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EXACT COMPUTATION OF THE ASYMPTOTIC EFFICIENCY OF MAXIMUM LIKELIHOOD ESTIMATORS OF A DISCONTINUOUS SIGNAL IN A GAUSSIAN WHITE NOISE

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In this paper, the problem of computing the exact value of the asymptotic efficiency of maximum likelihood estimators of a discontinuous signal in a Gaussian white noise is considered. A method based on constructing difference equations for the appropriate moments is presented and used to show that the exact variance of the Pitman estimator is $16\zeta(3)$, where ζ is the Riemann zeta function.

1. Introduction. Consider the problem of estimating a one-dimensional parameter θ based on observations of the process $X(t)$ satisfying the stochastic differential equation

$$dX(t) = \frac{1}{\varepsilon} S(t - \theta)\, dt + dW(t), \qquad t \in [0, 1],$$

where S is a function possessing at least one discontinuity of the first kind in the interval of observations, ε is a small parameter and $W(t)$ is a standard Wiener process. This estimation problem may also be referred to as estimation of a change-point as it is the continuous analog of the classical change-point problems in the regression context.

The problem considered by many comes from considering the asymptotic situation, in which S can be taken to be constant except for one discontinuity, and instead of using [0, 1] we use $(-\infty, \infty)$. In this case the problem is location invariant, and Pitman (1938) showed that the best invariant procedure for such a problem is the formal Bayes procedure with a uniform "prior" on the entire real line.

As this is not a regular estimation problem, the maximum likelihood estimator is, as is usually the case here, not asymptotically efficient. It is natural, therefore, to ask to compare the variances of the maximum likelihood estimator and that of the best invariant estimator.

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The formal setup is that specified in Ibragimov and Has'minskii (1981). We wish to obtain the ratio $\kappa = E\xi_2^2/E\xi_1^2$, where ξ_1 and ξ_2 are defined as follows:

$$\xi_1 = \arg\max_{t \in R_1} Z_0(t),$$

$$\xi_2 = \int_{-\infty}^{+\infty} tZ_0(t)\,dt \left(\int_{-\infty}^{+\infty} Z_0(t)\,dt\right)^{-1},$$

$$Z_0(t) = \exp\left(W(t) - \tfrac{1}{2}|t|\right),$$

where $W(t)$ is a two-sided Brownian motion defined as

$$W(t) = \begin{cases} W_1(t), & \text{if } t \geq 0, \\ W_2(-t), & \text{if } t < 0, \end{cases}$$

and $W_i(t)$ are independent standard Wiener processes defined for $t \geq 0$ and such that $W_i(0) = 0$.

The exact evaluation of $E\xi_1^2$ is not too difficult. Ibragimov and Has'minskii (1981) showed that $E\xi_1^2 = 26$. An attempt to evaluate $E\xi_2^2$ was also made in their monograph. Unfortunately, as they stated in the book, it seems that it is difficult to evaluate $E\xi_2^2$ explicitly. Instead they obtained a method for an approximate calculation of $E\xi_2^2$ and obtained through statistical simulation a value of 19.5 ± 0.5. Golubev (1979) proved that $E\xi_2^2$ is the second derivative of an improper integral of a composite function of modified Hankel and Bessel functions with respect to a parameter μ evaluated at 0. Again, the exact evaluation of the result has only been obtained by computer assistance. In this paper, we shall present a method based on constructing difference equations for the appropriate moments to compute the exact value of $E\xi_2^2$.

For an early discussion of the above problem, see Rubin (1961). A related problem of the Pitman estimator for the absolute error-loss function was considered by Paranjape and Rubin (1975). The exact distribution of the estimator was obtained in that paper. It may be worth mentioning that the problem of determining the distribution of the Pitman estimator for the quadratic loss function remains unsolved.

2. Main Result.

In this section, we first state the main result and prove it through a series of lemmas.

THEOREM 1.

$$E\xi_2^2 = 16\zeta(3),$$

where ζ is Riemann's zeta function defined as $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$.

Let X_λ, Y_λ and Z_λ be defined as

$$X_\lambda = \int_0^\infty \exp(W_1(t) - \lambda t)\, dt,$$

$$Y_\lambda = \int_0^\infty t \exp(W_1(t) - \lambda t)\, dt,$$

$$Z_\lambda = \int_0^\infty t^2 \exp(W_1(t) - \lambda t)\, dt,$$

where $\lambda > 0$.

LEMMA 1. *The reciprocal of the random variable X_λ has a gamma distribution with density defined by*

$$f(x) = \frac{2^{2\lambda} x^{2\lambda - 1} \exp(-2x)}{\Gamma(2\lambda)},$$

where $\lambda \geq \frac{1}{2}$.

PROOF. Consider the random process

$$X(t) = \exp(-W(t) - \lambda t) \int_{-\infty}^t \exp(W(s) + \lambda s)\, ds.$$

Observe that for any t the distribution of $X(t)$ is the same and coincides with the distribution of X_λ [see Ibragimov and Has'minskii (1981)]. Using Itô's formula, we obtain the following stochastic differential for the process $X(t)$:

$$dX(t) = -X(t)\, dW(t) + \left(1 - \left(\lambda - \tfrac{1}{2}\right) X(t)\right) dt.$$

It follows that the stationary density $g(x)$ of the process $X(t)$ satisfies the differential equation

$$\frac{1}{2} \frac{d^2}{dx^2}(x^2 g) - \frac{d}{dx}(1 - (\lambda - 1/2)x) g = 0,$$

subject to the constraint

$$\int_0^\infty g(x)\, dx = 1.$$

Solving the above equation, we find

$$g(x) = \frac{2^{2\lambda} x^{-(2\lambda + 1)} \exp(-2/x)}{\Gamma(2\lambda)},$$

which completes the proof. □

LEMMA 2. *Let p be any nonnegative integer. Then $EY_\lambda / X_\lambda^p < \infty$, where* $\lambda > \frac{1}{2}$.

PROOF. For any given $\lambda > \frac{1}{2}$, choose a τ such that $\frac{1}{2} < \tau < \lambda$, then

$$\frac{X_\tau - X_\lambda}{\lambda - \tau} > Y_\lambda.$$

By Lemma 1, X_λ has finite moments of all orders $k < 2\lambda$ and X_τ has finite moments of all orders $k < 2\tau$. Let $k = 1 + \delta$, where $0 < \delta < 2\tau - 1$. Then Minkowski's inequality implies

$$E(X_\tau - X_\lambda)^{1+\delta} \le 2^{1+\delta} E X_\tau^{1+\delta}.$$

Therefore Y_λ has finite moments of all orders $1 + \delta$. An application of Hölder's inequality finishes the proof. □

LEMMA 3. *Let* $A_{\alpha,n}(\lambda) = EX_\lambda^\alpha (Y_\lambda / X_\lambda)^n$, *$\alpha$ is a nonpositive integer*, $\lambda > \frac{1}{2}$. *Then* $A_{\alpha,1}(\lambda)$ *satisfies the following difference equation*:

$$\tfrac{1}{2}\alpha(\alpha - 2\lambda) A_{\alpha,1}(\lambda) + (\alpha - 1) A_{\alpha-1,1}(\lambda) + A_{\alpha,0}(\lambda) = 0; \tag{1}$$

and the unique solution of the equation for $\alpha = -q$ *is given by*

$$A_{-q,1}(\lambda) = \frac{2\Gamma(q + 2\lambda)}{2^q \Gamma(2\lambda)} \int_0^1 \frac{t^{2\lambda-1}}{1 - t} \frac{(1 - t^q)}{q}\, dt, \tag{2}$$

where Γ *is the gamma function and* q *is a nonnegative integer.*

PROOF. Note that, by Lemma 2, $A_{-q,1}(\lambda) = EY_\lambda / X_\lambda^{(q+1)} < \infty$ for all q. For any arbitrary small $\varepsilon > 0$, let

$$S_1 = \int_0^\varepsilon \exp(W_1(t) - \lambda t)\, dt,$$

$$S_2 = \int_0^\varepsilon t \exp(W_1(t) - \lambda t)\, dt,$$

$$T = \exp(W_1(\varepsilon) - \lambda\varepsilon).$$

Then we have

$$X_\lambda = S_1 + T X'_\lambda,$$
$$Y_\lambda = S_2 + T(Y'_\lambda + \varepsilon X'_\lambda).$$

where X'_λ and Y'_λ are independent of T, S_1 and S_2; (X'_λ, Y'_λ) and (X_λ, Y_λ) have the same joint distribution. Let $g(s) = 1/[s + TX'_\lambda]^{(q+1)}$, Taylor expansion of $g(s)$ at $s = 0$ yields

$$\begin{aligned} \frac{Y_\lambda}{X_\lambda^{q+1}} &= [S_2 + T(Y'_\lambda + \varepsilon X'_\lambda)] g(S_1) \\ &= \frac{Y'_\lambda}{T^q (X'_\lambda)^{q+1}} + \frac{\varepsilon}{T^q (X'_\lambda)^q} - \frac{(q+1) S_1 Y'_\lambda}{T^{q+1} (X'_\lambda)^{q+2}} + R, \end{aligned} \tag{3}$$

where

$$R = \frac{S_2}{(TX'_\lambda)^{q+1}} - \frac{(q+1)S_1S_2}{(TX'_\lambda)^{q+2}} - \frac{(q+1)\varepsilon S_1}{(TX'_\lambda)^{q+1}} + \frac{(q+1)(q+2)S_1^2S_2}{2(\theta+TX'_\lambda)^{q+3}} + \frac{(q+1)(q+2)TY'_\lambda S_1^2}{2(\theta+TX'_\lambda)^{q+3}} + \frac{(q+1)(q+2)\varepsilon TX'_\lambda S_1^2}{2(\theta+TX'_\lambda)^{q+3}}$$

and $0 < \theta < S_1$. Taking expectation on both sides of equation (3) gives

$$A_{-q,1}(\lambda) = A_{-q.1}(\lambda)ET^{-q} + \varepsilon ET^{-q}A_{-q,0}(\lambda) - (q+1)ES_1T^{-(q+1)}A_{-(q+1),1}(\lambda) + ER.$$

Therefore,

$$(ET^{-q} - 1)A_{-q,1}(\lambda) + \varepsilon ET^{-q}A_{-q,0}(\lambda) - (q+1)ES_1T^{-(q+1)}A_{-(q+1),1}(\lambda) + ER = 0.$$

Note that $ET^{-q} = \exp(q(q+2\lambda)\varepsilon/2)$ and $\lim_{\varepsilon\to 0} ES_1T^{-(q+1)}/\varepsilon = 1$; let $\varepsilon \to 0$. We have, from the above equation,

$$\frac{1}{2}q(q+2\lambda)A_{-q,1}(\lambda) + A_{-q,0}(\lambda) - (q+1)A_{-(q+1),1}(\lambda) + \lim_{\varepsilon\to 0} E\frac{R}{\varepsilon} = 0.$$

It remains to show that $\lim_{\varepsilon\to 0} ER/\varepsilon = 0$, but this is true by noting that $S_2 \le \varepsilon S_1$ and $\lim_{\varepsilon\to 0} E(T^{-1}S_1/\epsilon)^m = 1$, where m is any positive integer, and by an application of the Cauchy–Schwarz inequality. To solve (1), observe that, for $\alpha = -q$, (1) becomes

$$(4) \qquad (q+1)A_{-(q+1),1}(\lambda) - \frac{q(q+2\lambda)}{2}A_{-q,1}(\lambda) = A_{-q,0}(\lambda).$$

The factor $(q+2\lambda)/2$ suggests that $A_{-q,1}(\lambda)$ should be considered with the factor $\Gamma(q+2\lambda)/2^q$ removed. Since the density in Lemma 1 has the normalizing factor $\Gamma(2\lambda)$, we can put this in as well. Call the resulting function H_q, that is,

$$H_q = \frac{2^q\Gamma(2\lambda)}{\Gamma(q+2\lambda)}A_{-q,1}(\lambda).$$

Then (4) becomes

$$(q+1)H_{q+1} - qH_q = \frac{2}{q+2\lambda},$$

so this would say that qH_q should be $2\psi(q+2\lambda) + C$, where $\psi(z) = d\log\Gamma(z)/dz$ and C is a constant. Since the equation holds also for $q = 0$, $C = -2\psi(2\lambda)$. The result is then given by the standard integral form of the difference of the ψ function at two values, which yields formula (2). □

LEMMA 4. *Let $A_{\alpha,n}(\lambda)$ be defined as in Lemma 3. Then $A_{\alpha,1}(\frac{1}{2}) < \infty$ and it satisfies the following difference equation*:

$$\tfrac{1}{2}\alpha(\alpha-1)A_{\alpha,1}(\tfrac{1}{2}) + (\alpha-1)A_{\alpha-1,1}(\tfrac{1}{2}) + A_{\alpha,0}(\tfrac{1}{2}) = 0;$$

and the unique solution of the equation for $\alpha = -q$ is given by

$$A_{-q,1}\left(\frac{1}{2}\right) = \frac{2\Gamma(q+1)}{2^q}\int_0^1 t^{q-1}[-\ln(1-t)]\,dt. \tag{5}$$

PROOF. It is enough to prove that $A_{-q,1}(\frac{1}{2}) < \infty$ since we can repeat the same argument as given in Lemma 3 to derive the difference equation and use integration by parts to obtain (5). By Fatou's lemma, we find

$$\begin{aligned} A_{-q,1}\left(\frac{1}{2}\right) &= E \liminf_{\lambda\to 1/2^+} X_\lambda^{-q}\left(\frac{Y_\lambda}{X_\lambda}\right) \le \liminf_{\lambda\to 1/2^+} A_{-q,1}(\lambda) \\ &= \frac{\Gamma(q+1)}{2^{q-1}}\int_0^1 \frac{1-t^q}{(1-t)q}\,dt, \end{aligned}$$

which shows that $A_{-q,1}(\frac{1}{2})$ is finite. □

REMARK 1. The fact that $A_{-q,1}(\frac{1}{2}) < \infty$ can be obtained directly by a completely different method using the reflection principle for Brownian motion (defining appropriate stopping time) and Fubini's theorem. Instead we present the above proof in Lemma 3 and Lemma 4 because the truncation argument used in the proof is more intuitive and elementary.

REMARK 2. Formula (5) has only been obtained for positive q. However, the $A_{-q,1}$ are moments of a measure on $(0,\infty)$, as are the quantities given by (5). Since the moments grow slowly enough, the measure is determined by the moments of positive integer order, and thus (5) holds for all $q > -1$. In particular, it holds for $q = 0$, which is needed to continue the process.

LEMMA 5. *Let $B_{\alpha,n}(\lambda) = EX_\lambda^\alpha(Z_\lambda/X_\lambda)^n$. Then $B_{\alpha,1}(\frac{1}{2})$ satisfies the following difference equation*:

$$\tfrac{1}{2}\alpha(\alpha-1)B_{\alpha,1}(\tfrac{1}{2}) + (\alpha-1)B_{\alpha-1,1}(\tfrac{1}{2}) + 2A_{\alpha,1}(\tfrac{1}{2}) = 0;$$

and the unique solution of the equation for $\alpha = -q$ is given by

$$B_{-q,1}\left(\frac{1}{2}\right) = \frac{8\Gamma(q+1)}{2^q}\int_0^1 t^{q-1}\int_0^t \frac{1}{1-s}\int_s^1 \frac{-\ln(1-u)}{u^2}\,du\,ds\,dt$$

The proof of Lemma 5 is similar to the proofs of Lemmas 3 and 4. We omit the proof.

For notational simplicity, we omit the subscripts on X, Y and Z throughout the rest of the paper, with the understanding that $\lambda = \frac{1}{2}$. Observe that the random variable ξ_2 can be written as

$$\xi_2 = \frac{Y^{(2)} - Y^{(1)}}{X^{(1)} + X^{(2)}}$$

where

$$X^{(i)} = \int_0^\infty \exp(W_i(t) - t/2)\,dt$$

and

$$Y^{(i)} = \int_0^\infty t\exp(W_i(t) - t/2)\,dt,\ i = 1, 2.$$

LEMMA 6.

$$E\left(\frac{Y^{(2)} - Y^{(1)}}{X^{(1)} + X^{(2)}}\right)^2 = E\frac{Z}{X^{(1)} + X^{(2)}}.$$

PROOF. A direct application of Theorem 3 of Golubev (1979). □

Now we shall prove Theorem 1.

PROOF OF THEOREM 1. Let $\psi(X) = E(Z/X \mid X)$. Then, by Lemma 5, $EX^{-q}\psi(X) = B_{-q,1}(\frac{1}{2})$. Observe that the growth of $B_{-q,1}(\frac{1}{2})$ as a function of q is sufficiently slow, approximately in the order of $2^{-q}q!$, so that the above moment problem has a unique solution, which is

$$\psi(x) = 8\int_0^1\int_0^t \frac{1}{1-s}\int_s^1\left[\frac{-\ln(1-u)}{u^2}\right]du\,ds\,t^{-2}\exp\left(-\frac{2(1-t)}{xt}\right)dt.$$

Observe that the joint density of $(X^{(1)}, X^{(2)})$ is

$$f(x_1, x_2) = 4(x_1x_2)^{-2}\exp\left(-2\left(\frac{1}{x_1} + \frac{1}{x_2}\right)\right).$$

Using Lemma 6, we find

$$E\xi_2^2 = E\left[\frac{X^{(1)}}{X^{(1)} + X^{(2)}}E\left(\frac{Z}{X^{(1)}}\middle|X^{(1)}\right)\right] = \int_0^\infty\int_0^\infty \frac{x_1}{x_1 + x_2}f(x_1, x_2)\psi(x_1)\,dx_1\,dx_2.$$

Writing $x_1 = 1/(twz)$ and $x_2 = 1/[(1-w)z]$, we obtain

$$= 32\int_0^1\int_0^t\int_s^1\left[\frac{-\ln(1-u)}{u^2}\right]\frac{1}{1-s}\,du\,ds\,t^2$$

$$\times\int_0^\infty\int_0^1\left[\frac{1-w}{1+(t-1)w}\right]tz\exp(-2z)\,dw\,dz\,dt$$

$$= 8\int_0^1\frac{1-t+t\ln(t)}{t(1-t)^2}\int_0^t\int_s^1\frac{-\ln(1-u)}{u^2(1-s)}\,du\,ds\,dt.$$

Note that $d[1 - t + \ln(t)]/(1-t) = [1 - t + t\ln(t)]/[t(1-t)^2]\,dt$, integrating by parts gives

$$= -8\int_0^1\frac{1-t+\ln(t)}{(1-t)^2}\int_t^1\frac{-\ln(1-u)}{u^2}\,du\,dt.$$

Writing $v = 1 - t$, we obtain

$$= -8\int_0^1 \frac{v + \ln(1-v)}{v^2} \int_0^v \frac{-\ln(u)}{(1-u)^2}\, du\, dv.$$

Observe again that $d[(1-v)\ln(1-v)/v] = -[v + \ln(1-v)]/v^2\, dv$, integrating by parts yields

$$= 8\int_0^1 \frac{\ln(v)\ln(1-v)}{v(1-v)}\, dv$$

$$= 16\int_0^1 \frac{\ln(1-v)\ln(v)}{v}\, dv$$

$$= 16\zeta(3). \qquad \square$$

Thus it follows from Theorem 1 that the asymptotic efficiency of maximum likelihood estimators of discontinuous signal in a Gaussian white noise is

$$\kappa = \frac{E\xi_2^2}{E\xi_1^2} = \frac{8}{13}\zeta(3).$$

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