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THE LARGE-SAMPLE POWER OF PERMUTATION TESTS FOR RANDOMIZATION MODELS

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The permutation test using the usual F-statistic from a randomized block experiment is considered under a randomization model. Alternative hypotheses assuming additive treatment effects are considered. It is shown that the critical value of the test statistic tends to a constant in probability as the number of blocks becomes large. The large-sample power of the test is calculated for a sequence of alternatives arising naturally from the randomization model.

1. Introduction. The physical randomization process in a randomized block design consists of assigning p treatments at random to the p plots in each of n blocks. This produces random variables whose distributions, under the null hypothesis of no treatment effects, are invariant under a group of permutations. This enables a permutation test, based on this group of permutations, to be used as an exact test of the null hypothesis. It is important to distinguish the two types of permutations which are used. A random permutation is used to obtain the design of the experiment and then, in order to obtain a permutation test, permutations of the actual yields are considered. When the null hypothesis is true, it is necessary only to observe that the physical randomization produces random variables whose distributions are invariant under a group of permutations. However, when it is assumed that treatments do have an effect, it is necessary to consider both the random permutations produced by the randomization process and the permutations used to obtain the permutation test.

It is the purpose of this note to obtain the power of the permutation test for large n, under alternatives arising naturally from the randomization model. Hoeffding (1952) considered a similar problem, but he assumed that the plot errors were independently distributed with equal variance.

The methods and notation used have been selected to conform as closely as possible to those of Hoeffding (1952). We will consider the randomization model under the assumption that treatment effects are completely additive. If $y_n = (y_{n11}, \cdots, y_{n1p}, \cdots, y_{nn1}, \cdots, y_{nnp})$, are the plot errors, or the plot yields when no treatments are applied, then an appropriate model for the design is

$$X_{nij} = t_{nj} + Y_{nij} ,$$

where t_{nj} is the effect of treatment j and $Y_{nij} = y_{niS_{ij}}$, where $(S_{i1}, \cdots, S_{ip})$, $i = 1, \cdots, n$, are n independent random vectors whose values are the p! equally

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probable permutations of $(1, \cdots, p)$. In the sequel the first subscript n will usually be omitted for convenience of notation.

We will consider tests based on the function

$$t(x) = \frac{\sum_{j=1}^{p} u_j(x)^2}{n^{-1}(p-1)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} (x_{ij} - x_{i\cdot})^2},$$

where $u_j(x) = n^{-\frac{1}{2}} \sum_{i=1}^{n} (x_{ij} - x_{i\cdot})$ and the arithmetic mean over a subscript is denoted by replacing the subscript with a dot.

Let $\mathbf{G}$ be the group of $(p!)^n = M$ permutations of

$$X = (X_{11}, \cdots, X_{1p}, \cdots, X_{n1}, \cdots, X_{np})$$

such that if $g \in \mathbf{G}$, then $gX = (X_{1j_{11}}, \cdots, X_{1j_{1p}}, \cdots, X_{nj_{n1}}, \cdots, X_{nj_{np}})$, where $(j_{i1}, \cdots, j_{ip})$, $i = 1, \cdots, n$, are any permutations of $(1, \cdots, p)$. Let $t^{(1)}(x) \leqq \cdots \leqq t^{(M)}(x)$ be the ordered values of $t(gx)$ for all g in $\mathbf{G}$. Let $k = M - [M\alpha]$, where α is the required size of the test and $[M\alpha]$ is the integer part of $M\alpha$. We are interested in a test which rejects the null hypothesis when $t(X) > t^{(k)}(X)$ and accepts it otherwise. This test is of size $[M\alpha]/M$. We wish to consider the power of the test, which is $P\{t(X) > t^{(k)}(X)\}$, under sequences of alternatives given by the sequences t_{nj} and Y_{nij}. We do this by showing that under certain conditions $t^{(k)}(X) \to \lambda$ in probability, where $P[\chi^2_{p-1} > \lambda] = \alpha$, if χ^2_{p-1} is a chi-squared variate with $p-1$ degrees of freedom, and that $P\{t(X) < y\}$ tends to zero or to the distribution function of a noncentral chi-squared variate with $p-1$ degrees of freedom and with noncentrality parameter given as a limit of a sequence of functions of t_{nij} and y_{nij}. In the usual analysis of variance model, the y_{nij} are assumed to be independent samples from a normal distribution with means b_{ni} and variances σ^2. In this case, it is noted that the permutation test is asymptotically as powerful as the usual F-test.

2. Convergence of the critical value of the test.

DEFINITION. Let

$$\sigma_n^2 = n^{-1}(p-1)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} (y_{ij} - y_{i\cdot})^2$$

and

$$\delta_n^2 = \sigma_n^{-2}(p-1)^{-1} \sum_{j=1}^{p} (t_j - t_\cdot)^2 .$$

LEMMA 1.

$$\sigma_n^{-2}(1 + \delta_n^2)^{-1} n^{-1}(p-1)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} (X_{ij} - X_{i\cdot})^2 \to 1 \quad \textit{in probability.}$$

REMARK. The probability measures considered here are those which apply to the randomization procedure. That is, they are implicit in the definitions of the random variables S_{ij} defined in Section 1.

PROOF OF LEMMA 1.

$$(1) \qquad \sum_{j=1}^{p} (X_{ij} - X_{i\cdot})^2 = \sum_{j=1}^{p} (y_{ij} - y_{i\cdot})^2 + 2 \sum_{j=1}^{p} (Y_{ij} - y_{i\cdot})(t_j - t_\cdot) + \sum_{j=1}^{p} (t_j - t_\cdot)^2 .$$

Also

$$E\{\sum_{j=1}^{p}(Y_{ij} - y_{i\cdot})(t_j - t_\cdot)\} = 0$$

and

$$\begin{aligned}\sigma_n^{-4}(1+\delta_n^2)^{-2}n^{-2}(p-1)^{-2}\sum_{i=1}^{n}\operatorname{var}\{\sum_{j=1}^{p}(Y_{ij}-y_i)(t_j-t_\cdot)\}\\ = \sigma_n^{-4}(1+\delta_n^2)^{-2}n^{-2}(p-1)^{-3}\sum_{i=1}^{n}\sum_{j=1}^{p}(y_{ij}-y_{i\cdot})^2\sum_{j=1}^{p}(t_j-t_\cdot)^2\\ = n^{-1}(p-1)^{-1}(1+\delta_n^2)^{-2}\delta_n^2 \to 0\,,\end{aligned}$$

so since the Y_{ij} are independent for different i, it follows from Chebychev's inequality that

$$\sigma_n^{-2}(1+\delta_n^2)^{-1}n^{-1}(p-1)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{p}(Y_{ij}-y_{i\cdot})(t_j-t_\cdot) \to 0$$

in probability. So the lemma follows immediately by summing (1) over i and dividing by $n(p-1)\sigma_n^2(1+\delta_n^2)$.

Let G and G' be independent identically distributed random variables whose values are the M elements g of $\mathbf{G}$, each element having probability M^{-1}. Then the following lemma is an immediate consequence of Lemma 1.

LEMMA 2. *If $t'(X) = \sigma_n^{-2}(1+\delta_n^2)^{-1}\sum_{j=1}^{p}u_j(x)^2$ and if the joint distribution of $(t'(GX), t'(G'X))$ approaches a limit, then the joint distribution of $(t(GX), t(G'X))$ approaches the same limit.*

It is convenient to state a lemma which is contained in Theorem 3.2 of Hoeffding (1952).

LEMMA 3. *If $t(GX)$ and $t(G'X)$ have the limiting joint distribution $F(y)F(y')$, where $F(y)$ is a distribution function, and if the equation $F(y) = 1-\alpha$ has a unique solution $y=\lambda$, then $t^{(k)}(X)\to\lambda$ in probability.*

THEOREM 1. *If, for any $\eta > 0$,*

$$\sum_{|\beta_i|>\eta}\beta_i^2 \to 0 \qquad \text{as } n\to\infty\,, \tag{2}$$

where

$$\beta_i^2 = \frac{\sum_{j=1}^{p}\{(y_{ij}-y_{i\cdot})^2+(t_j-t_\cdot)^2\}}{\sum_{i=1}^{n}\sum_{j=1}^{p}\{(y_{ij}-y_{i\cdot})^2+(t_j-t_\cdot)^2\}}\,, \tag{3}$$

then the limiting distribution of $(t(GX), t(G'X))$ is that of a pair of independent chi-squared variates with $p-1$ degrees of freedom.

PROOF. Let $U_j = u_j(GX)$ and $U_j' = u_j(G'X)$. The random vector

$$n^{\frac12}U = n^{\frac12}(U_1, \cdots, U_p, U_1', \cdots, U_p')$$

is the sum of n independent random vectors $Z_i = (Z_{iR_{i1}}, \cdots, Z_{iR_{ip}}, Z_{iR'_{i1}}, \cdots, Z_{iR'_{ip}})$, where $(Z_{i1}, \cdots, Z_{ip})$ has the distribution of $(Y_{i1} - y_{i\cdot} + t_1 - t_\cdot, \cdots, Y_{ip} - y_{i\cdot} + t_p - t_\cdot)$ and $(R_{i1}, \cdots, R_{ip})$, $(R'_{i1}, \cdots, R'_{ip})$, $i = 1, \cdots, n$, are $2n$ independent random vectors whose values are the $p!$ equally probable permutations of $(1, \cdots, p)$. Let (c_{jk}) be an orthogonal $p\times p$ matrix with $c_{p1} = \cdots = c_{pp}$, let

$$W_{ij} = \sum_{k=1}^{p} c_{jk}Z_{iR_{ik}} \quad\text{and}\quad W'_{ij} = \sum_{k=1}^{p} c_{jk}Z_{iR'_{ik}}$$

and let

$$W_j = n^{-\frac{1}{2}} \sum_{i=1}^n W_{ij} \quad \text{and} \quad W_j' = n^{-\frac{1}{2}} \sum_{i=1}^n W'_{ij} .$$

Now

$$W_j = \sum_{k=1}^p c_{jk} U_k \quad \text{and} \quad W_j' = \sum_{k=1}^p c_{jk} U_k' ,$$

so, since (c_{jk}) is orthogonal,

$$\sum_{k=1}^p U_k^2 = \sum_{j=1}^{p-1} W_j^2 \quad \text{and} \quad \sum_{k=1}^p U_k'^2 = \sum_{j=1}^{p-1} W_j'^2 .$$

Also for $j, j' \leqq p - 1$,

$$EW_j = EW_j' = 0 , \qquad EW_j W'_{j'} = 0 ,$$

and

$$\begin{aligned} EW_j W_{j'} &= n^{-1} \sum_{i=1}^n EW_{ij} W_{ij'} \\ &= n^{-1}p^{-1} \sum_{i=1}^n \{\sum_{k=1}^p c_{jk} c_{j'k} \sum_{l=1}^p E(Y_{il} - y_{i}. + t_l - t.)^2\} \\ &\quad + n^{-1}p^{-1}(p-1)^{-1} \sum_{i=1}^n \sum_{k \neq k'} c_{jk} c_{j'k'} \\ &\quad \times \sum_{l \neq l'} E(Y_{il} - y_{i}. + t_l - t.)(Y_{il'} - y_{i}. + t_{l'} - t.) \\ &= \delta_{jj'} n^{-1}p^{-1} \sum_{i=1}^n \{\sum_{l=1}^p (y_{il} - y_{i}.)^2 + \sum_{l=1}^p (t_l - t.)^2\} \\ &\quad + n^{-1}p^{-1}(p-1)^{-1} \sum_{i=1}^n \sum_{k=1}^p c_{jk} c_{j'k} \sum_{l=1}^p E(Y_{il} - y_{i}. + t_l - t.)^2 \\ &= \delta_{jj'} \sigma_n^2(1 + \delta_n^2) , \end{aligned}$$

where $\delta_{jj'} = 0$ if $j \neq j'$ and 1 if $j = j'$. Similarly, $EW_j'W_j' = \delta_{jj'}\sigma_n^2(1 + \delta_n^2)$.

If the vectors

$$W_i^* = \sigma_n^{-1}(1 + \delta_n^2)^{-\frac{1}{2}}(W_{i1}, \cdots, W_{i,p-1}, W'_{i1}, \cdots, W'_{i,p-1})$$

satisfy a generalized Lindeberg condition, namely, for any $\varepsilon > 0$

$$L_n = n^{-1} \sum_{i=1}^n \int_{w > \varepsilon n^{\frac{1}{2}}} w^2 \, dP[|W_i^*| < w] \to 0$$

as $n \to \infty$, where $|W_i^*|^2 = \sigma_n^{-2}(1 + \delta_n^2)^{-1}[W_{i1}^2 + \cdots + W'^2_{i,p-1}]$, then using Theorem 21a of Cramér (1962), we have that the limiting distribution of $W^* = \sigma_n^{-1}(1 + \delta_n^2)^{-\frac{1}{2}}(W_1, \cdots, W'_{p-1})$ is that of a vector of $2p - 2$ independent standard normal variates. Now

$$\sigma_n^2(1 + \delta_n^2)\,|W_i^*|^2 = 2 \sum_{j=1}^p Z_{ij}^2 \leqq 4 \sum_{j=1}^p (y_{ij} - y_{i}.)^2 + 4 \sum_{j=1}^p (t_j - t.)^2 .$$

So if

$$E = \{i : \sum_{j=1}^p [(Y_{ij} - Y_{i}.)^2 + (t_j - t.)^2] > \tfrac{1}{4} \varepsilon^2 n \sigma_n^2(1 + \delta_n^2)\}$$

then

$$\begin{aligned} L_n &\leqq n^{-1} 4 \sigma_n^{-2}(1 + \delta_n^2)^{-1} \sum_{i \in E} \sum_{j=1}^p \{(y_{ij} - y_{i}.)^2 + (t_j - t.)^2\} \\ &= 4(p-1) \sum_{|\beta_i| > \eta} \beta_i^2 , \end{aligned}$$

where β_i are defined in (3) and $\eta^2 = \frac{1}{4}(p-1)^{-1}\varepsilon^2$. Thus Condition (2) is sufficient for the generalised Lindeberg condition. So the limiting distribution of $(t'(GX), t'(G'X))$, and hence from Lemma 2 of $(t(GX), t(G'X))$, is that of a pair of independent chi-squared variates with $p - 1$ degrees of freedom.

REMARK. Simpler conditions which imply (2) are $0 < c' < \sum_{j=1}^p (y_{ij} - y_{i}.)^2 < c''$ for all i.

The next theorem is an immediate consequence of Theorem 1 and Lemma 3.

THEOREM 2. *If condition* (2) *holds for any* $\eta > 0$, *then* $t^{(k)}(X) \to \lambda$ *in probability.*

3. The asymptotic power of the test. Consider sequences of alternative hypotheses given by the sequences t_{nj}, Y_{nij} $j = 1, \cdots, p$, $n = 1, 2, \cdots$. Let

$$\rho_n = n\sigma_n^{-2} \sum_{j=1}^{p} (t_{nj} - t_{n\cdot})^2 .$$

Then we can show that the asymptotic power of the test depends on the limit of the sequence $\{\rho_n\}$.

THEOREM 3. *If, for any* $\eta > 0$

$$\sum_{|\gamma_i|>\eta} \gamma_i^2 \to 0 \qquad \text{as } n \to \infty , \tag{4}$$

where $\gamma_i^2 = n^{-1}\sigma_n^{-2} \sum_{j=1}^{p} (y_{ij} - y_{i\cdot})^2$, *then*

(i) *if* $\rho_n \to 0$, $t(X)$ *tends in distribution to a chi-squared variate with* $p - 1$ *degrees of freedom;*

(ii) *if* $\rho_n \to \rho$, *a constant,* $t(X)$ *tends in distribution to a noncentral chi-squared variate with* $p - 1$ *degrees of freedom and noncentrality parameter* ρ;

(iii) *if* $\rho_n \to \infty$, $P\{t(X) < y\} \to 0$ *for all* $y > 0$.

PROOF. Let $V_{ij} = \sum_{k=1}^{p} c_{jk} X_{ik}$ and $V_j = n^{-\frac{1}{2}} \sum_{i=1}^{n} V_{ij}$. Then for $j, j' \leqq p - 1$,

$$E(V_{ij}) = \sum_{k=1}^{p} c_{jk} t_k ,$$

$$\operatorname{Var}(V_{ij}) = (p-1)^{-1} \sum_{j=1}^{p} (y_{ij} - y_{i\cdot})^2 \quad \text{and} \quad \operatorname{Cov}(V_{ij}, V_{ij'}) = 0 , \qquad j \neq j' .$$

In the same way as in Theorem 1, it may be shown that (4) is sufficient for the vectors $\sigma_n^{-1}(V_{i1} - E(V_{i1}), \cdots, V_{i,p-1} - E(V_{i,p-1}))$ to satisfy a generalized Lindeberg condition. So the limiting distribution of $\sigma_n^{-1}(V_1 - E(V_1), \cdots, V_{p-1} - E(V_{p-1}))$ is that of a vector of $p - 1$ independent standard normal variates. Now $t'(X) = \sigma_n^{-2}(1 + \delta_n^2)^{-1} \sum_{j=1}^{p-1} V_j^2$ and from Lemma 1, $t(X)$ and $t'(X)$ have the same asymptotic distribution.

In cases (i) and (ii), $\delta_n^2 \to 0$. If $\rho_n \to 0$ then $E(V_j) \to 0$ for $j = 1, \cdots, p - 1$, so (i) follows immediately. In case (ii), $\sigma_n^{-2} \sum_{j=1}^{p-1} [E(V_j)]^2 = \rho_n \to \rho$, so (ii) follows. If $\rho_n \to \infty$, for at least one $j = 1, \cdots, p - 1$, $\sigma_n^{-1}(1 + \delta_n^2)^{-\frac{1}{2}} |E(V_j)| \to \infty$, but $\sigma_n^{-2}(1 + \delta_n^2)^{-1} \operatorname{Var}(V_j) < 1$, so (iii) is true.

It follows from Theorem 2 and case (i) of Theorem 3, that if the null hypothesis of no treatment effects holds, then $P\{t(X) < t^{(k)}(X)\} \to 1 - \alpha$. This also follows from the definition of k, when it is observed that the distribution of X is invariant under the permutations of **G**. Also, under a sequence of alternate hypotheses corresponding to the sequence ρ_n, the power of the test tends to 1 if $\rho_n \to \infty$ and to $H(\lambda)$ if $\rho_n \to \rho$, where λ is defined in the statement of Theorem 2 and $1 - H(y)$ is the distribution function of a noncentral chi-squared variate with $p - 1$ degrees of freedom and noncentrality parameter ρ.

It is possible to consider the case when the y_{ij} are a particular realization of some random errors. Then if Condition (2) is satisfied with probability 1, it will follow again that $t^{(k)}(X) \to \lambda$ in probability and if condition (4) is satisfied with probability 1, the results of Theorem 3 will hold as stated.

In the usual analysis of variance model it is assumed that $y_{ij} = b_i + e_{ij}$, where the b_i are constants and the e_{ij} are independent normal variates with zero means and variances σ^2. Here, $\sigma_n^2 \to \sigma^2$ and Conditions (2) and (4) are satisfied, with probability 1. So a permutation test applied to this model has the asymptotic properties given above. The usual F-statistic for testing the null hypothesis is an increasing function of $t(X)$. For ρ_n defined by these y_{nij}, t_{nj} we have that $\rho_n \to \rho$ with probability 1 and that $t(X)$ tends in distribution to a noncentral chi-squared variate with $p - 1$ degrees of freedom and noncentrality parameter ρ. So in this case, the permutation test is asymptotically as powerful as the conventional test of the same size.

REMARK. The same procedure as this could be applied to the randomization model for the completely randomized design. Here it would be necessary to use a multivariate form of the Wald–Wolfowitz theorem to prove normality of a vector of the form of W^*, but the results would be very similar to those given above.

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