The Annals of Statistics
1994, Vol. 22, No. 1, 357–368

EXTREMAL PROBABILISTIC PROBLEMS AND HOTELLING'S T^2 TEST UNDER A SYMMETRY CONDITION

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We consider the Hotelling T^2 statistic for an arbitrary d-dimensional sample. If the sampling is not too deterministic or inhomogeneous, then under the zero-means hypothesis the limiting distribution for T^2 is χ_d^2. It is shown that a test for the orthant symmetry condition introduced by Efron can be constructed which does not differ essentially from the one based on χ_d^2 and at the same time is applicable not only to large random homogeneous samples but to all multidimensional samples. The main results are not limit theorems, but exact inequalities corresponding to the solutions to certain extremal problems. The following auxiliary result itself may be of interest: $\chi_d - \sqrt{d-1}$ has a monotone likelihood ratio.

1. Introduction. For an arbitrary sample $X_1, \ldots, X_n$ in $\mathbb{R}^d$, let $\overline{X} = n^{-1}\sum_{i=1}^n X_i$ and $\mathcal{C} = n^{-1}\sum_{i=1}^n X_i X_i^T - \overline{X}\,\overline{X}^T$, where the superscript T denotes matrix transposition; it is assumed that $\mathbb{R}^d = \mathcal{M}_{d,1}$, where $\mathcal{M}_{d,n}$ stands for the set of all real $d \times n$ matrices. The Hotelling statistic is defined by the formula $T^2 = \cot^2 \theta$, as in Efron (1969) and Eaton and Efron (1970), where θ is the angle between the vector $\nu = (1, \ldots, 1) \in \mathcal{M}_{1,n}$ and the linear hull $L(X)$ of the rows of the matrix $X \in \mathcal{M}_{d,n}$ whose columns are $X_1, \ldots, X_n$. This definition of T^2 differs by the factor $n-1$ from the usual one, $(n-1)\overline{X}^T \mathcal{C}^{-1}\overline{X}$, suitable for Gaussian independent identically distributed (iid) random vectors X_i in $\mathbb{R}^d$.

Following Efron (1969) and Eaton and Efron (1970), consider

$$R^2 = \frac{T^2}{1+T^2} = \cos^2\theta = \frac{1}{n}\nu\Pi_X\nu^T,$$

where Π_X is the matrix of the orthoprojector from $\mathcal{M}_{1,n}$ onto $L(X)$. If $X_1, X_2, \ldots$ are, for example, iid and nondegenerate in $\mathbb{R}^d$, then both nT^2 and nR^2 tend to χ_d^2 in distribution.

In what follows, as in Efron (1969) and Eaton and Efron (1970), the sample is only assumed to satisfy "the orthant symmetry condition":

$$(X_1, \ldots, X_n) =_D (\varepsilon_1 X_1, \ldots, \varepsilon_n X_n),$$

where ε_i are iid with $\mathbb{P}(\varepsilon_i = \pm 1) = \frac{1}{2}$, independent of X, and $=_D$ means equality in distribution. Put $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. Then [see Eaton and Efron (1970)]

$$nR^2 =_D \varepsilon\Pi_X\varepsilon^T. \tag{1.1}$$

Received March 1991; revised February 1993.

AMS 1991 *subject classifications.* Primary 62H15; secondary 60E15, 62F04, 62F35, 62G10, 62G15.

Key words and phrases. Hotelling's T^2 test, probability inequalities, extremal probability problems, monotone likelihood ratio, stochastic ordering.

Efron (1969) and Eaton and Efron (1970) discovered that the distribution of nR^2 is conservative with respect to the limiting χ_d^2 distribution in the sense that

$$\mathbb{E}f(\sqrt{n}R) \le \mathbb{E}f(\chi_d), \tag{1.2}$$

for $f(u) = u^{2m}$, $m = 1, 2, \ldots$, where $R = \sqrt{R^2}$, $\chi_d = \sqrt{\chi_d^2}$ and χ_d^2 is a random variable having the χ_d^2 distribution.

In this paper, we first show (Theorem 2.1) that it is possible to extend (1.2) to the class of functions considered in Eaton (1970, 1974). This enables us to extract the bound

$$\mathbb{P}(\sqrt{n}R > u) < c\mathbb{P}(\chi_d > u), \qquad u > 0, \tag{1.3}$$

where $c = 2e^3/9$ is the best possible constant that can be obtained from (1.2). Technically, it is of course the transition from (1.2) to (1.3) that is the most difficult part of this paper.

2. Statement of results. Let C^2_{conv} denote the class of all even real functions on $\mathbb{R}$ having a finite convex second derivative f'' (see also Proposition A.1 in the Appendix).

THEOREM 2.1. *For any convex* $f \in C^2_{\mathrm{conv}}$, (1.2) *is true.*

THEOREM 2.2. *With* $c = 2e^3/9$, (1.3) *is true.*

(See also Proposition A.2 and the preamble to it in the Appendix.) These theorems are particular cases of the following two.

Let A stand for any nonnegative definite matrix in $\mathcal{M}_{n,n}$, and set $\xi = (\xi_1, \ldots, \xi_n)$, where the ξ_i's are any independent symmetrically distributed random variables with $\mathbb{E}\xi_i^2 = 1$.

THEOREM 2.3. *For any* $f \in C^2_{\mathrm{conv}}$,

$$\mathbb{E}f\left(\sqrt{\varepsilon A \varepsilon^T}\right) \le \mathbb{E}f\left(\sqrt{\xi A \xi^T}\right).$$

THEOREM 2.4. *If* Π *is an orthoprojector* $n \times n$ *matrix, then*

$$\mathbb{P}(\varepsilon \Pi \varepsilon^T > u) < c\mathbb{P}(\chi_r^2 > u), \qquad u > 0,$$

where $c = 2e^3/9$, $r = \operatorname{rank} \Pi$.

Setting $A = x^T x$, where $x = (x_1, \ldots, x_n) \in \mathcal{M}_{1,n}$, $xx^T = 1$, one obtains the following two corollaries.

COROLLARY 2.5. *For any $f \in C^2_{\mathrm{conv}}$,*

$$\mathbb{E}f(\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n) \le \mathbb{E}f(\xi_1 x_1 + \cdots + \xi_n x_n).$$

COROLLARY 2.6. *For any $u > 0$,*

$$\mathbb{P}(|\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n| > u) < 2c(1 - \Phi(u)),$$

where $c = 2e^3/9$, $\Phi(u) = \int_{-\infty}^{u} \varphi(t)\,dt$, and $\varphi(t) = (2\pi)^{-1/2} e^{-t^2/2}$.

In turn, Corollary 2.5 implies the following corollary.

COROLLARY 2.7. *If $f \in C^2_{\mathrm{conv}}$ and $xx^T = 1$, then, for $\xi \sim N(0,1)$,*

$$\mathbb{E}f(\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n) \le \mathbb{E}f(\xi).$$

Another approach to the last inequality, based upon the majorization technique, was given in Eaton (1970). Corollary 2.6 is an improvement of the conjecture in Eaton (1974): If $xx^T = 1$, then

$$\mathbb{P}(|\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n| > u) < c\frac{\varphi(u)}{u}, \qquad u > \sqrt{2}.$$

In view of an inequality of Hunt (1955) [see also Eaton (1974)], Theorems 2.3 and 2.4 and Corollaries 2.5–2.7 remain true for any independent zero-mean random variables $\eta_1, \dots, \eta_n$ with $|\eta_i| \le 1$, instead of $\varepsilon_1, \dots, \varepsilon_n$.

Using Theorem 2.2, it is possible to obtain rather precise information on the quantiles of the Hotelling statistic under the orthant symmetry condition.

For any real-valued random variable ζ, denote

$$x_\delta(\zeta) = \inf\{u \in \mathbb{R}: \mathbb{P}(\zeta \ge u) \le \delta\}, \qquad 0 < \delta < 1.$$

Set

$$c = 2e^3/9, \qquad x_\delta = x_\delta(\chi_d), \qquad \tilde{x}_\delta = x_\delta(R),$$

where R is the statistic defined in the Introduction.

THEOREM 2.8. *If $\delta \le 0.5$, then $x_\delta > (d-1)^{1/2}$ and*

$$\tilde{x}_\delta < x_{\delta/c}, \tag{2.1}$$

$$\tilde{x}_\delta < x_\delta + \frac{\ln c}{x_\delta - (d-1)/x_\delta}, \tag{2.2}$$

$$\tilde{x}_\delta < x_\delta + o(1), \tag{2.3}$$

$$\tilde{x}_\delta < x_\delta\left[1 + \frac{1}{\sqrt{d}}o(1)\right], \tag{2.4}$$

$$\tilde{x}_\delta < x_\delta(1 + o(1)), \tag{2.5}$$

where $o(1) \to 0$ *uniformly in* $d, n, X_1, \dots, X_n$ *when* $\delta \downarrow 0$.

(See also the numerical example and the preamble to it at the end of the Appendix.)

In the proof of Theorem 2.8, we use the following theorem.

THEOREM 2.9. *The family* (Z_d: $1 \le d \le \infty$), *where* $Z_d = \chi_d - (d-1)^{1/2}$, *for* $1 \le d < \infty$, *and* $Z_\infty \sim N(0, \frac{1}{2})$, *has a monotone likelihood ratio*; *more exactly*,

$$f_d(t)f_r(s) \ge f_r(t)f_d(s), \qquad 1 \le d \le r \le \infty, \ -\infty < s < t < \infty,$$

where f_d *is the density of* Z_d.

Specifically, we use the following corollary.

COROLLARY 2.10. *This family*, (Z_d: $1 \le d \le \infty$), *is stochastically decreasing*:

$$\mathbb{P}(Z_d > t) \ge \mathbb{P}(Z_r > t), \qquad 1 \le d \le r \le \infty, t \in \mathbb{R}.$$

In particular, for $r = \infty$, this gives

$$\mathbb{P}(\chi_d - \sqrt{d-1} \ge t) > 1 - \Phi(t\sqrt{2}), \qquad t \in \mathbb{R}, d \ge 1. \tag{2.6}$$

3. Proofs.

PROOF OF THEOREM 2.1. Set $A = \Pi_X$, $\xi_i \sim N(0,1)$ in Theorem 2.3 (proved later), use (1.1) and note that

$$[\forall\, i\, \xi_i \sim N(0,1)] \quad \Rightarrow \quad \xi\Pi\xi^{\mathsf{T}} =_D \chi_r^2, \tag{3.1}$$

where Π is any orthoprojector $n \times n$ matrix and $r = \operatorname{rank}\Pi$. □

PROOF OF THEOREM 2.2. See (1.1) and Theorem 2.4, proved later. □

PROOF OF THEOREM 2.3. Let $u_+ := \max(u, 0)$. For $f \in C^2_{\text{conv}}$, consider the Taylor expansion

$$f(u) = f(0) + f''(0)\frac{u^2}{2} + \frac{1}{6}\int_{t\ge 0} (|u| - t)_+^3 \, df'''(t), \tag{3.2}$$

where f''' is the right derivative of the convex function f'' (note that f''' is nondecreasing). We need two lemmas.

LEMMA 3.1. *If* $f \in C^2_{\text{conv}}$, $b \ge 0$ *and* $g_{b,f}(u) = f\big((u^2+b)^{1/2}\big)$, *then* $g_{b,f} \in C^2_{\text{conv}}$.

PROOF. In view of (3.2), any $f \in C^2_{\rm conv}$ is a mixture of functions of the following three types: (I) the constants; (II) $u \mapsto au^2, a \in \mathbb{R}$; (III) $u \mapsto a(|u|-t)^3_+$, $t \geq 0$, $a \geq 0$. Hence, we can assume that f belongs to one of these types. Only functions f of type (III), if any , need nontrivial treatment in this context, so we must prove only that g''' is nondecreasing, where $g(u) := (z^{1/2}-t)^3_+, z := u^2+b$, $b \geq 0$, $t \geq 0$. Calculations show the following:

(i) $z < t^2 \Rightarrow g^{(4)}(u) = 0$, where $g^{(4)}$ is the fourth derivative of g;
(ii) $z > t^2 \Rightarrow g^{(4)}(u) = 9bz^{-7/2}[(z-5t^2)b + 4t^2z] \geq 9bz^{-5/2}\min(z-t^2, 4t^2) \geq 0$;
(iii) $z = t^2 \neq 0 \Rightarrow g'''(u+0) - g'''(u-0) = 6t^{-3}|u|^3 \geq 0$;
(iv) $z = t = 0 \Rightarrow b = u = 0, g'''(u+0) - g'''(u-0) = 12 \geq 0$. □

LEMMA 3.2. *If $f \in C^2_{\rm conv}$, then $\mathbb{E}f(\varepsilon_1) \leq \mathbb{E}f(\xi_1)$, where ξ_1 is the same as in Theorem* 2.3.

PROOF. Remarks in Eaton (1974) and (3.2) imply $C^2_{\rm conv} \subseteq F$, where F is the class of all even differentiable functions f such that, for $w(x) = f(a + b\sqrt{x}) + f(a - b\sqrt{x})$, $w'(x)$ is nondecreasing in $x > 0$; $a, b \in \mathbb{R}$ (in fact, $C^2_{\rm conv} = F$; see Proposition A.1 in the Appendix). Hence, w is convex on $[0, \infty)$, and, for $a = 0$, $b = 1$, one has $2\mathbb{E}f(\varepsilon_1) = w(1) \leq \mathbb{E}w(\xi_1^2) = 2\mathbb{E}f(\xi_1)$, by Jensen's inequality. □

Now we can complete the proof of Theorem 2.3. Let $f \in C^2_{\rm conv}$. Note that $\varepsilon A \varepsilon^T = (\alpha\varepsilon_1 + \beta)^2 + b$, where the values of $b \geq 0$, α, β do not depend on ε_1. Observe that if a function g belongs to $C^2_{\rm conv}$, then so does the function $u \mapsto g(\alpha u + \beta) + g(\alpha u - \beta)$, for all $\alpha, \beta \in \mathbb{R}$. Hence, the function $h(u) := g_{b,f}(\alpha u + \beta) + g_{b,f}(\alpha u - \beta)$ belongs to $C^2_{\rm conv}$, by Lemma 3.1. By virtue of Lemma 3.2,

$$\mathbb{E}f\left(\left(\varepsilon A \varepsilon^T\right)^{1/2}\right) = \mathbb{E}h(\varepsilon_1) \leq \mathbb{E}h(\xi_1) = \mathbb{E}f\left(\left(\tilde{\varepsilon} A \tilde{\varepsilon}^{\rm T}\right)^{1/2}\right),$$

where $\tilde{\varepsilon} := (\xi_1, \varepsilon_2, \ldots, \varepsilon_n)$. By successively replacing the remaining ε_i's by ξ_i's, we complete the proof of Theorem 2.3. □

PROOF OF THEOREM 2.4. This proof is based on the series of Lemmas 3.3–3.6. We need the following notation:

$$C_r = \frac{1}{\int_0^\infty s^{r-1}e^{-s^2/2}I\{s > 0\}\,ds};$$

$$\gamma = \gamma(u) = \gamma_r(u) = \frac{\mathbb{E}(\chi_r - u)^3_+}{C_r}$$

$$= \int_u^\infty (s-u)^3 s^{r-1}e^{-s^2/2}I\{s > 0\}\,ds;$$

$I\{\mathcal{A}\} := 1$ if $\mathcal{A}$ is true and equals 0 otherwise;

$$q = q(u) = q_r(u) = -\frac{1}{6}\gamma'''(u) = \frac{\mathbb{P}(\chi_r \geq u)}{C_r}$$
$$= \int_u^\infty s^{r-1} e^{-s^2/2} I\{s > 0\}\, ds;$$
$$\gamma^{(j)} = \gamma^{(j)}(u) = \frac{d^j \gamma(u)}{du^j} \qquad (\gamma^{(0)} = \gamma);$$
$$Q_r(u) = \inf\left\{\frac{\mathbb{E}f(\chi_r)}{f(u)} : f \in C^2_{\mathrm{conv}},\ f \text{ is convex and strictly positive on } \mathbb{R}\right\};$$
$$W_r(u) = \inf\left\{\frac{\mathbb{E}(\chi_r - t)^3_+}{(u-t)^3} : t \in (0, u)\right\};$$
$$\Lambda_r(u) = \frac{Q_r(u)}{\mathbb{P}(\chi_r \geq u)};$$
$$\mu_r = \frac{\mathbb{E}\chi_r^3}{\mathbb{E}\chi_r^2}.$$

Now we can state the following auxiliary results.

LEMMA 3.3. *For all $r > 0, j = 0, 1, 2, 3, 4$,*

(3.3) $$(-1)^j \gamma^{(j)}(u) \geq 0, \qquad u \in \mathbb{R},$$

(3.4) $$(-1)^j \gamma^{(j)}(u) \sim 6u^{r-5+j} e^{-u^2/2}, \qquad u \to \infty,$$

(3.5) $$\gamma^{(4)}(u) = 6u^{r-1} e^{-u^2/2} I\{u \geq 0\} = -6q'(u), \qquad u \neq 0;$$

here and in what follows, $a \sim b$ means $a/b \to 1$.

PROOF. Equalities (3.5) are trivial. The first of them and L'Hôpital's rule, used successively four times, imply (3.4). Since $\gamma^{(4)} \geq 0$ and $\gamma^{(j)}(u) \to 0$ as $u \to \infty$, (3.3) is also true. □

LEMMA 3.4. *For $r \geq 1$ and $u > (r-1)^{1/2}$, $q(u)q''(u) < q'(u)^2$.*

PROOF. Set $h = q - q'^2/q''$. Then $h \to 0$ as $u \to \infty$,

$$h' = -(q'')^{-2} q'^3 \left[1 + \frac{r-1}{u^2}\right] > 0,$$

according to Lemma 3.3. Hence, $h < 0$. □

LEMMA 3.5. *For all* $u \geq 0$,

$$\mathbb{P}(\varepsilon\Pi\varepsilon^T \geq u^2) \leq Q_r(u) \tag{3.6}$$

$$= \min\left[1, \frac{r}{u^2}, W_r(u)\right] \tag{3.7}$$

$$= \begin{cases} 1, & \text{if } 0 \leq u \leq \sqrt{r}, \\ \dfrac{r}{u^2}, & \text{if } \sqrt{r} \leq u \leq \mu_r, \\ W_r(u), & \text{if } u \geq \mu_r. \end{cases} \tag{3.8}$$

Furthermore,

$$\mu_r \in \left(\sqrt{r+1}, \sqrt{r+2}\right). \tag{3.9}$$

PROOF. Inequality (3.6) follows from Theorem 2.3 (cf. the proof of Theorem 2.1) and Chebyshev's inequality. Using (3.2) and arguments similar to (2.8) and (2.9) of Eaton (1974), one can obtain (3.7). Observe that, for any nonnegative random variable ξ (in particular, for $\xi = \chi_r$), the function $p \mapsto \ln \mathbb{E}\xi^p$ is convex on $(0, \infty)$. This, together with the identity

$$\mathbb{E}\chi_r^j = (r + j - 2)\mathbb{E}\chi_r^{j-2}, \qquad j > 2 - r \tag{3.10}$$

(used with $j = 3$ and $j = 4$), implies (3.9).

It remains to prove (3.8). Define

$$\mu(t) = t - \frac{3\gamma(t)}{\gamma'(t)}.$$

Then $\mu'(t) = 2(\beta_3\beta_1 - \beta_2^2)/\beta_2^2 > 0$, again by the log-convexity, this time by that of $\beta_p := \mathbb{E}(\chi_r - t)_+^p$ in p. In view of (3.4), $\mu(t) \to \infty$ when $t \to \infty$; so

$$t \leftrightarrow u = \mu(t)$$

is a one-to-one increasing correspondence, under which, in particular, the numbers $t \geq 0$ correspond to $u \geq \mu(0) = \mu_r$, and vice versa. Set

$$F(t, u) = \frac{\mathbb{E}(\chi_r - t)_+^3}{(u - t)^3} = C_r \frac{\gamma(t)}{(u - t)^3}, \qquad t < u,$$

where C_r is the constant defined in the beginning of the proof of Theorem 2.3. Since $(\partial/\partial t)F(t, u) = C_r(u - t)^{-4}\gamma'(t)(u - \mu(t))$ and $\gamma' < 0$,

$$\begin{aligned} W_r(u) &= F(\mu^{-1}(u), u) \\ &= \min\{F(t, u) \colon t \in (-\infty, u)\}, \qquad u \geq \mu_r. \end{aligned} \tag{3.11}$$

In particular, $W_r(u) \leq F(0, u) = u^{-3}r\mu_r \leq \min(1, r/u^2)$, $\forall u \geq \mu_r$, in view of (3.9). This implies (3.8) for $u \geq \mu_r$. If now $u \leq \mu_r$, then $W_r(u) = F(0, u) \geq r/u^2$, which completes the proof of (3.8), as well as that of the lemma. □

The most difficult step in this paper is to prove the following lemma.

LEMMA 3.6. *For any* $u \geq 0$, $\Lambda_r(u) < 2e^3/9$.

PROOF. First take $u \geq \mu_r$. The crucial observation is that [in view of (3.8) and (3.11)]

$$Q_r(u) \leq F(\tau, u), \tag{3.12}$$

where

$$\tau := \tau(u) := u + \frac{3q}{q'}$$

and, as before, $q = q(u)$, $q' = q'(u)$. Consider the Taylor expansion

$$\gamma(\tau) = \gamma(u) - (u-\tau)\gamma'(u) + \frac{1}{2}(u-\tau)^2\gamma''(u) - \frac{1}{6}(u-\tau)^3\gamma'''(u) + \frac{1}{6}(u-\tau)^4\int_0^1 \theta^3\gamma^{(4)}(\tau + (u-\tau)\theta)\,d\theta.$$

If $0 < s < u$, then $s/u < \exp[(s-u)/u]$, $u^2 - s^2 < (u-s)2u$; hence, using the first equality in (3.5), one has, for $s = \tau + (u-\tau)\theta$, $0 < \theta < 1$,

$$\begin{aligned}\frac{\gamma^{(4)}(s)}{\gamma^{(4)}(u)} &\leq \left(\frac{s}{u}\right)^{r-1}\exp\left(\frac{u^2-s^2}{2}\right)\\ &< \exp\left\{-(s-u)\left[u - \frac{r-1}{u}\right]\right\}\\ &= \exp\left\{(1-\theta)a_u\right\},\end{aligned}$$

by virtue of the definition of τ and the identity $q''(u) = -[u-(r-1)/u]q'(u)$, where

$$a_u := \frac{3qq''}{q'^2}.$$

Thus,

$$\gamma(\tau) < \sum_{j=0}^{3}\frac{(u-\tau)^j}{j!}\left|\gamma^{(j)}(u)\right| + \frac{(u-\tau)^4}{3!}\gamma^{(4)}(u)J(a_u), \tag{3.13}$$

where

$$J(a) := \int_0^1 \theta^3 e^{(1-\theta)a}\,d\theta.$$

Set

$$f_j = (-1)^j \left[\gamma^{(j)} + \frac{6q^{4-j}}{(q')^{3-j}} \right], \qquad j = 0, 1, 2.$$

Lemma 3.4 and (3.9) imply (1) $q\,q'' < q'^2$ on $[\mu_r, \infty)$; furthermore (2) $f_j \to 0$ as $u \to \infty$, and (3) $\gamma''' = -6q$. Using the last three facts, one obtains the following, step by step:

$$0 < f_2', \quad f_2 < 0, \quad 0 < -f_2 < f_1', \quad f_1 < 0, \quad 0 < -f_1 < f_0', \quad f_0 < 0.$$

So, $f_j < 0$, that is,

$$|\gamma^{(j)}(u)| < 6q \cdot \left(\frac{u - \tau}{3}\right)^{3-j} \tag{3.14}$$

for $j = 0, 1, 2$; in view of the definitions of γ, τ and q, (3.14) holds for $j = 3, 4$ (when it turns into an equality).

By Lemma 3.4, $a_u[= 3q\,q''/q'^2] < 3$, for $u > \sqrt{r-1}$; also, $J(a) \uparrow$ since $e^{(1-\theta)a} \uparrow$ in a, for each $\theta \in (0,1)$; hence $J(a_u) < J(3) = 2(e^3 - 13)/27$ if $u \geq \mu_r[> \sqrt{r-1}$, see (3.9)]. Gathering now (3.12)–(3.14) and the definitions of $\Lambda_r(u)$, $F(t,u)$ and τ, we complete the proof of the lemma for the case $u \geq \mu_r$.

Consider next the case $r^{1/2} \leq u \leq \mu_r$. Then, by Lemma 3.5,

$$\Lambda_r(u) = \frac{r}{C_r u^2 q(u)},$$

where C_r was defined in the beginning of the proof of Theorem 2.4. Set

$$g(u) = -uq(u)\lambda'(u) = 2q(u) - u^r e^{-u^2/2},$$

where $\lambda(u) := \ln \Lambda_r(u)$. Calculations show that $g'(u) < 0$, for $u \in \left(0, (r+2)^{1/2}\right)$; so this is true for $u \in (0, \mu_r]$, according to (3.9). Hence, $uq(u)\lambda'(u) \uparrow$ on $[r^{1/2}, \mu_r]$. Therefore, Λ_r attains its maximum on $[r^{1/2}, \mu_r]$ at one of the endpoints [if Λ_r attains its maximum on $[r^{1/2}, \mu_r]$ at some $u_0 \in (r^{1/2}, \mu_r)$, then $\lambda'(u_0) = 0$ and $\lambda'(u) > u_0 q(u_0)\lambda'(u_0)/[uq(u)] = 0$, for $u \in (u_0, \mu_r)$; this is a contradiction].

By what has been proved, $\Lambda_r(\mu_r) < 2e^3/9$. Using the identity $q_{r+2}(0) = rq_r(0)$, one obtains, by induction,

$$[q(0) :=] q_r(0) < \frac{e^3}{9}\left(\frac{r}{e}\right)^{r/2}.$$

We can assume that $g(r^{1/2}) > 0$, that is,

$$q(r^{1/2}) > \frac{1}{2}\left(\frac{r}{e}\right)^{r/2}$$

[otherwise, for $u \in (r^{1/2}, \mu_r]$, one has $g(u) < 0$ (since $g'(u) < 0$) and so $\lambda'(u) > 0$, $\Lambda_r(u) \leq \Lambda_r(\mu_r) < 2e^3/9$]. These remarks show that the only remaining possibility is $q(0) < (2e^3/9)q(r^{1/2})$, or, equivalently (see the definitions again), $\Lambda_r(r^{1/2}) < 2e^3/9$, which completes the consideration of the case $r^{1/2} \leq u \leq \mu_r$.

Consider, finally, the case $0 \le u \le r^{1/2}$. Then, by Lemma 3.5,

$$\Lambda_r(u) = \frac{1}{\mathbb{P}(\chi_r \ge u)} \le \Lambda_r(\sqrt{r}) < \frac{2e^3}{9},$$

as just shown. □

Now, we can easily complete the proof of Theorem 2.4. It follows immediately from (3.6), Lemma 3.6 and the definition of Λ_r. □

PROOF OF COROLLARY 2.5–2.7. The proof is trivial.

PROOF OF THEOREM 2.8. Inequality (2.1) is implied by Theorem 2.2. Inequality (2.6), a particular case of Corollary 2.10 (proved later), shows that $x_\delta > (d-1)^{1/2}$ when $\delta \le 0.5$. Set $g(u) = cq(u+h) - q(u)$, where

$$c = \frac{2e^3}{9}, \qquad h = h(u) = \frac{\ln c}{u - (d-1)/u}, \qquad u > \sqrt{d-1}.$$

Then

$$\begin{aligned} g'(u) &= -q'(u) + cq'(u+h)(1+h') \\ &> -q'(u) + cq'(u+h), \\ g'(u)u^{1-d}e^{u^2/2} &> 1 - c\left(1 + \frac{h}{u}\right)^{d-1} \exp\left\{\frac{u^2}{2} - \frac{(u+h)^2}{2}\right\} \\ &> 1 - c\exp\left\{-\left(u - \frac{d-1}{u}\right)h\right\} = 0. \end{aligned}$$

Hence $g'(u) > 0$, and so, $g(u) < 0$, $\forall u > (d-1)^{1/2}$, which gives (2.2).

Again using (2.6), one has $x_\delta - (d-1)^{1/2} \to \infty$ as $\delta \downarrow 0$ and thus obtains (2.3) and (2.4). Finally, (2.5) is obvious. □

PROOF OF THEOREM 2.9. Take $1 \le d \le r < \infty$ and set $a = (d-1)^{1/2}$ and $b = (r-1)^{1/2}$. Then

$$\left(\ln \frac{f_r(u)}{f_d(u)}\right)' = \frac{(d-r)u^2}{(a+b)(u+a)(u+b)} \le 0$$

if $u > -a$; hence, $f_d(t)f_r(s) \ge f_r(t)f_d(s)$ when $-a \le s < t$; if $s < -a$, then $f_d(s) = 0$, so this case is trivial. It remains to note that $f_r(t) \to f_\infty(t)$ as $r \to \infty$, $\forall t \in \mathbb{R}$. □

PROOF OF COROLLARY 2.10. If a family has a monotone likelihood ratio, it is stochastically monotone [see Keilson and Sumita (1982)]. □

APPENDIX

Consider, as defined in Eaton (1970, 1974), the class F of all even differentiable functions f such that, for $w(x) = f(a + b\sqrt{x}) + f(a - b\sqrt{x})$, $w'(x)$ is nondecreasing in $x > 0$; $a, b \in \mathbb{R}$.

PROPOSITION A.1. $C^2_{\mathrm{conv}} = F$.

PROOF. Remarks in Eaton (1974) and (3.2) imply $C^2_{\mathrm{conv}} \subseteq F$, so we need to show only that $F \subseteq C^2_{\mathrm{conv}}$. Take any $f \in F$. Set $f_m(u) = \mathbb{E}f(u + \xi/m)$, $m = 1, 2, \ldots$, where ξ is a bounded symmetrically distributed random variable with a sufficiently smooth density. If, say, $\xi_1 \sim N(0, 1)$, then $\mathbb{E}\xi_1^4 = 3$, and the Taylor expansion gives

$$\mathbb{E}f_m(a + b\xi_1) - f_m(a) - f_m''(a)b^2/2 = [f_m^{(4)}(a) + o(1)]3b^4/24, \tag{A.1}$$

where $b \to 0$. Since f_m is a mixture of functions belonging to F, the argument in the beginning of the proof of Lemma 3.2 shows that $\mathbb{E}f_m(a + b\xi_1) \geq \mathbb{E}f_m(a + b\varepsilon_1)$, $a, b \in \mathbb{R}$. Comparing (A.1) with the analogous formula with ε_1 instead of ξ_1, we see that $f_m^{(4)} \geq 0$, and so $f_m'''(u)$ is nondecreasing in u. Hence, setting $\Delta^3 f(u) = f(u + 3) - 3f(u + 2) + 3f(u + 1) - f(u)$, one has, $\forall u \geq 0\, \exists \theta \in (0, 3)$, $\Delta^3 f(u) = f_m'''(u + \theta) \geq f_m'''(u) \geq f_m'''(0) = 0$. Note that $f_m(u) \to f(u)$, $\forall u \in \mathbb{R}$ as $m \to \infty$. Thus, $\Delta^3 f_m(u)$ is bounded in m for any $u \in \mathbb{R}$, and, therefore, so is $f_m'''(u)$. By Helly's theorem, $f_m''' \to h$ weakly on each compact set in $\mathbb{R}$ for a subsequence of values of $m \to \infty$ and a nondecreasing function h. This and (3.2) with f_m in place of f imply that

$$f_m(u) \to f(0) + \frac{B}{2}u^2 + \frac{1}{6}\int_{t \geq 0} (|u| - t)_+^3\, dh(t), \qquad u \in \mathbb{R},$$

for some $B \in \mathbb{R}$. Since, at the same time, $f_m(u) \to f(u)$, it is easy to observe that $dh(t) = df'''(t)$, and so f''' is nondecreasing, that is, $f \in C^2_{\mathrm{conv}}$. □

It can be seen that $Q_r(u)$, defined in the proof of Theorem 2.4, is the smallest upper bound for $\mathbb{P}(\sqrt{n}R > u)$ that can be extracted from Theorem 2.1. The following proposition therefore means that the constant $2e^3/9 = 4.463\ldots$ in (1.3) is the best possible that can be obtained from (1.2).

PROPOSITION A.2. $Q_r(u) \sim (2e^3/9)\mathbb{P}(\chi_r > u)$, $\qquad u \to \infty$.

PROOF. By Lemma 3.5, $Q_r(u) = W_r(u)$, for $u \geq \mu_r$. Now set $t = \mu^{-1}(u)$, so that $u = \mu(t)$, where $\mu(t)$ was defined in the proof of Lemma 3.5. Using (3.11), the definition of $F(t, u)$ and (3.4), one can conclude that

$$\begin{aligned} \frac{Q_r(u)}{\mathbb{P}(\chi_r > u)} &= \frac{F(t, u)}{C_r q(u)} \sim \frac{6}{27}\left(\frac{t}{u}\right)^{r-2} \exp\left\{\frac{u^2 - t^2}{2}\right\} \\ &\sim \frac{2}{9}\exp[(u - t)u] = \frac{2}{9}\exp\left\{-\frac{3\gamma(t)}{\gamma'(t)}\mu(t)\right\} \to \frac{2e^3}{9}. \end{aligned}$$

□

TABLE 1

d	1	2	5	10	20	50	∞
x_δ	1.96	2.45	3.33	4.28	5.61	8.22	$\sqrt{d} + 1.16$
$x_{\delta/c}$	2.54	3.00	3.85	4.78	6.10	8.69	$\sqrt{d} + 1.61$

Inequality (2.4) means that the larger the dimension is, the better (2.1) works (the same tendency takes place when δ decreases); Table 1 compares the values of x_δ and $x_{\delta/c}$ computed for $\delta = 0.05$.

Acknowledgments. The author is pleased to thank an Associate Editor and the referees for many valuable suggestions, having led to improvement of the presentation. He is grateful to Professor Kumar Joag-Dev and a referee for the reference to Keilson and Sumita (1982).

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