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A NOTE ON COMPARISON OF GENETIC EXPERIMENTS[1]

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Comparison of some normal experiments involving intraclass and interclass correlation is reduced to comparison of one-way random normal experiments. In consequence, the main result in Shaked and Tong is generalized and completed.

1. Introduction. Shaked and Tong (1992) consider a genetic experiment involving intraclass and interclass correlation. After reparametrization, the experiment can be presented as follows.

Let $k = (k_1, \dots, k_r)$ be a vector of allocation of $n = \sum k_i$ individuals in r nonempty groups. Denote by $\mathcal{E}(n, r; k_1, \dots, k_r)$ the experiment corresponding to the allocation and obtained by observing a normal random vector X with expectation $EX = \mu J_n$ and covariance matrix

$$\text{(1)} \qquad \operatorname{Cov} X = \sigma\Big[I_n + \rho \operatorname{diag}\big(J_{k_1}J'_{k_1}, \dots, J_{k_r}J'_{k_r}\big) + \lambda J_n J'_n\Big],$$

where J_k is the column of k 1's while $\mu \in \Re$, $\sigma > 0$, $\rho \geq 0$ and $\lambda \geq 0$ are parameters. We note that by setting $\lambda = 0$ in (1), one can reduce such an experiment to the one-way random normal experiment $\mathcal{N}(\mu J_n, \sigma[I_n + \rho \operatorname{diag}(J_{k_1}J'_{k_1}, \dots, J_{k_r} J'_{k_r})])$ [cf. Stępniak (1982)]. A subexperiment of $\mathcal{E}(n, r; k_1, \dots, k_r)$ may also be induced by a prior information that $\sigma = \sigma_0$, $\rho = \rho_0$ and $\lambda = \lambda_0$. Let us denote such an experiment by $\mathcal{E}(n, r; k_1, \dots, k_r/\sigma_0, \rho_0, \lambda_0)$. Consider also another experiment $\mathcal{E}(n^*, r^*; k_1^*, \dots, k_{r^*}^*/\sigma_0, \rho_0, \lambda_0)$ corresponding to an allocation $k^* = (k_1^*, \dots, k_{r^*}^*)$ of $n^* = \sum k_i^*$ individuals in r^* groups. Shaked and Tong (1992) have shown that if $n^* = n$, $r^* = r$ and k^* majorizes k (denoted by $k^* \succ k$), then the experiment $\mathcal{E}(n, r; k_1, \dots, k_r/\sigma_0, \rho_0, \lambda_0)$ is at least as informative as the experiment $\mathcal{E}(n^*, r^*; k_1^*, \dots, k_{r^*}^*/\sigma_0, \rho_0, \lambda_0)$ [denoted by $\mathcal{E}(n, r; k_1, \dots, k_r/\sigma_0, \rho_0, \lambda_0) \geq \mathcal{E}(n^*, r^*; k_1^*, \dots, k_{r^*}^*/\sigma_0, \rho_0, \lambda_0)$].

It appears that the condition $k^* \succ k$ is not necessary for the relation $\mathcal{E}(n, r; k_1, \dots, k_r/\sigma_0, \rho_0, \lambda_0) \geq \mathcal{E}(n^*, r^*; k_1^*, \dots, k_{r^*}^*/\sigma_0, \rho_0, \lambda_0)$ and, moreover, some more general results in the subject can be proved in a simpler way.

2. The results. The following is a generalization of Shaked and Tong [(1992), Theorem 3].

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THEOREM 1. *The experiment* $\mathcal{E}(n, r; k_1, \ldots, k_r/\sigma_0, \rho_0, \lambda_0)$ *is at least as informative as the experiment* $\mathcal{E}(n^*, r^*; k_1^*, \ldots, k_{r^*}^*/\sigma_0, \rho_0, \lambda_0)$ *if and only if*

$$\sum_{i=1}^{r} \frac{k_i}{1 + \rho_0 k_i} \geq \sum_{i=1}^{r^*} \frac{k_i^*}{1 + \rho_0 k_i^*}. \tag{2}$$

PROOF. The smoothing theorem for linear normal experiments with known covariance matrices [cf. Torgersen (1991), page 420; Stępniak (1987), Theorem 1; or Stępniak, Wang and Wu (1984), Remark following Theorem 1] yields that $\mathcal{N}(A\beta, V) \geq \mathcal{N}(B\beta, W)$ if and only if $\mathcal{N}(A\beta, V + \gamma AA') \geq \mathcal{N}(B\beta, W + \gamma BB')$ for a given but arbitrary $\gamma > 0$. By taking $A = B = J_n$ and $\beta = \mu$, the comparison of experiments $\mathcal{E}(n, r; k_1, \ldots, k_r/\sigma_0, \rho_0, \lambda_0)$ and $\mathcal{E}(n^*, r^*; k_1^*, \ldots, k_{r^*}^*/\sigma_0, \rho_0, \lambda_0)$ reduces to the same problem for the one-way random normal experiments $\mathcal{N}(\mu J_n, \sigma_0[I_n + \rho_0 \operatorname{diag}(J_{k_1}J'_{k_1}, \ldots, J_{k_r}J'_{k_r})])$ and $\mathcal{N}(\mu J_{n^*}, \sigma_0[I_{n^*} + \rho_0 \operatorname{diag}(J_{k_1^*}J'_{k_1^*}, \ldots, J_{k_r^*}J'_{k_r^*})])$. Now, for the case $n^* = n$ and $r^* = r$, Theorem 1 follows directly by Stępniak [(1982), Lemma 3.1]. On the other hand, observing the proof of the lemma one can see that the assumptions $n^* = n$ and $r^* = r$ are redundant. □

A consequence of the theorem is as follows.

COROLLARY 1 [Shaked and Tong (1992), Theorem 3]. *If* $k^* = (k_1^*, \ldots, k_r^*)$ *majorizes* $k = (k_1, \ldots, k_r)$, *then the experiment* $\mathcal{E}(n, r; k_1, \ldots, k_r/\sigma_0, \rho_0, \lambda_0)$ *is at least as informative as the experiment* $\mathcal{E}(n, r; k_1^*, \ldots, k_r^*/\sigma_0, \rho_0, \lambda_0)$ *for all* $\sigma_0 > 0$, $\rho_0 \geq 0$ *and* $\lambda_0 \geq 0$.

For proof we only need to use the fact that function $f_\rho(k_1, \ldots, k_r) = -\sum_{i=1}^{r} k_i/(1 + \rho k_i)$ is Schur-convex.

REMARK 1. When $r \geq 3$ the condition $k^* \succ k$ is not necessary for the relation $\mathcal{E}(n, r; k_1, \ldots, k_r/\sigma_0, \rho_0, \lambda_0) \geq \mathcal{E}(n, r; k_1^*, \ldots, k_r^*/\sigma_0, \rho_0, \lambda_0)$, as shown in Stępniak (1989) by example with $k = (2, 8, 12)$ and $k^* = (1, 10, 11)$.

Now we return to the initial experiment $\mathcal{E}(n, r; k_1, \ldots, k_r)$ without any prior information about σ and ρ (but with possible information about λ). The following theorem is a direct consequence of Stępniak [(1982), Corollary 4.1].

THEOREM 2. *The experiment* $\mathcal{E}(n, r; k_1, \ldots, k_r)$ *is at least as informative as the experiment* $\mathcal{E}(n^*, r^*; k_1^*, \ldots, k_{r^*}^*)$ *if and only if* $r^* = r$ *and* $k_1^*, \ldots, k_r^*$ *is a permutation of* $k_1, \ldots, k_r$.

Therefore only equivalent experiments are comparable in this case.

REMARK 2. Hauke and Markiewicz (1993) extend the result by Shaked and Tong (1992) to experiments with more complex covariance matrices.

REFERENCES

HAUKE, J. and MARKIEWICZ, A. (1993). Comparison of experiments via a group majorization ordering. Unpublished manuscript.

SHAKED, M. and TONG, Y. L. (1992). Comparison of experiments via dependence of normal variables with a common marginal distribution. *Ann. Statist.* **20** 614–618.

STĘPNIAK, C. (1982). Optimal allocation of observations in one-way random normal model. *Ann. Inst. Statist. Math. A* **34** 175–180.

STĘPNIAK, C. (1987). Reduction problems in comparison of linear models. *Metrika* **34** 211–216.

STĘPNIAK, C. (1989). Stochastic ordering and Schur-convex functions in comparison of linear experiments. *Metrika* **36** 291–298.

STĘPNIAK, C., WANG, S. G. and WU, C. F. J. (1984). Comparison of linear experiments with known covariances. *Ann. Statist.* **12** 358–365.

TORGERSEN, E. (1991). *Comparison of Statistical Experiments*. Cambridge Univ. Press.

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