H_i(g)$, $0 \leqq g < \infty$, $1 \leqq i \leqq n$. Then, we have the following.

LEMMA 4.9. *Under* (3.8), $\max_{1\leqq i\leqq n} n^{-\frac{1}{2}} g(X_i) = o_p(1)$ *as* $n \to \infty$.

PROOF. Let us write $g_i = g(X_i)$, $i = 1, \cdots, n$, and let

$$(4.33) \qquad g_n^{(j)} = \max\{g_j, g_{j+m+1}, \cdots, g_{j+(n_j-1)(m+1)}\}, \qquad j = 1, \cdots, m+1,$$

where n_j are defined by (3.11). Then, $\max_{1\leqq i\leqq n} g(X_i) = \max_{1\leqq j\leqq m+1} g_n^{(j)}$, and hence, for every $\varepsilon > 0$,

$$(4.34) \qquad P\{\max_{1\leqq i\leqq n} g(X_i) > \varepsilon n^{\frac{1}{2}}\} \leqq \sum_{j=1}^{m+1} P\{g_n^{(j)} > \varepsilon n^{\frac{1}{2}}\}.$$

Now, for every $j(= 1, \cdots, m+1)$, by Lemma 3.1 of [10], $P\{g_n^{(j)} > \varepsilon n^{\frac{1}{2}}\} \to 0$ as $n \to \infty$, and therefore the lemma follows.

LEMMA 4.10. *Under* (ii) *of Assumption* III *of Section* 3, *as* $n \to \infty$

$$(4.35) \qquad \max_{1\leqq i\leqq n} |\psi(X_i, \bar{F}_{(n)}(X_i)) - \psi(X_i, S_n(X_i))| = o_p(1).$$

PROOF. The left-hand side of (4.35) is bounded above by

$$(4.36) \qquad [\max_{1\leqq i\leqq n} g(X_i)][\max_{1\leqq i\leqq n} |\bar{F}_{(n)}(X_i) - S_n(X_i)|],$$

and hence, the lemma directly follows from Lemmas 4.2 and 4.9.

5. The proof of Theorem 3.1. Let us recall that $Y_{ni} = \psi(X_i, \bar{F}_{(n)}(X_i))$, $Z_{ni} = \psi(X_i, S_n(X_i))$, $1 \leqq i \leqq n$, and $Z_n^* = \max_{1\leqq i\leqq n} Z_{ni}$, $Y_{n,n} = \max_{1\leqq i\leqq n} Y_{ni} = \psi(X_{n,r}, \bar{F}_{(n)}(X_{n,r}))$, where the random variable r is a positive integer $\leqq n$. Also, let $a_n^{(1)}$ and $a_n^{(2)}$ be so defined that

$$(5.1) \qquad h_n(x_n^0 - a_n^{(1)}) = h_n(x_n^0 + a_n^{(2)}) = h_n^0 - cn^{-\frac{1}{2}} \log n, \qquad c > 0.$$

Then, both $a_n^{(1)}$ and $a_n^{(2)} \to 0$ as $n \to \infty$, and by Lemma 4.1

$$(5.2) \qquad P\{X_{n,r} \in [x_n^0 - a_n^{(1)}, x_n^0 + a_n^{(2)}]\} \to 1 \qquad \text{as } n \to \infty.$$

Consider now three random subsets

$$(5.3) \qquad S_n^{(1)} = \{X_i : x_n^0 - a_n^{(1)} < X_i < x_n^0 + a_n^{(2)}, i = 1, \cdots, n\},$$

$$(5.4) \qquad S_n^{(2)} = \{X_i : X_i \notin [x_n^0 - a_n^{(1)}, x_n^0 + a_n^{(2)}] \text{ but } |X_i - x_n^0| < \rho, i = 1, \cdots, n\},$$

$$(5.5) \qquad S_n^{(3)} = \{X_i : |X_i - x_n^0| > \rho, i = 1, \cdots, n\},$$

where $\rho(> 0)$ is small, and it leads to the satisfaction of the Conditions II, III and IV of Section 3. Using then the Conditions I, II, III and IV of Section 3 along with the Lemma 4.6, we obtain that

$$(5.6) \qquad \sup_i \{|n^{\frac{1}{2}}(Z_{ni} - Y_{ni}) - \psi_{01}(x_n^0, p_n) W_n^*| : i \in S_n^{(1)}\} = o_p(1),$$

as $n \to \infty$. By (5.2), $\max_i \{Y_{ni}, i \in S_n^{(1)}\} = Y_{n,n}$, in probability, as $n \to \infty$, and hence, by (5.6), as $n \to \infty$

$$(5.7) \qquad Z_{n,1}^* = \max_i \{Z_{ni} : i \in S_n^{(1)}\} = Y_{n,n} + n^{-\frac{1}{2}} \psi_{01}(x_n^0, p_n) W_n^* + o_p(n^{-\frac{1}{2}}).$$

Also, by Lemma 4.1, $|n^{\frac{1}{2}}(Y_{n,n} - h_n^0)| = o_p(n^{-\frac{1}{2}})$, and hence,

(5.8) $$Z_{n,1}^* = h_n^0 + n^{-\frac{1}{2}}\xi_n W_n^* + o_p(n^{-\frac{1}{2}}) , \qquad \text{as } n \to \infty ,$$

where $\xi_n = \psi_{01}(x_n^0, p_n)$. Consequently, by (3.16) and Lemma 4.8,

(5.9) $$\mathscr{L}(n^{\frac{1}{2}}[Z_{n,1}^* - h_n^0]/\gamma_{n,m}) \to \mathscr{N}(0, 1) \qquad \text{as } n \to \infty ,$$

provided $\inf_n \gamma_{n,m} > 0$. Hence, for every $c > 0$,

(5.10) $$P\{Z_{n,1}^* \geqq h_n^0 - \tfrac{1}{2}cn^{-\frac{1}{2}} \log n\} \to 1 \qquad \text{as } n \to \infty .$$

Thus, to prove the theorem, it suffices to show that as $n \to \infty$,

(5.11) $$\textstyle\sum_{j=2}^3 P\{\max_i [Z_{ni} : i \in S_n^{(j)}] \geqq h_n^0 - \tfrac{1}{2}cn^{-\frac{1}{2}} \log n\} \to 0 ,$$

(which implies that $n^{\frac{1}{2}}(Z_n^* - Z_{n,1}^*) = 0$, in probability, as $n \to \infty$). By (3.2), and the continuity of $g(x)$ in the closed interval $[x_n^0 - \rho, x_n^0 + \rho]$, we have $\sup_n [\sup_x \{g(x) : x \in [x_n^0 - \rho, x_n^0 + \rho]\}] < \infty$, and hence, by Lemma 4.2,

(5.12) $$\begin{aligned} &\sup_i \{|n^{\frac{1}{2}}(Z_{ni} - Y_{ni})| : i \in S_n^{(2)}\} \\ &\quad \leqq [\sup_x \{g(x) : x \in [x_n^0 - \rho, x_n^0 + \rho]\}] \\ &\qquad \times [\sup_x \{n^{\frac{1}{2}}|\bar{F}_{(n)}(x) - S_n(x)| : x \in [x_n^0 - \rho, x_n^0 + \rho]\}] \\ &\quad = O_p(1) . \end{aligned}$$

On the other hand, by (5.1) and (5.4), $\max_i \{Y_{ni} : i \in S_n^{(2)}\} \leqq h_n^0 - cn^{-\frac{1}{2}} \log n$. Hence, by (5.12), $\max_i \{Z_{ni} : i \in S_n^{(2)}\} \leqq h_n^0 - \frac{1}{2}cn^{-\frac{1}{2}} \log n$, in probability, as $n \to \infty$. Finally, by (3.9), (3.10) and Lemma 4.10, it follows that for every $\rho > 0$, there exists a $\rho^*(> 0)$, such that

(5.13) $$\max_i \{Z_{ni} : i \in S_n^{(3)}\} \leqq h_n^0 - \rho^* , \qquad \text{in probability, as } n \to \infty .$$

Consequently, (5.11) holds, and the theorem follows.

6. Moment convergence of Z_n^*. We impose the following additional regularity conditions:

(a) $\psi(x, y)$ is convex in $y (0 \leqq y \leqq 1)$ i.e.,

(6.1) $$\psi(x, y) \leqq (1 - y)\psi(x, 0) + y\psi(x, 1) , \qquad \forall y \in (0, 1), \text{ where}$$

(b) $\psi(x, \delta)$, $\delta = 0, 1$ are nonnegative,

(6.2) $$\psi(x, 0) \text{ is } \uparrow \text{ in } x \text{ and } \psi(x, 1) \text{ is } \downarrow \text{ in } x: \quad -\infty < x < \infty, \text{ and}$$

(c)

(6.3) $$\sup_n \textstyle\int_0^\infty \psi^2(x, \delta)\, d\bar{F}_{(n)}(x) \leqq \mu_2(\delta) < \infty \qquad \text{for } \delta = 0, 1 .$$

THEOREM 6.1. *Under the conditions of Theorem* 3.1, *and* (6.1)—(6.3), $\lim_{n\to\infty} |E(Z_n^*) - h_n^0| = 0$.

PROOF. Let $X_{n,1} \leqq \cdots \leqq X_{n,n}$ be the order statistics, and let $Z_n^{(i)} = \psi(X_{n,i}, S_n(X_{n,i}))$, $1 \leqq i \leqq n$, so that $Z_n^* = \max_{1\leqq i\leqq n} Z_n^{(i)}$. Then, by (6.1)

(6.4) $$Z_n^{(i)} \leqq [(n - i)/n]\psi(X_{n,i}, 0) + (i/n)\psi(X_{n,i}, 1) , \qquad 1 \leqq i \leqq n .$$

Now, by (6.2),

$$(6.5) \qquad \frac{n-i}{n}\psi(X_{n,i}, 0) \leqq \frac{1}{n}\sum_{j=i}^{n}\psi(X_{n,j}, 0) \leqq \frac{1}{n}\sum_{j=1}^{n}\psi(X_{n,j}, 0) = \frac{1}{n}\sum_{j=1}^{n}\psi(X_j, 0) ;$$

$$(6.6) \qquad \frac{i}{n}\psi(X_{n,i}, 1) \leqq \frac{1}{n}\sum_{j=1}^{i}\psi(X_{n,j}, 1) \leqq \frac{1}{n}\sum_{j=1}^{n}\psi(X_{n,j}, 1) = \frac{1}{n}\sum_{j=1}^{n}\psi(X_j, 1) .$$

Therefore, on writing $\bar{\psi}_n = n^{-1}\sum_{i=1}^{n}\{\psi(X_i, 0) + \psi(X_i, 1)\}$, we have

$$(6.7) \qquad 0 \leqq Z_n^* \leqq \bar{\psi}_n .$$

Let $\Phi_n(x)$ be the cdf of $\bar{\psi}_n$. Then, by (6.3), $\bar{\psi}_n$ is uniformly integrable, i.e., the identity function is uniformly integrable with respect to $\{\Phi_n\}$. Now, by Theorem 3.1, Z_n^* converges in probability to h_n^0. Hence, the theorem follows by using the dominated convergence theorem.

The next problem is to study conditions under which h_n^0 may be replaced by $E(Z_n^*)$ in Theorem 3.1. This is of interest as often we want to study the rate of convergence of the bias of Z_n^* relative to its asymptotic standard deviation. This problem is studied in the case of i.i.d. rv's in Theorem 7.1 (see also the remark following it).

In the remainder of the paper, we consider a simple ψ where all these results apply directly.

7. A special case. In (1.1), B_n has been introduced to describe the strength of a bundle of parallel filaments. Let $B_n^* = n^{-1}B_n = \max_{1\leqq i\leqq n}\{[1-(i-1)/n]X_{n,i}\}$, where $X_{n,i}$, $i = 1, \cdots, n$, are the ordered values of the nonnegative random variables $X_i, \cdots, X_{ii}$ representing the strength of the individual filaments in the bundle. Observe that $B_n^* = \max_{1\leqq i\leqq n}\{X_i[1 - S_n(X_i -)]\} = \sup_{x\geqq 0}\{x[1 - S_n(x)]\}$ (see [10]). Now, if the X_i are i.i.d. rv's with $E(X^2) < \infty$, it is easily seen that $n^{\frac{1}{2}}|B_n^* - \max_{1\leqq i\leqq n} X_i[1 - S_n(X_i)]| \to_p 0$ as $n \to \infty$. Thus, we are led to consider a simple ψ where $\psi(x, y) = x(1-y)$, $0 \leqq x < \infty$, $0 \leqq y \leqq 1$. We define $h_0 = \sup_{x\geqq 0}\{x[1 - F(x)]\}$, so that by assumption, $h_0 = x_0[1 - F(x_0)]$ where h_0 and x_0 are both unique. For i.i.d. rv's, Daniels [4] has given a very elaborate deduction of the asymptotic normality of the standardized form of B_n^*. It would be very complicated to extend his method if the variables are not identically distributed or independent. On the other hand, our Theorem 3.1 yields a very simple proof valid under more general conditions. Further, it is shown in [10] that when the X_i are i.i.d. rv's with finite expectation, $\{B_n^*\}$ forms a reverse sub-martingale sequence, and when $E(X_i^2) < \infty$, $B_n^* \to h_0$ a.s. and $E(B_n^*) \to h_0$ as $n \to \infty$. Hence, $Z_n^* \to h_0$ a.s., as $n \to \infty$. Moreover, (6.1)—(6.3) hold for this particular case, and by Theorem 6.1, $E(Z_n^*) \to h_0$ as $n \to \infty$, which also follows from the fact that $E(B_n^*) \to h_0$ as $n \to \infty$.

For a sequence of estimators $\{T_n\}$ of a parameter θ, the ARB (asymptotic relative bias) is defined as the limit of the ratio of the bias of T_n to its asymptotic standard deviation. In addition to the criterion of consistency, it is quite natural to seek that the ARB of an estimator should be equal to 0, so that all the conclusions derived from the asymptotic distribution of the standardized form of T_n remain valid no matter whether we substitute θ or ET_n. We shall show that in the case of i.i.d. rv's, the ARB of $\{Z_n^*\}$ is equal to 0. For simplicity of proof, we first consider the particular case of $\psi(x, F(x)) = x[1 - F(x)]$.

THEOREM 7.1. *If* $\mu_2' = E(X_1^2) < \infty$, $x[1 - F(x)]$ *assumes a unique maximum* h_0 *at* $x = x_0$, *and if for some* $x_1(> x_0)$, $x[1 - F(x)]$ *is* $\downarrow$ *for* $x > x_1$, *then*

$$\lim_{n\to\infty} \{n^{\frac{1}{2}}|E(B_n^*) - h_0|\} = 0 .$$

PROOF. Let $M_n = x_0 n^{\frac{1}{2}}[S_n(x_0) - F(x_0)]$. Then, by definition,

$$B_n^* = \sup_x x[1 - S_n(x)] \geqq x_0[1 - S_n(x_0)] = h_0 - n^{-\frac{1}{2}} M_n , \tag{7.1}$$

where $E(M_n) = 0$ and $E(M_n^2) \leqq \frac{1}{4}x_0^2$, $\forall n \geqq 1$. Hence,

$$n^{\frac{1}{2}}(B_n^* - h_0) + M_n \geqq 0 \qquad \text{and} \qquad EB_n^* \geqq h_0 , \qquad \forall n \geqq 1 . \tag{7.2}$$

Also, as a special case, we obtain from Theorem 3.1, (5.8) and (5.11) that

$$n^{\frac{1}{2}}(B_n^* - h_0) + M_n \to_p 0 , \qquad \text{as } n \to \infty . \tag{7.3}$$

Therefore, the proof of the theorem will follow from (7.2), (7.3) and the dominated convergence theorem (cf. [8] page 125), provided we can show that there is some nonnegative random variable W_n such that for all n,

$$n^{\frac{1}{2}}(B_n^* - h_0) \leqq W_n \qquad \text{where } W_n \text{ is uniformly integrable.} \tag{7.4}$$

With this end in view, we choose a positive number $c(> x_1)$ such that $c > 1$,

$$\sup_{x>c} x[1 - F(x)] = c[1 - F(c)] = h_1 < \tfrac{1}{3}h_0 , \qquad \text{and} \qquad F(c) > \tfrac{1}{2} . \tag{7.5}$$

Then, we have

$$\begin{aligned} B_n^* &\leqq \max [\sup_{x\leqq c} x[1 - S_n(x)], \sup_{x>c} x[1 - S_n(x)]] \\ &= \max [h_0 + \sup_{x\leqq c} x[F(x) - S_n(x)], \sup_{x>c} x[1 - S_n(x)]] \\ &\leqq \max [h_0 + c \sup_x [F(x) - S_n(x)], \sup_{x>c} x[1 - S_n(x)]] , \end{aligned} \tag{7.6}$$

and as $\sup_x [F(x) - S_n(x)] \geqq 0$ (the equality always holds at $x = 0$), we have

$$\begin{aligned} &n^{\frac{1}{2}}(B_n^* - h_0) \\ &\quad\leqq \max \{c \sup_x n^{\frac{1}{2}}[F(x) - S_n(x)], \max [0, n^{\frac{1}{2}}\{\sup_{x>c} x[1 - S_n(x)] - h_0\}]\} \\ &\quad\leqq c\{\sup_x n^{\frac{1}{2}}[F(x) - S_n(x)]\} + \max [0, n^{\frac{1}{2}}\{\sup_{x>c} x[1 - S_n(x)] - h_0\}] \\ &\quad\leqq c\{\sup_x n^{\frac{1}{2}}[F(x) - S_n(x)]\} + \max [0, n^{\frac{1}{2}}\{\sup_{x>c} x[F(x) - S_n(x)] - \tfrac{2}{3}h_0\}] \\ &\quad= W_n^{(1)} + W_n^{(2)} = W_n , \quad \text{say,} \end{aligned} \tag{7.7}$$

where both $W_n^{(1)}$ and $W_n^{(2)}$ are nonnegative. Now $\sup_x n^{\frac{1}{2}}[F(x) - S_n(x)]$ has the same distribution as of $\sup_x n^{\frac{1}{2}}[S_n(x) - F(x)]$, and hence, by Lemma 2, page 646, of Dvoretsky, Kiefer and Wolfowitz [6], we have

$$P\{\sup_x n^{\frac{1}{2}}[F(x) - S_n(x)] > r\} \leqq c_1 e^{-2r^2}, \qquad \text{for all } r \geqq 0, \tag{7.8}$$

where c_1 does not depend on n. (7.8) implies that $W_n^{(1)}$ is uniformly integrable.

Since $\mu_2' < \infty$, there exists an $h(> 0)$, such that

$$x[F(x+h) - F(x)] \leqq \int_x^{x+h} x\, dF(x) \leqq \tfrac{1}{6}h_0 \qquad \text{for all } x \geqq x_1 \tag{7.9}$$

Define then a countable set of points

$$b_j = c + hj, \qquad j = 0, 1, 2, \cdots, \infty, \tag{7.10}$$

where we let $h < \frac{1}{2}$. Also, let

$$U_{nj} = b_j[F(b_j) - S_n(b_j)], \qquad j = 0, 1, 2, \cdots. \tag{7.11}$$

Then, for every $x \in [b_j, b_{j+1}]$, $|x[F(x) - S_n(x)]| \leqq b_j^{-1}(b_{j+1})\{\max[|U_{nj}|, |U_{nj+1}|] + b_j[F(b_{j+1}) - F(b_j)]\} \leqq (1+h)\{\max[|U_{nj}|, |U_{nj+1}|] + \frac{1}{6}h_0\} \leqq \frac{3}{2}\{\max[|U_{nj}|, |U_{nj+1}|]\} + \frac{1}{4}h_0$, for $j = 0, 1, 2, \cdots$. Thus, $W_n^{(2)} \leqq \max[0, n^{\frac{1}{2}}\{\max_{j\geqq 0} \frac{3}{2}|U_{nj}| - \frac{1}{3}h_0\}]$. Hence,

$$P\{W_n^{(2)} \geqq \omega\} \leqq P\{\max_{j>0} |U_{nj}| > \tfrac{2}{9}h_0 + \tfrac{2}{3}n^{-\frac{1}{2}}\omega\}, \qquad \text{for all } \omega \geqq 0. \tag{7.12}$$

Now, some standard computations yield that $E\{U_{nj} \mid U_{n0}, \cdots, U_{nj-1}\} = U_{nj-1}\{b_j[1 - F(b_j)]/b_{j-1}[1 - F(b_{j-1})]\} = \psi_j U_{nj-1}$, say, $j \geqq 1$, where by the assumed monotonicity of $x[1 - F(x)]$ (for $x > x_1$),

$$\psi_j = b_j[1 - F(b_j)]/b_{j-1}[1 - F(b_{j-1})] \leqq 1 \qquad \text{for all } j \geqq 1. \tag{7.13}$$

Thus, $E\{|U_{nj}| \mid U_{n0}, \cdots, U_{nj-1}\} \geqq \psi_j|U_{nj-1}|$, $\forall j \geqq 1$, and $E[U_{nj}^2] = n^{-1}b_j^2 F(b_j)[1 - F(b_j)]$, $j \geqq 0$. Hence, by Theorem 2.1 of Birnbaum and Marshall [3], we obtain that for all $\omega \geqq 0$,

$$\begin{aligned} &P\{\max_{j\geqq 0} |U_{nj}| > \tfrac{2}{9}h_0 + \tfrac{2}{3}n^{-\frac{1}{2}}\omega\} \\ &\quad\leqq n^{-1}(\tfrac{2}{9}h_0 + \tfrac{2}{3}n^{-\frac{1}{2}}\omega)^{-2}\{\textstyle\sum_{j=1}^{\infty}[b_j^2 F(b_j)\{1 - F(b_j)\} \\ &\qquad - \psi_j^2 b_{j-1}^2 F(b_{j-1})\{1 - F(b_{j-1})\}] + b_0^2 F(b_0)[1 - F(b_0)]\} \\ &\quad= (\tfrac{2}{9}n^{\frac{1}{2}}h_0 + \tfrac{2}{3}\omega)^{-2}\{\textstyle\sum_{j=1}^{\infty} b_j^2[F(b_j) - F(b_{j-1})][1 - F(b_j)]/[1 - F(b_{j-1})] \\ &\qquad + b_0^2 F(b_0)[1 - F(b_0)]\} \\ &\quad\leqq (\tfrac{2}{9}n^{\frac{1}{2}}h_0 + \tfrac{2}{3}\omega)^{-2}\{\textstyle\sum_{j=1}^{\infty} b_j^2[F(b_j) - F(b_{j-1})] + b_0^2[1 - F(b_0)]\} \\ &\quad\leqq (\tfrac{2}{9}n^{\frac{1}{2}}h_0 + \tfrac{2}{3}\omega)^{-2}\{2\textstyle\int_c^{\infty} x^2\, dF(x)\} \\ &\quad< 41\mu_2'\{n^{\frac{1}{2}}h_0 + 3\omega\}^{-2}. \end{aligned} \tag{7.14}$$

This implies that $W_n^{(2)}$ is also uniformly integrable. The proof of the theorem follows then from (7.4), (7.7), (7.8) and (7.14). □

REMARKS. (i) The assumption of monotonicity of $x[1 - F(x)]$ in the tail, though not restrictive, can be removed at the cost of the existence of $\mu_r' = E(X_1^r)$ for some $r > 2$; for brevity, the proof is omitted. (ii) For general $\psi(x, F(x))$, by (III) of Section 3, $|\psi(x, S_n(x)) - \psi(x, F(x))| \leqq g(x)|S_n(x) - F(x)|$, and hence, whenever $\max_{1\leqq i\leqq n} \psi(X_i, S_n(X_i)) = \sup_{x\geqq 0} \psi(x, S_n(x))$, Theorem 7.1 holds if $x[1 -$

$F(x)]$ be replaced by $g(x)[1 - F(x)]$ i.e., if $E\{g^2(X_1)\} < \infty$ and $g^2(x)[1 - F(x)]$ is $\downarrow$ in $x \geqq x_1$. The proof is analogous.

8. Some possible generalizations of Theorem 3.1. (i) *Multivariate case.* Let $\mathbf{X}_i = (X_{i1}, \cdots, X_{ip})'$, $i = 1, 2, \cdots, n$ be independent stochastic p-vectors, $(p \geqq 1)$, with continuous df's $F_1, \cdots, F_n$, respectively. For the jth variate, let $F_{i[j]}(x)$ be the df of X_{ij}, $i = 1, \cdots, n$; $j = 1, \cdots, p$, and let

$$\bar{F}_{n[j]}(x) = n^{-1} \sum_{i=1}^{n} F_{i[j]}(x) , \qquad S_{n(j)}(x) = n^{-1} \sum_{i=1}^{n} c(x - X_{ij}) , \qquad j = 1, \cdots, p . \tag{8.1}$$

Consider a real nonnegative valued function

$$h_n(\mathbf{x}) = \psi(\mathbf{x}, F_{n[1]}(x_1), \cdots, F_{n[p]}(x_p)) , \qquad \mathbf{x} \in R^p . \tag{8.2}$$

As in Sections 2 and 3, we assume that $h_n(\mathbf{x})$ has a unique (finite) maximum h_n^0 at x_n^0 where h_n^0 satisfies (3.2) and $||\mathbf{x}_n^0||$ satisfies (3.2); $||\mathbf{x}||$ being the Euclidean norm of $\mathbf{x}$. Let then $Z_n^* = \max_{1 \leq i \leq n} Z_{ni}$, where

$$Z_{ni} = \psi(\mathbf{X}_i, S_{n[1]}(X_{il}), \cdots, S_{n[p]}(X_{ip})) , \qquad i = 1, \cdots, n . \tag{8.3}$$

Replacing y in $\psi(x, y)$ by a vector $\mathbf{y} = (y_1, \cdots, y_p)$ and x by $\mathbf{x}$, we impose the same conditions as in Section 3, and denote the partial derivatives of ψ with respect to $y_1, \cdots, y_p$, evaluated at $(\mathbf{x}_n^0, F_{n[1]}(x_{n1}^0), \cdots, F_{n[p]}(x_{np}^0))$, by $\xi_n^{(1)}, \cdots, \xi_n^{(p)}$ which are assumed to satisfy (3.7). Let then

$$\mathbf{W}_n^* = (W_{n,1}^*, \cdots, W_{n,p}^*)' ; \qquad W_{n,j}^* = n^{\frac{1}{2}}[S_{n(j)}(x_{nj}^0) - \bar{F}_{n[j]}(x_{nj}^0)] , \qquad j = 1, \cdots, p , \tag{8.4}$$

$\boldsymbol{\xi}_n = (\xi_n^{(1)}, \cdots, \xi_n^{(p)})'$ and let $\boldsymbol{\nu}_n$ be the covariance matrix of $\mathbf{W}_n^*$. Finally, let

$$\gamma_n^2 = \boldsymbol{\xi}_n' \boldsymbol{\nu}_n \boldsymbol{\xi}_n \quad \text{and assume that} \quad \inf_n \gamma_n > 0 . \tag{8.5}$$

Then, by the same technique as in Theorem 3.1, it can be shown that

$$|n^{\frac{1}{2}}(Z_n^* - h_n^0) - \boldsymbol{\xi}_n' \mathbf{W}_n^*| \to_p 0 \qquad \text{as } n \to \infty , \tag{8.6}$$

and hence, as $n \to \infty$,

$$\mathscr{L}(n^{\frac{1}{2}}(Z_n^* - h_n^0)/\gamma_n) \to \mathscr{N}(0, 1) . \tag{8.7}$$

(ii) *Vector case.* Suppose now that $\mathbf{h}_n(x) = [h_n^{(1)}(x), \cdots, h_n^{(p)}(x)]$, $x \in R$, where $h_n^{(j)}(x) = \psi_{(j)}(x, \bar{F}_{(n)}(x))$ satisfies the conditions of Sections 2 and 3; the vector of sample and population maxima are denoted by $\mathbf{Z}_n^*$ and $\mathbf{h}_n^0$, respectively. Then, by the same technique as in Theorem 3.1, we have

$$\mathscr{L}(n^{\frac{1}{2}}[\mathbf{Z}_n^* - \mathbf{h}_n^0]) \to \mathscr{N}_p(0, \mathbf{\Gamma}_n) , \tag{8.8}$$

where $\mathbf{\Gamma}_n$ is the dispersion matrix of $n^{\frac{1}{2}}\psi_{(j)01}(x_{nj}^0, \bar{F}_{(n)}(x_{nj}^0))[S_n(x_{nj}^0) - \bar{F}_{(n)}(x_{nj}^0)]$, $j = 1, \cdots, p$, and $h_{nj}^0 = \psi_{(j)}(x_{nj}^0, \bar{F}_{(n)}(x_{nj}^0))$, $j = 1, \cdots, p$.

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