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WEAK CONSISTENCY OF EXTREME VALUE ESTIMATORS IN $C[0, 1]$

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We prove that when the distribution of a stochastic process in $C[0, 1]$ is in the domain of attraction of a max-stable process, then natural estimators for the extreme-value index (which is now a continuous function) and for the mean measure of the limiting Poisson process are consistent in the appropriate topologies. The ultimate goal, estimating probabilities of small (failure) sets, will be considered later.

1. Introduction. Multivariate extreme value theory and its statistical implications are by now well understood [Resnick (1987), Smith (1990) and de Haan and de Ronde (1998) just to mention a few references]. Giné, Hahn and Vatan (1990) have characterized max-stable stochastic processes in $C[0, 1]$. This seems to be the most sensible extension of extreme-value theory to infinite-dimensional spaces. De Haan and Lin (2001) have characterized the domain of attraction of max-stable processes in $C[0, 1]$. The aim of the present paper is to initiate making these results useful for statistical application by proving consistency of natural estimators for the main "parameters" of the max-stable process based on observations from a process which is in its domain of attraction. The result is stated in Theorem 2.1.

Infinite-dimensional extreme-value theory seems to be useful in a problem of coast protection [cf. de Haan and Lin (2001)].

Next we explain the framework of our results. Consider a sequence of i.i.d. random processes $\xi_1, \xi_2, \dots$ in $C[0, 1]$. Suppose the sequence of processes

$$\left\{\frac{\max_{1\le i\le n} \xi_i(t) - b_t(n)}{a_t(n)}\right\}_{t\in[0,1]} \tag{1.1}$$

converges in $C[0, 1]$ to a stochastic process η with nondegenerate marginals. Here $a_t(n) > 0$ and $b_t(n)$ are nonrandom normalizing constants chosen in such a way that the marginal limit distributions are standard extreme-value distributions of the form $\exp\{-(1 + \gamma x)^{-1/\gamma}\}$ for some $\gamma \in \mathbb{R}$, $1 + \gamma x > 0$ and defined by continuity for $\gamma = 0$.

We note the following structural property.

The probability distribution of the limit process η is determined by the continuous function γ (the extreme-value index) plus a measure ν on the

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space $C[0,1]$ which is homogeneous; that is, for any Borel set A and a positive a,

$$\nu(aA) = a^{-1}\nu(A). \tag{1.2}$$

For each function $f \in C[0,1]$, $f > 0$, we have

$$\begin{aligned} &-\log P\{(1+\gamma(t)\eta(t))^{1/\gamma(t)} < f(t) \text{ for all } t\} \\ &\quad = \nu\{g \in C^+[0,1]; g(t) \geq f(t) \text{ for some } t\} \end{aligned} \tag{1.3}$$

[Giné, Hahn and Vatan (1990), Proposition 3.2, part (iv), where the measure ν is given in its "polar" representation].

We need the following result from de Haan and Lin (2001), Theorems 2.4 and 2.10.

PROPOSITION 1.1. *The sequence of stochastic processes* (1.1) *converges in $C[0,1]$ to a stochastic process η with nondegenerate marginals if and only if the following hold*:

(i)

$$\frac{1}{n}\bigvee_{i=1}^{n} \zeta_i \to \bar{\eta} \tag{1.4}$$

in $C[0,1]$ with $\zeta_i(t) := \frac{1}{1-F_t(\xi_i(t))}$ and $\bar{\eta}(t) := (1+\gamma(t)\eta(t))^{1/\gamma(t)}$ for $t \in [0.1]$.

An equivalent statement is: there is a measure ν on $C^+[0,1] := \{f \in C[0,1]; f \geq 0, f \not\equiv 0\}$ such that for each $c > 0$ the restriction of the measure ν_s defined by

$$\nu_s(\cdot) := sP\left\{\frac{1}{s}\zeta_1(\cdot) \in \cdot\right\}$$

to $S_c := \{f \in C^+[0,1]; \|f\|_\infty \geq c\}$ converges weakly $(s \to \infty)$ to the restriction of ν to S_c. The relation between η and ν is as in (1.3).

(ii)

$$\lim_{s\to\infty} \frac{U_t(sx) - U_t(s)}{a_t(s)} = \frac{x^{\gamma(t)} - 1}{\gamma(t)} \tag{1.5}$$

and

$$\lim_{s\to\infty} \frac{a_t(sx)}{a_t(s)} = x^{\gamma(t)} \tag{1.6}$$

locally uniformly for $x \in (0,\infty)$ and uniformly for $t \in [0,1]$, where $a_t(s) := a_t([s])$ [from (1.1)].

Now we proceed as in the finite-dimensional case: we fit the appropriate limit distribution to the tail part of the distribution of the original process [cf. (1.9)]. Next this limit distribution enables us to extend the original probability distribution beyond the range of the available data as follows:

A failure region F defined, for example, by $F = \{\xi(t) \geq f(t) \text{ for some } 0 \leq t \leq 1\}$ with f a continuous function which is extreme with respect to the sample in the sense that $\tilde{\xi}_i(t) < f(t)$ for $i = 1, 2, \dots, n$ and $0 \leq t \leq 1$, where $\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n$ are the observed realizations. Since this feature is essential to the problem, we want to keep it in our asymptotic approximation. This implies that f must depend on n, the sample size, that is, $f = f_n$, and in fact we assume that

$$f(t) = U_t\left(\frac{nc_n}{k_n} h(t)\right) \tag{1.7}$$

with h a fixed positive continuous function, c_n a sequence of positive constants and

$$U_t(x) := \left(\frac{1}{1 - F_t}\right)^{\leftarrow}(x) \tag{1.8}$$

with $F_t(x) := P\{\xi(t) \leq x\}$ [cf. de Haan and Sinha (1999), relation (1.5)].

We shall now attempt to explain the intuitive reasoning that leads to a way to estimate $P(F)$: for $n \to \infty, k = k(n) \to \infty, k(n)/n \to 0$,

$$\begin{aligned}
&P\{\xi(t) \geq f_n(t) \text{ for some } t \in [0, 1]\} \\
&\quad = P\left\{\frac{1}{1 - F_t(\xi(t))} \geq \frac{1}{1 - F_t(f_n(t))} \text{ for some } t \in [0, 1]\right\} \\
&\quad \overset{(1.7)}{=} P\left\{\frac{1}{1 - F_t(\xi(t))} \geq \frac{nc_n}{k} h(t) \text{ for some } t \in [0, 1]\right\} \\
&\quad \approx \frac{k}{n}\left\{-\log P^{n/k}\left\{\frac{1}{(n/k)\{1 - F_t(\xi(t))\}} < c_n h(t) \text{ for } 0 \leq t \leq 1\right\}\right\} \qquad (1.9) \\
&\quad \overset{(*)}{\approx} \frac{k}{n}\{-\log P\{(1 + \gamma(t)\eta(t))^{1/\gamma(t)} < c_n h(t) \text{ for } 0 \leq t \leq 1\}\} \\
&\quad \overset{(1.3)}{=} \frac{k}{n}\nu\{g \in C^+[0, 1]; g(t) \geq c_n h(t) \text{ for some } 0 \leq t \leq 1\} \\
&\quad \overset{(1.2)}{=} \frac{k}{nc_n}\nu\{g \in C^+[0, 1]; g(t) \geq h(t) \text{ for some } 0 \leq t \leq 1\}.
\end{aligned}$$

The approximate equation $(*)$ follows from the convergence of (1.1) (see Proposition 1.1).

Since $nP\{\xi(t) \geq f_n(t)$ for some $t \in [0,1]\}$ is the mean number of realizations that fall in the failure set and since we want this to go to zero (this is the "essential feature" mentioned above), we need to assume $c_n \to \infty$ and $k(n) = o(c_n)$, $n \to \infty$.

Now to turn this into a useful statistics tool we need to estimate the measure ν. Moreover, we need to estimate the unknown function h which can be evaluated approximately as follows:

$$h(t) = \frac{k}{nc_n\{1 - F_t(f_n(t))\}} \approx \frac{k}{nc_n}\left(1 + \gamma(t)\frac{f_n(t) - b_t(n/k)}{a_t(n/k)}\right)^{1/\gamma(t)}. \tag{1.10}$$

Hence we also need to estimate the functions $\gamma(t), a_t(n/k)$ and $b_t(n/k)$. The estimation of these four objects is the purpose of this paper. The actual estimation of $P(F)$ is the subject of future research.

Finally we remark that all our results still hold if the time parameter runs through an arbitrary compact set in $\mathbb{R}^d$, not just $[0, 1]$.

2. Result. Suppose that $\{\xi_i, i \geq 1\}$ are i.i.d. random elements of $C[0, 1]$ and that $F_t(x)$, the marginal distribution function of $\xi_i(t)$, is a continuous and increasing function of x for each t.

Assume that

$$P\left\{\inf_t \xi_1(t) > 0\right\} > 0 \tag{2.1}$$

(this can be achieved by applying a shift).

Define, for $x > 1$,

$$U_t(x) := F_t^{\leftarrow}\left(1 - \frac{1}{x}\right)$$

and the function $a_t(\cdot)$ by $a_t(s) := a_t([s])$ for $s > 0$ [cf. (1.1)].

From (2.1) we can suppose

$$\inf_{0 \leq t \leq 1} U_t(2) \geq 0.$$

We assume weak convergence of the maxima in C-space:

$$\frac{\bigvee_{i=1}^n \xi_i(t) - U_t(n)}{a_t(n)} \to \eta(t), \tag{2.2}$$

where $a_\cdot(n)$ is positive and in $C[0, 1]$ and η is a random element of $C[0, 1]$ satisfying

$$P\big(\eta(t) \leq x\big) = \exp\{-\big(1 + \gamma(t)x\big)^{-1/\gamma(t)}\}$$

for each $t \in [0, 1]$, $\gamma \in C[0, 1]$; the extreme-value index of $\xi_1(t)$ is $\gamma(t)$ for each t. Let $\xi_{1,n}(t) \leq \xi_{2,n}(t) \leq \cdots \leq \xi_{n,n}(t)$ be the order statistics of $\xi_i(t), i = 1, 2, \ldots, n$.

We define the sample functions

$$M_n^{(j)}(t) = \frac{1}{k}\sum_{i=0}^{k-1}\bigl(\log \xi_{n-i,n}(t) - \log \xi_{n-k,n}(t)\bigr)^j, \qquad j = 1, 2. \tag{2.3}$$

Now we define estimators for $\gamma(t)$, $a_t(n/k)$ and $b_t(n/k)$ as in Dekkers, Einmahl and de Haan (1989):

$$\hat{\gamma}_n^+(t) = M_n^{(1)}(t) \tag{2.4}$$

(Hill estimator);

$$\hat{\gamma}_n^-(t) = 1 - \frac{1}{2}\left\{1 - \frac{(M_n^{(1)})^2}{M_n^{(2)}}\right\}^{-1}; \tag{2.5}$$

$$\hat{\gamma}_n(t) = \hat{\gamma}_n^+(t) + \hat{\gamma}_n^-(t) \tag{2.6}$$

(moment estimator);

$$\hat{U}_t\left(\frac{n}{k}\right) = \xi_{n-k,n}(t), \tag{2.7}$$

$$\hat{a}_t\left(\frac{n}{k}\right) = \xi_{n-k,n}(t)\hat{\gamma}_n^+(t)\bigl(1 - \hat{\gamma}_n^-(t)\bigr) \tag{2.8}$$

(location and shift estimators).

For fixed t these are well-known one-dimensional estimators [cf., e.g., de Haan and Rootzén (1993)].

Next we denote, for $i = 1, 2, \dots, n$,

$$\hat{\zeta}_i^{(1)}(t) := \frac{n}{k}\left\{1 + \hat{\gamma}_n(t)\left[\frac{\xi_i(t) - \hat{U}_t(n/k)}{\hat{a}_t(n/k)} \vee \left(-\frac{1}{\hat{\gamma}_n^+(t)}\right)\right]\right\}^{1/\hat{\gamma}_n(t)}$$

and

$$\hat{\zeta}_i^{(2)}(t) := \frac{1}{1 - \hat{F}_{n,t}(\xi_i(t))}$$

with $1 - \hat{F}_{n,t}(x) := \frac{1}{n}\sum_{i=1}^n I_{\{\xi_i(t) > x\}}$.

Define the estimators

$$\hat{\nu}_{n,k}^{(1)}(\cdot) = \frac{1}{k}\sum_{i=1}^n I\left\{\frac{k}{n}\hat{\zeta}_i^{(1)} \in \cdot\right\}, \tag{2.9}$$

$$\hat{\nu}_{n,k}^{(2)}(\cdot) = \frac{1}{k}\sum_{i=1}^n I\left\{\frac{k}{n}\hat{\zeta}_i^{(2)} \in \cdot\right\}. \tag{2.10}$$

The latter estimator has been inspired by Huang (1992), the former by de Haan and Resnick (1993).

THEOREM 2.1. *As $k \to \infty$, $k/n \to 0$, we have*

$$\sup_{0 \le t \le 1} |\gamma_n^+(t) - \gamma^+(t)| \xrightarrow{P} 0 \qquad \text{with } \gamma^+(t) = \gamma(t) \vee 0, \tag{2.11}$$

$$\sup_{0 \le t \le 1} |\gamma_n^-(t) - \gamma^-(t)| \xrightarrow{P} 0 \qquad \text{with } \gamma^-(t) = \gamma(t) \wedge 0, \tag{2.12}$$

$$\sup_{0 \le t \le 1} |\hat{\gamma}_n(t) - \gamma(t)| \xrightarrow{P} 0, \tag{2.13}$$

$$\sup_{0 \le t \le 1} \left| \frac{\hat{U}_t(n/k) - U_t(n/k)}{a_t(n/k)} \right| \xrightarrow{P} 0, \tag{2.14}$$

$$\sup_{0 \le t \le 1} \left| \frac{\hat{a}_t(n/k)}{a_t(n/k)} - 1 \right| \xrightarrow{P} 0, \tag{2.15}$$

$$\hat{\nu}_{n \cdot k}^{(1)}|_{S_c} \xrightarrow{d} \nu|_{S_c}, \tag{2.16}$$

$$\hat{\nu}_{n \cdot k}^{(2)}|_{S_c} \xrightarrow{d} \nu|_{S_c} \tag{2.17}$$

in the space of finite measures on $C^+[0, 1]$, with $c > 0$ and

$$S_c := \{f \in C^+[0, 1]; \|f\|_\infty \ge c\}.$$

REMARK 2.2. Relations (2.16) and (2.17) mean that

$$\hat{\nu}_{n \cdot k}^{(i)}(E) \to \nu(E), \qquad i = 1, 2,$$

in probability, where $E \subset S_c$ is a Borel ν-continuous set (cf. proof of Lemma 3.1).

3. Proof.

We first prove some auxiliary results.

LEMMA 3.1. *Define the random measures*

$$\bar{\nu}_{n,k}(\cdot) = \frac{1}{k} \sum_{i=1}^{n} I\left\{\frac{k}{n}\zeta_i \in \cdot\right\}$$

with $\zeta_i(t) := \frac{1}{1 - F_t(\xi_i(t))}$.

As $k \to \infty$, $k/n \to 0$ and $c > 0$,

$$\bar{\nu}_{n,k}|_{S_c} \xrightarrow{d} \nu|_{S_c} \tag{3.1}$$

in the space of finite measures on $C^+[0, 1]$.

PROOF. According to Daley and Vere-Jones [(1988), Theorem 9.1.VI], for (3.1) we only need to prove, for any Borel ν-continuous sets $E_1, E_2, \dots, E_m \subset S_c$, $(\bar{\nu}_{n,k}(E_1), \bar{\nu}_{n,k}(E_2), \dots, \bar{\nu}_{n,k}(E_m)) \to (\nu(E_1), \nu(E_2), \dots, \nu(E_m))$ in probability. Since the limit is not random, this is equivalent to the following: for any Borel ν-continuous set $E \subset S_c$,

$$\bar{\nu}_{n,k}(E) \to \nu(E) \qquad \text{in probability.} \tag{3.2}$$

Using characteristic functions we know that this is equivalent to

$$\frac{n}{k} P\left(\frac{k}{n}\zeta_i \in E\right) \to \nu(E), \tag{3.3}$$

which is the same as

$$\nu_{n/k}|_{S_c} \to \nu|_{S_c}. \tag{3.4}$$

This has been proved in Proposition 1.1(i). □

Next we show the convergence of the tail empirical distribution functions.

LEMMA 3.2. *For each t, let $\zeta_{1,n}(t) \le \zeta_{2,n}(t) \le \cdots \le \zeta_{n,n}(t)$ be the order statistics of $\zeta_i(t)$, $i = 1, 2, \dots, n$, and define*

$$1 - G_{n,t}(x) = \frac{1}{k}\sum_{i=1}^{n} I_{\{\zeta_i(t) > (n/k)x\}}.$$

Then, for any positive c,

$$\sup_{0 \le t \le 1, x \ge c} \left| 1 - G_{n,t}(x) - \frac{1}{x} \right| \xrightarrow{P} 0 \tag{3.5}$$

and

$$\sup_{0 \le t \le 1, x \ge c} \left| \frac{k}{n}\zeta_{n-kx,n}(t) - \frac{1}{x} \right| \xrightarrow{P} 0. \tag{3.6}$$

Also, suppose μ and λ are continuous functions defined on $[0, 1]$ *with $\mu < 1$, $\lambda < 1$, $\mu + \lambda < 1$. Then*

$$\sup_{0 \le t \le 1} \left| \frac{1}{k}\sum_{i=0}^{k-1} \frac{(\zeta_{n-i,n}(t)/\zeta_{n-k,n}(t))^{\mu(t)} - 1}{\mu(t)} - \frac{1}{1-\mu(t)} \right| \xrightarrow{P} 0 \tag{3.7}$$

and

$$\begin{aligned} \sup_{0 \le t \le 1} \Bigg| \frac{1}{k}\sum_{i=0}^{k-1} & \frac{(\zeta_{n-i,n}(t)/\zeta_{n-k,n}(t))^{\mu(t)} - 1}{\mu(t)} \\ & \times \frac{(\zeta_{n-i,n}(t)/\zeta_{n-k,n}(t))^{\lambda(t)} - 1}{\lambda(t)} \\ & - \frac{2 - \mu(t) - \lambda(t)}{(1-\mu(t)-\lambda(t))(1-\mu(t))(1-\lambda(t))} \Bigg| \xrightarrow{P} 0. \end{aligned} \tag{3.8}$$

PROOF. Fix $c > 0$. From Lemma 3.1 and use of Skorohod construction we can suppose

$$\bar{\nu}_{n,k}|_{S_c} \to \nu|_{S_c} \qquad \text{a.s.},$$

where $S_c = \{f \in C^+[0,1], \|f\|_\infty \geq c\}$. Note that this is convergence of finite random measures. For any finite measures ν, μ, define the metric [cf. Daley and Vere-Jones (1988), (A2.5.1)]

$$\begin{aligned} d(\nu,\mu) := \inf\big\{\varepsilon > 0 \colon & \text{ for all closed sets } F \in C^+[0,1], \\ & \nu(F) \leq \mu(F^\varepsilon) + \varepsilon \text{ and } \mu(F) \leq \nu(F^\varepsilon) + \varepsilon\big\}, \end{aligned}$$

where $F^\varepsilon := \{f \in C^+[0,1], \|f - g\|_\infty \leq \varepsilon \text{ for some } g \in F\}$.

Now for any positive ε eventually

$$d\big(\bar{\nu}_{n,k}|_{S_c}, \nu|_{S_c}\big) \leq \varepsilon \qquad \text{a.s.}$$

Next define the closed set

$$E_{x,t} = \{f \in C^+[0,1];\ f(t) \geq x\}.$$

Note that in our situation the set $E^\varepsilon_{x,t}$ is the same as $E_{x-\varepsilon,t}$. Also $\nu(E_{x,t}) = 1/x$ [Giné, Hahn and Vatan (1990), pages 150 and 151]. It follows that for $x > c$, $0 \leq t \leq 1$,

$$\begin{aligned} 1 - G_{n,t}(x) &= \bar{\nu}_{n,k}\big(f \in C^+[0,1];\ f(t) > x\big) \\ &\leq \bar{\nu}_{n,k}(E_{x,t}) \leq \nu(E_{x-\varepsilon,t}) + \varepsilon = \frac{1}{x-\varepsilon} + \varepsilon \end{aligned}$$

and

$$1 - G_{n,t}(x) \geq \bar{\nu}_{n,k}(E_{x+\varepsilon,t}) \geq \nu(E_{x+2\varepsilon,t}) - \varepsilon = \frac{1}{x+2\varepsilon} - \varepsilon.$$

This proves (3.5). Statement (3.6) follows because the uniform convergence of the function $1 - G_{n,t}(x)$ to $1/x$ is equivalent to the uniform convergence of its inverse $(k/n)\xi_{n-kx,n}(t)$ to the same function.

For (3.7), observe that

$$\begin{aligned} &\frac{1}{k}\sum_{i=0}^{k-1} \frac{(\zeta_{n-i,n}(t)/\zeta_{n-k,n}(t))^{\mu(t)} - 1}{\mu(t)} \\ &\quad = \left(\frac{n}{k}\frac{1}{\zeta_{n-k,n}(t)}\right)^{\mu(t)} \frac{1}{k}\sum_{i=0}^{k-1} \frac{(\zeta_{n-i,n}(t)(k/n))^{\mu(t)} - (\zeta_{n-k,n}(t)(k/n))^{\mu(t)}}{\mu(t)} \\ &\quad = \left(\frac{n}{k}\frac{1}{\zeta_{n-k,n}(t)}\right)^{\mu(t)} \int_{(k/n)\zeta_{n-k+1,n}(t)}^{\infty} \left(\int_{(k/n)\zeta_{n-k+1,n}(t)}^{x} s^{\mu(t)-1}\,ds\right) dG_{n,t}(x) \end{aligned}$$

$$= \left(\frac{n}{k}\frac{1}{\zeta_{n-k,n}(t)}\right)^{\mu(t)} \int_{(k/n)\zeta_{n-k+1,n}(t)}^{\infty} (1 - G_{n,t}(x)) x^{\mu(t)-1}\, dx$$

$$= \left(\frac{n}{k}\frac{1}{\zeta_{n-k,n}(t)}\right)^{\mu(t)} \int_{1}^{\infty} (1 - G_{n,t}(x)) x^{\mu(t)-1}\, dx$$

$$+ \left(\frac{n}{k}\frac{1}{\zeta_{n-k,n}(t)}\right)^{\mu(t)} \int_{(k/n)\zeta_{n-k+1,n}(t)}^{1} (1 - G_{n,t}(x)) x^{\mu(t)-1}\, dx.$$

From (3.5) and (3.6) the second part converge to 0. So we only need to prove

$$\sup_{0\le t\le 1}\left|\int_1^\infty (1 - G_{n,t}(x)) x^{\mu(t)-1}\, dx - \frac{1}{1-\mu(t)}\right| \xrightarrow{P} 0. \tag{3.9}$$

Let $Y_i := \sup_t \zeta_i(t)$, $i = 1, 2, \dots, n$. These are i.i.d. r.v.'s. From Proposition 1.1 we have

$$\bigvee_{i=1}^n \frac{1}{n} Y_i = \bigvee_1^n \frac{1}{n} \sup_t \zeta_i(t) = \sup_t \bigvee_1^n \frac{1}{n}\zeta_i(t) \to \sup_t \bar{\eta}(t) =: Y$$

in distribution, where we know

$$P(Y \le x) = \exp\left\{-\left(\frac{x}{c}\right)^{-1}\right\}$$

for some $c > 1$.

Let $Y_{n-i,n}$, $i = 1, 2, \dots, n$, be the order statistics of Y_i, $i = 1, 2, \dots, n$, and

$$1 - F_n(x) := \frac{1}{k}\sum_1^n I\left(Y_{i,n} > x\frac{n}{k}\right).$$

We have

$$1 - G_{n,t}(x) \le 1 - F_n(x).$$

Hence for any $y > 0$ by one-dimensional results,

$$\int_y^\infty (1 - G_{n,t}(x)) x^{\mu(t)-1}\, dx \le \int_y^\infty (1 - F_n(x)) x^{\mu(t)-1}\, dx$$

$$\to \int_y^\infty \frac{c}{x} x^{\mu(t)-1}\, dx = \frac{c y^{\mu(t)-1}}{1-\mu(t)}.$$

Moreover,

$$\int_1^y (1 - G_{n,t}(x)) x^{\mu(t)-1}\, dx \xrightarrow{P} \int_1^y x^{\mu(t)-2}\, dx = \frac{1 - y^{\mu(t)-1}}{1-\mu(t)} \qquad \text{uniformly in } t.$$

By letting $y \to \infty$ we get (3.9). Hence we have proved (3.7). The proof of (3.8) is similar. □

LEMMA 3.3. *Suppose* $a_t(s) > 0, a_t^{-1}(s), g_t(s)$ *are locally bounded in* $t \in [0,1], 0 \le s < \infty, \gamma(t) \in C[0,1]$ *and*

$$\frac{g_t(sx) - g_t(s)}{a_t(s)} \to \frac{x^{\gamma(t)} - 1}{\gamma(t)}, \tag{3.10}$$

$$\frac{a_t(sx)}{a_t(s)} \to x^{\gamma(t)} \tag{3.11}$$

locally uniformly for $x > 0, 0 \le t \le 1$.

Then for any positive ε, *there exists* s_0 *such that for* $s > s_0$, $sx > s_0$ *we have*

$$\left|\frac{a_t(sx)}{a_t(s)} - x^{\gamma(t)}\right| < \varepsilon x^{\gamma(t)} \exp\{\varepsilon |\log x|\} \tag{3.12}$$

or, alternatively,

$$(1-\varepsilon)x^{\gamma(t)} \exp\{-\varepsilon|\log x|\} < \frac{a_t(sx)}{a_t(s)} < (1+\varepsilon)x^{\gamma(t)} \exp\{\varepsilon|\log x|\} \tag{3.13}$$

and

$$\left|\frac{g_t(sx) - g_t(s)}{a_t(s)} - \frac{x^{\gamma(t)} - 1}{\gamma(t)}\right| < \varepsilon\bigl(1 + x^{\gamma(t)} \exp\{\varepsilon|\log x|\}\bigr). \tag{3.14}$$

PROOF. The proof is not much more complicated than, and has the same structure as, the proof of the same result with t fixed. We refer to de Haan and Pereira [(1999), Appendix], which contains a simplification, applicable in this case, of a result of Drees [(1998), Lemma 2.1]. □

LEMMA 3.4. *With the same conditions as in Lemma* 3.3 *and* $g_t(s) > 0$, *we have, for* $s \to \infty$,

$$\frac{a_t(s)}{g_t(s)} \to \gamma^+(t) \qquad \textit{uniformly in } t. \tag{3.15}$$

For any positive ε, *there exists an* $s_0 > 0$ *such that if* $s, sx > s_0$, *we have*

$$\left|\frac{\log g_t(sx) - \log g_t(s)}{a_t(s)/g_t(s)} - \frac{x^{\gamma^-(t)} - 1}{\gamma^-(t)}\right| < \varepsilon\bigl(1 + x^{\gamma^-(t)} \exp\{\varepsilon|\log x|\}\bigr). \tag{3.16}$$

PROOF. For (3.15), we need to prove, for any $t_n \to t_0$ and $s_n \to \infty$,

$$\frac{g_{t_n}(s_n)}{a_{t_n}(s_n)} \to \begin{cases} \gamma(t_0)^{-1}, & \text{for } \gamma(t_0) > 0, \\ \infty, & \text{for } \gamma(t_0) \le 0. \end{cases} \tag{3.17}$$

(i) For $\gamma(t_0) > 0$, from Lemma 3.3 for any $\varepsilon \in (0, \gamma(t_0))$, we can find $s_0 > 0$ such that for $s_n > s_0$ we have

$$\left| \frac{g_{t_n}(s_n) - g_{t_n}(s_0)}{a_{t_n}(s_n)} - \frac{1-(s_0/s_n)^{\gamma(t_n)}}{\gamma(t_n)} \right| < \varepsilon\left(1 + \left(\frac{s_0}{s_n}\right)^{\gamma(t_n)-\varepsilon}\right),$$

which implies

$$\lim_{n\to\infty} \frac{g_{t_n}(s_n) - g_{t_n}(s_0)}{a_{t_n}(s_n)} = \frac{1}{\gamma(t_0)}. \tag{3.18}$$

From Lemma 3.3 we get

$$a_t(s) \to \infty,$$

$s \to \infty$, uniformly for $t \in \{t : \gamma(t) > \frac{1}{2}\gamma(t_0)\}$, which implies

$$\lim_{n\to\infty} \frac{g_{t_n}(s_0)}{a_{t_n}(s_n)} = 0. \tag{3.19}$$

From (3.18) and (3.19) we get the first part of (3.17).

(ii) For $\gamma(t_0) \le 0$, first define

$$\psi_t(s) := \int_1^\infty g_t(sx) - g_t(s) \frac{dx}{x^2} = s\int_s^\infty g_t(x)\,\frac{dx}{x^2} - g_t(s) \qquad \text{for } s > 0 \tag{3.20}$$

for $t \in E_c := \{0 \le t \le 1, \gamma(t) \le c\}$ with any $c < 1$.

Then we get

$$\frac{d(g_t(s) + \psi_t(s))}{ds} = \int_s^\infty g_t(x)\,\frac{dx}{x^2} - \frac{g_t(s)}{s} = \frac{\psi_t(s)}{s}.$$

This implies, for any positive s_0,

$$g_t(s) = \int_{s_0}^s \frac{\psi_t(x)}{x}\,dx - \psi_t(s) + s_0\int_{s_0}^\infty \frac{g_t(x)}{x^2}\,dx \qquad \text{for } s \ge s_0. \tag{3.21}$$

Next we will prove

$$\lim_{s\to\infty} \frac{\psi_t(s)}{a_t(s)} \to \frac{1}{1-\gamma(t)} \qquad \text{uniformly with } t \in E_c. \tag{3.22}$$

Note that

$$\frac{\psi_t(s)}{a_t(s)} = \int_1^\infty \frac{g_t(sx) - g_t(s)}{a_t(s)}\,\frac{dx}{x^2}.$$

From Lemma 3.3 we know for any $\varepsilon \in (0, 1-c)$ there exists a positive s_0 such that

$$\left| \frac{g_t(s) - g_t(sx)}{a_t(s)} x^{-2} \right| < x^{-2}\left(\left| \frac{x^{\gamma(t)} - 1}{\gamma(t)} \right| + \varepsilon\left(1 + x^{\gamma(t)+\varepsilon}\right) \right)$$

$$\text{for } s > s_0, x \in [1, \infty).$$

Since the right-hand side is integrable on $x \in [1, \infty)$, we get

$$\lim_{s\to\infty} \int_1^\infty \frac{g_t(sx) - g_t(s)}{a_t(s)} \frac{dx}{x^2} = \lim_{s\to\infty} \int_1^\infty \frac{x^{\gamma(t)} - 1}{\gamma(t)} \frac{dx}{x^2} = \frac{1}{1 - \gamma(t)}.$$

This leads to (3.22).

Now back to the proof of (3.17) for $\gamma(t_0) \le 0$. From (3.22) we get

$$\frac{g_{t_n}(s_n)}{a_{t_n}(s_n)} \sim \frac{g_{t_n}(s_n)}{\psi_{t_n}(s_n)} \frac{1}{1 - \gamma(t_n)}.$$

Since $g_t(s) > 0$, from (3.22), (3.21) and Lemma 3.3, we get, for any $\varepsilon > 0$,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{g_{t_n}(s_n)}{a_{t_n}(s_n)} \\
&\quad \ge \liminf_{n\to\infty} \frac{1}{1 - \gamma(t_n)} \left(\int_{s_0/s_n}^1 \frac{\psi_{t_n}(s_n u)}{\psi_{t_n}(s_n)} \frac{du}{u} - 1 \right) \\
&\quad \ge \liminf_{n\to\infty} \frac{1}{1 - \gamma(t_n)} \int_{s_0/s_n}^1 (1 - 2\varepsilon) u^{\gamma(t_n) - 1 + 2\varepsilon} \, du - \frac{1}{1 - \gamma(t_0)} \\
&\quad \ge \liminf_{n\to\infty} \frac{1}{1 - \gamma(t_n)} \int_{s_0/s_n}^1 (1 - 2\varepsilon) u^{\varepsilon - 1} \, du - \frac{1}{1 - \gamma(t_0)} \\
&\quad = \liminf_{n\to\infty} \frac{1 - 2\varepsilon}{1 - \gamma(t_n)} \frac{1 - (s_0/s_n)^\varepsilon}{\varepsilon} - \frac{1}{1 - \gamma(t_0)} \\
&\quad = \frac{1 - 2\varepsilon}{(1 - \gamma(t_0))\varepsilon} - \frac{1}{1 - \gamma(t_0)}.
\end{aligned}$$

Letting $\varepsilon \to 0$ we get the second part of (3.17). The proof of the first part of the lemma is complete.

For (3.16), according to (3.15) and Lemma 3.3 we only need to prove

$$\frac{\log g_t(sx) - \log g_t(s)}{a_t(s)/g_t(s)} \to \frac{x^{\gamma^-(t)} - 1}{\gamma^-(t)}$$

locally uniformly in t and x. That is, for any $t_n \to t_0$, $x_n \to x_0 > 0$, $s_n \to \infty$,

$$\lim_{n\to\infty} \frac{\log g_{t_n}(s_n x_n) - \log g_{t_n}(s_n)}{a_{t_n}(s_n)/g_{t_n}(s_n)} = \frac{x_0^{\gamma^-(t_0)} - 1}{\gamma^-(t_0)}.$$

For $\gamma(t_0) > 0$, from (3.10)–(3.15), we get

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\log g_{t_n}(s_n x_n) - \log g_{t_n}(s_n)}{a_{t_n}(s_n)/g_{t_n}(s_n)} \\
&\quad = \lim_{n\to\infty} \frac{\log([(g_{t_n}(s_n x_n) - g_{t_n}(s_n))/a_{t_n}(s_n)](a_{t_n}(s_n)/g_{t_n}(s_n)) + 1)}{a_{t_n}(s_n)/g_{t_n}(s_n)}
\end{aligned}$$

$$= \frac{\log(\gamma(t_0)(x_0^{\gamma(t_0)} - 1)/\gamma(t_0) + 1)}{\gamma(t_0)} = \log x_0;$$

for $\gamma(t_0) \le 0$ since $a_{t_n}(s_n)/g_{t_n}(s_n) \to 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\log g_{t_n}(s_n x_n) - \log g_{t_n}(s_n)}{a_{t_n}(s_n)/g_{t_n}(s_n)} \\
&\quad = \lim_{n\to\infty} \frac{\log([(g_{t_n}(s_n x_n) - g_{t_n}(s_n))/a_t(s_n)](a_{t_n}(s_n)/g_{t_n}(s_n)) + 1)}{a_{t_n}(s_n)/g_{t_n}(s_n)} \\
&\quad = \lim_{n\to\infty} \frac{g_{t_n}(s_n x_n) - g_{t_n}(s_n)}{a_{t_n}(s_n)} = \frac{x_0^{\gamma(t_0)} - 1}{\gamma(t_0)}.
\end{aligned}$$

This completes the proof of this lemma. □

PROOF OF THEOREM 2.1. We first prove (2.14). Note $\{U_t(\zeta_i(t)), i = 1, 2, \ldots\} = \{\xi_i(t), i = 1, 2, \ldots\}$. Then

$$\frac{\hat{U}_t(n/k) - U_t(n/k)}{a_t(n/k)} = \frac{\xi_{n-k,n}(t) - U_t(n/k)}{a_t(n/k)} \stackrel{d}{=} \frac{U_t(\zeta_{n-k,n}(t)) - U_t(n/k)}{a_t(n/k)}.$$

From (1.5) and Lemma 3.2 we get (2.14).

Next we consider (2.11), (2.12) and (2.13). For any $\varepsilon > 0$,

$$\begin{aligned}
\frac{M_n^{(1)}(t)}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))} &= \frac{1}{k}\sum_{i=0}^{k-1} \frac{\log \xi_{n-i,n}(t) - \log \xi_{n-k,n}(t)}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))} \\
&\stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1} \frac{\log U_t(\zeta_{n-i,n}(t)) - \log U_t(\zeta_{n-k,n}(t))}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))}.
\end{aligned}$$

From Lemma 3.4,

$$\begin{aligned}
&\left|\frac{M_n^{(1)}(t)}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))} - \frac{1}{k}\sum_{i=0}^{k-1} \frac{(\zeta_{n-i,n}(t)/\zeta_{n-k,n}(t))^{\gamma^-(t)} - 1}{\gamma^-(t)}\right| \\
&\quad < \varepsilon\left(1 + \frac{1}{k}\sum_{i=0}^{k-1}\left(\frac{\zeta_{n-i,n}(t)}{\zeta_{n-k,n}(t)}\right)^{\gamma^-(t)+\varepsilon}\right) \\
&\quad = \varepsilon\left(2 + \left(\gamma^-(t) + \varepsilon\right)\frac{1}{k}\sum_{i=0}^{k-1} \frac{(\zeta_{n-i,n}(t)/\zeta_{n-k,n}(t))^{\gamma^-(t)+\varepsilon} - 1}{\gamma^-(t) + \varepsilon}\right).
\end{aligned}$$

Using Lemma 3.2 we get

$$\sup_{0\le t\le 1}\left|\frac{M_n^{(1)}(t)}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))} - \frac{1}{1-\gamma^-(t)}\right| \stackrel{P}{\to} 0. \tag{3.23}$$

Similarly we get

$$\sup_{0\le t\le 1}\left|\frac{M_n^{(2)}(t)}{(a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t)))^2} - \frac{2}{(1-\gamma^-(t))((1-2\gamma^-(t))}\right| \xrightarrow{P} 0. \tag{3.24}$$

From Lemma 3.4 we get

$$\lim_{s\to\infty}\sup_{0\le t\le 1}\left|\frac{a_t(s)}{U_t(s)} - \gamma^+(t)\right| = 0.$$

Hence

$$\sup_{0\le t\le 1}\left|\frac{a_t(\zeta_{n-k,n}(t))}{U_t(\zeta_{n-k,n}(t))} - \gamma^+(t)\right| \xrightarrow{P} 0. \tag{3.25}$$

From (3.23) and (3.25) we get (2.11).

From (3.23)–(3.25) we get

$$\sup_{0\le t\le 1}\left|\frac{(M_n^{(1)}(t))^2}{M_n^{(2)}(t)} - \frac{1-2\gamma^-(t)}{1-\gamma^-(t)}\right| \xrightarrow{P} 0, \tag{3.26}$$

which implies (2.12). Hence (2.13) is obtained.

Now we prove (2.15). Note

$$\frac{\hat{a}_t(n/k)}{a_t(n/k)} = \frac{a_t(\zeta_{n-k,n}(t))}{a_t(n/k)}\frac{M_n^{(1)}(t)}{a_t(\zeta_{n-k,n}(t))/U_t(\zeta_{n-k,n}(t))}(1-\hat{\gamma}^-(t));$$

from (3.12), (3.6), (3.23) and (2.12) we get (2.15).

Finally we shall prove (2.16) and (2.17). A similar argument as in the proof of Lemma 3.1 shows we only need to prove that, for any ν-continuous set $E \in S_c$ with any $a > 0$,

$$\hat{\nu}_{n,k}^{(1)}(E) \xrightarrow{P} \nu(E) \tag{3.27}$$

and

$$\hat{\nu}_{n,k}^{(2)}(E) \xrightarrow{P} \nu(E). \tag{3.28}$$

For (3.28), we only need to prove that, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\{d(\hat{\nu}_{n,k}^{(2)}|_{S_c}, \nu_{n,k}|_{S_c}) \le \varepsilon\} = 1 \tag{3.29}$$

with d, a metric of finite measures, defined in the proof of Lemma 3.2.

Note

$$\begin{aligned}\frac{k}{n}\hat{\zeta}_i^{(2)} &= \frac{1}{(1/k)\sum_{j=1}^n I_{\{\xi_j(t)>\xi_i(t)\}}} = \frac{1}{(1/k)\sum_{j=1}^n I_{\{\zeta_j(t)>\zeta_i(t)\}}}\\ &= \frac{1}{1-G_{n,t}}\left(\frac{k}{n}\zeta_i(t)\right).\end{aligned} \tag{3.30}$$

From (3.5) we get, for any $0 < c < b$,

$$\sup_{0 \le t \le 1, c \le x \le b} \left| \frac{1}{1 - G_{n,t}}(x) - x \right| \xrightarrow{P} 0. \tag{3.31}$$

Since $\frac{1}{1-G_{n,t}}(x) \ge 0$ is a monotone function of x we get

$$\sup_{0 \le t \le 1, 0 \le x \le b} \left| \frac{1}{1 - G_{n,t}}(x) - x \right| \xrightarrow{P} 0. \tag{3.32}$$

Note $\nu(S_b) = \frac{C}{b}$ with a positive constant $C := \nu(S_1)$. We can find $b \gg \varepsilon$ such that $\nu(S_{b-2\varepsilon}) \le \varepsilon$.

Suppose given the conditions

$$\sup_{0 \le t \le 1, c \le x \le b} \left| \frac{1}{1 - G_{n,t}}(x) - x \right| \le \varepsilon \tag{3.33}$$

and

$$d\big(\bar{\nu}_{n,k}|_{S_c}, \nu|_{S_c}\big) \le \varepsilon \tag{3.34}$$

we have the following:

(i) For any closed set $E \in S_{[c,b]} := \{f \in C^+[0,1], \sup_{0 \le t \le 1} f(t) \in [c,b]\}$ we have

$$\begin{aligned} \hat{\nu}_{n,k}^{(2)}(E) &= \bar{\nu}_{n,k}\left(\frac{1}{1 - G_{n,t}}(E)\right) && \text{[from (3.30)]} \\ &\le \bar{\nu}_{n,k}(E^{\varepsilon}) && \text{[from (3.34)]} \end{aligned}$$

and similarly

$$\bar{\nu}_{n,k}(E) \le \hat{\nu}_{n,k}^{(2)}(E^{\varepsilon}).$$

These imply

$$d\big(\bar{\nu}_{n,k}|_{S_{[c,b]}}, \hat{\nu}_{n,k}^{(2)}|_{S_{[c,b]}}\big) \le \varepsilon.$$

Hence

$$\begin{aligned} &d\big(\hat{\nu}_{n,k}^{(2)}|_{S_{[c,b]}}, \nu|_{S_{[c,b]}}\big) \\ &\quad \le d\big(\bar{\nu}_{n,k}|_{S_{[c,b]}}, \hat{\nu}_{n,k}^{(2)}|_{S_{[c,b]}}\big) + d\big(\bar{\nu}_{n,k}|_{S_{[c,b]}}, \nu|_{S_{[c,b]}}\big) \le 2\varepsilon. \end{aligned} \tag{3.35}$$

(ii) For any closed set $E \in S_b$, note from (3.33) we get

$$\frac{1}{1 - G_{n,t}}(S_b) = \left\{ \frac{1}{1 - G_{n,t}} \circ f \,\middle|\, f(t) \in C^+[0,1], \sup_{0 \le t \le 1} f(t) \ge b \right\} \subset S_{b-\varepsilon}$$

and from (3.34) we get

$$\bar{\nu}_{n,k}(S_{b-\varepsilon}) \le \nu\left((S_{b-\varepsilon})^{\varepsilon}\right) + \varepsilon \le \nu(S_{b-2\varepsilon} + \varepsilon) \le 2\varepsilon;$$

then

$$\hat{\nu}^{(2)}_{n,k}(E) \le \hat{\nu}^{(2)}_{n,k}(S_b) = \bar{\nu}_{n,k}\left(\frac{1}{1-G_{n,t}}(S_b)\right) \le \bar{\nu}_{n,k}(S_{b-\varepsilon}) \le 2\varepsilon.$$

Note $\nu(E) \le \nu(S_b) \le \varepsilon$. We get

$$d(\hat{\nu}^{(2)}_{n,k}|_{S_b}, \nu|_{S_b}) \le 2\varepsilon. \tag{3.36}$$

Hence

$$d(\hat{\nu}^{(2)}_{n,k}|_{S_c}, \nu|_{S_c}) \le d(\hat{\nu}^{(2)}_{n,k}|_{S_{[c,b]}}, \nu|_{S_{[c,b]}}) + d(\hat{\nu}^{(2)}_{n,k}|_{S_b}, \nu|_{S_b}) \le 4\varepsilon.$$

Finally we have

$$\begin{aligned}\liminf_{n\to\infty} P\{d(\hat{\nu}^{(2)}_{n,k}|_{S_c}, \nu|_{S_c}) \le 5\varepsilon\}&\\ \ge \liminf_{n\to\infty} P\left\{d(\bar{\nu}_{n,k}|_{S_c}, \nu|_{S_c}) \le \varepsilon, \sup_{0\le t\le 1, c\le x\le b}\left|\frac{1}{1-G_{n,t}}(x) - x\right| \le \varepsilon\right\} &= 1.\end{aligned}$$

For the latter equality we use Lemma 3.1 and (3.38).

This completes the proof of (3.29).

For (3.27) note

$$\frac{k}{n}\hat{\zeta}^{(1)}_i(t) = \left[1+\hat{\gamma}_n(t)\left(\frac{U_t(\zeta_i(t)) - \hat{U}_t(n/k)}{\hat{a}_t(n/k)} \vee \left(-\frac{1}{\hat{\gamma}^+_n(t)}\right)\right)\right]^{1/\hat{\gamma}_n(t)} = H_{n,t}\left(\frac{k}{n}\zeta_i(t)\right)$$

with

$$\begin{aligned}H_{n,t}(x) &:= \left\{1+\hat{\gamma}_n(t)\left[\frac{U_t((n/k)x) - \hat{U}_t(n/k)}{\hat{a}_t(n/k)} \vee \left(-\frac{1}{\hat{\gamma}^+_n(t)}\right)\right]\right\}^{1/\hat{\gamma}_n(t)}\\ &= \left\{1+\hat{\gamma}_n(t)\left[\left(\frac{a_t(n/k)}{\hat{a}_t(n/k)}\right.\right.\right.\\ &\qquad \times \left(\frac{U_t((n/k)x) - U_t(n/k)}{a_t(n/k)}\right.\\ &\qquad\qquad \left.\left.\left.\left. + \frac{U_t(n/k) - \hat{U}_t(n/k)}{\hat{a}_t(n/k)}\right)\right) \vee \left(-\frac{1}{\hat{\gamma}^+_n(t)}\right)\right]\right\}^{1/\hat{\gamma}_n(t)}.\end{aligned}$$

From (2.13)–(2.15) and Proposition 1.1(ii) we get, for any $0 < c < b$,

$$\sup_{0\le t\le 1, c\le x\le b} |H_{n,t}(x) - x| \xrightarrow{P} 0. \tag{3.37}$$

Since $H_{n,t}(x) \ge 0$ is a monotone function of x we get

$$\sup_{0\le t\le 1, 0\le x\le b} |H_{n,t}(x) - x| \xrightarrow{P} 0. \tag{3.38}$$

The rest of the proof is similar to the proof of (3.28). The theorem has been proved. □

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REFERENCES

BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

DALEY, D. J. and VERE-JONES, D. (1988). *An Introduction to the Theory of Point Processes*. Springer, Berlin.

DEKKERS, A. L. M., EINMAHL, J. H. J. and DE HAAN, L. (1989). A moment estimator for the index of an extreme-value distribution. *Ann. Statist.* **17** 1833–1855.

DREES, H. (1998). On smooth statistical tail functionals. *Scand. J. Statist.* **25** 187–210.

GINÉ, E., HAHN, M. and VATAN, P. (1990). Max-infinitely divisible and max-stable sample continuous processes. *Probab. Theory Related Fields* **87** 139–165.

DE HAAN, L. and LIN, T. (2001). On convergence towards an extreme value distribution in $C[0, 1]$. *Ann. Probab.* **29** 467–483.

DE HAAN, L. and PEREIRA, T. T. (1999). Estimating the index of a stable distribution. *Statist. Probab. Lett.* **41** 39–55.

DE HAAN, L. and RESNICK, S. I. (1993). Estimating the limit distribution of multivariate extremes. *Comm. Statist. Stochastic Models* **9** 275–309.

DE HAAN, L. and DE RONDE, J. (1998). Sea and wind: Multivariate extremes at work. *Extremes* **1** 7–45.

DE HAAN, L. and ROOTZÉN, H. (1993). On the estimation of high quantiles. *J. Statist. Plann. Inference* **35** 1–13.

DE HAAN, L. and SINHA, A. K. (1999). Estimating the probability of a rare event. *Ann. Statist.* **27** 732–759.

HUANG, X. (1992). Statistics of bivariate extremes. Dissertation. Tinbergen Research Series 22, Erasmus Univ. Rotterdam.

RESNICK, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.

SMITH, R. L. (1990). Extreme value theory. In *Handbook of Applicable Mathematics* (W. Ledermann, ed.) **7**. Wiley, New York.

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