CONE ORDER ASSOCIATION AND STOCHASTIC CONE ORDERING WITH APPLICATIONS TO ORDER-RESTRICTED TESTING[1]

By Arthur Cohen and H. B. Sackrowitz

Rutgers University

Cohen, Sackrowitz and Samuel-Cahn introduced the notion of cone order association and established a necessary and sufficient condition for a normal random vector to be cone order associated (COA). In this paper we provide the following: (1) a necessary and sufficient condition for a multinomial distribution to be COA when the cone is a pairwise contrast cone; (2) a relationship between COA and regular association; (3) a notion of stochastic cone ordering (SCO) of random vectors along with two preservation theorems indicating monotonicity properties of expectations as functions of parameters; and (4) applications to unbiasedness of tests and monotonicity of power functions of tests in cone order–restricted hypothesis-testing problems. In particular, the matrix order alternative hypothesis-testing problem is treated when the underlying distributions are independent Poisson or the joint distribution is multinomial.

1. Introduction and summary. A convex cone is a subset $\mathscr{K} \subseteq \mathbb{R}^k$ such that if $\mathbf{x}, \mathbf{y} \in \mathscr{K}$, then $\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y} \in \mathscr{K}$ for all $\lambda_1, \lambda_2 \geq 0$. A closed convex cone $\mathscr{K}$ induces a partial ordering $\leq [\mathscr{K}]$ as follows: $\mathbf{x} \leq [\mathscr{K}] \mathbf{y}$ if and only if $\mathbf{y} - \mathbf{x} \in \mathscr{K}$. The cone $\mathscr{K}$ is pointed if $\mathbf{x} \in \mathscr{K}$ and $-\mathbf{x} \in \mathscr{K}$ implies $\mathbf{x} = \mathbf{0}$. A function $W(\mathbf{x})$ is nondecreasing with respect to (w.r.t.) the cone $\mathscr{K}$, or is said to be cone order monotone w.r.t. $\mathscr{K}$ (COM $[\mathscr{K}]$), if $W(\mathbf{x}) \leq W(\mathbf{y})$ whenever $\mathbf{x} \leq [\mathscr{K}]\mathbf{y}$.

Cohen, Sackrowitz and Samuel-Cahn (1995b) recognized the important role COM functions played in the study of a large class of hypothesis-testing problems. It was seen that the establishment of stochastic properties of such functions yields useful tools in solving these problems. In particular, Cohen, Sackrowitz and Samuel-Cahn (1995a) introduced the notion of cone order association (COA). A $p \times 1$ random vector $\mathbf{X}$ is said to be COA w.r.t. $\mathscr{K}$, written COA $[\mathscr{K}]$, if, for any pair W_1, W_2 of COM $[\mathscr{K}]$ functions,

$$(1.1) \qquad EW_1(\mathbf{X})W_2(\mathbf{X}) \geq EW_1(\mathbf{X}) EW_2(\mathbf{X}),$$

whenever the preceding expectations exist. Note that if $\mathscr{K}$ is the first quadrant of $\mathbb{R}^k$, then COA $[\mathscr{K}]$ is simply association A, as introduced by Esary, Proschan and Walkup (1967). The notions of A and COA have a number of applications. See, for example, Esary, Proschan and Walkup (1967), Ahmed,

León and Proschan (1981), Cohen and Sackrowitz (1994) and Cohen, Sackrowitz and Samuel-Cahn (1995a).

Among the results in this paper, two are connected with COA. In order to state the first result, we need to know that the (positive) dual of a cone $\mathscr{K}$ is defined as

$$(1.2) \qquad \mathscr{K}^* = \left[\boldsymbol{\theta} \colon \mathbf{v}'\boldsymbol{\theta} \geq 0, \text{ all } \mathbf{v} \in \mathscr{K} \right].$$

Hereafter we always assume $\mathscr{K}$ is a closed convex pointed cone. We also let $\{\mathbf{b}_\nu\}$, ν lying in an index set Γ, $\mathbf{b}_\nu \in \mathscr{K}$, be a set of generators of the cone $\mathscr{K}$. That is, any $\mathbf{v} \in \mathscr{K}$ can be expressed as a nonnegative linear combination of the $\mathbf{b}_\nu$. Furthermore, no proper subset of $\mathbf{b}_\nu$'s can qualify.

THEOREM 1.1. *Consider a cone $\mathscr{K}$ and its dual $\mathscr{K}^*$. Let $\mathbf{X}^{p \times 1}$ be a random vector and let $Y_\nu = \mathbf{a}'_\nu \mathbf{X}$, where $\{\mathbf{a}_\nu\}$ are a set of generators of $\mathscr{K}$. Then $\mathbf{X}$ is COA $[\mathscr{K}^*]$ if and only if the Y_ν's are A.*

This result appears in Cohen, Sackrowitz and Samuel-Cahn (1995a) for the special case where $\mathscr{K}^*$ is a polyhedral cone with k linearly independent generators.

Theorem 1.1 has value in the following senses. First, it may be easier to establish Y_ν are A than to establish $\mathbf{X}$ is COA or vice versa. Second, and of greater importance, is the fact that the proof requires showing that any COM $[\mathscr{K}^*]$ function of $\mathbf{x}$ can be expressed as a nondecreasing function of the y_ν's. This proves to be extremely valuable in the construction of good tests of hypotheses in many practical problems. In fact, this is precisely what is done in Cohen, Sackrowitz and Samuel-Cahn (1995b). That is, good tests are determined by finding monotone functions of the y_ν's.

To state the second result concerning COA, we let the cone $\mathscr{K}_B$ be a polyhedral cone of the form

$$(1.3) \qquad \mathscr{K}_B = \{\boldsymbol{\theta} \colon \boldsymbol{\theta} \in \mathbb{R}^k, \, B\boldsymbol{\theta} \geq \mathbf{0}\},$$

where the rows of $B^{r \times k}$ are generators of $\mathscr{K}_B^*$. If the rows of B are contrasts (i.e., the sum of the row elements equals 0) with only two nonzero elements, $\mathscr{K}_B^*$ is called a pairwise contrast cone. Note that a pairwise contrast cone corresponds to a partial ordering on the set $\{1, 2, \ldots, k\}$. Also note that many cones of interest are pairwise contrast cones, for example, the simple order cone, the simple tree cone and the umbrella cone. See, for example, Robertson, Wright and Dykstra (1988). Let H be the equiangular line in $\mathbb{R}^k$. Our second result is as follows.

THEOREM 1.2. *Let $\mathbf{U}^{k \times 1}$ be a random vector with multinomial distribution with parameters $(n, k, (1/k)\mathbf{1})$, that is, $\mathbf{U} \sim \mathscr{M}(n, k, (1/k)\mathbf{1})$, $\mathbf{1}' = (1, 1, \ldots, 1)$. Let $\mathscr{K}_B^*$ be a pairwise contrast cone where B has rank $(k-1)$. Then $\mathbf{U}$ is COA $[\mathscr{K}_B^*]$ if and only if $\mathscr{K}_B^* \oplus H \supset \mathscr{K}_B$.*

Our next main result is concerned with the notion of stochastic cone ordering (SCO) of random vectors. A random vector $\mathbf{X}^{k \times 1}$ is stochastically cone order less than or equal to $\mathbf{Y}^{k \times 1}$ with respect to the cone $\mathscr{K}$, written $\mathbf{X} \leq_{\mathrm{st}} [\mathscr{K}]\mathbf{Y}$, if for every $h(\mathbf{X})$ which is COM $[\mathscr{K}]$,

$$(1.4) \qquad\qquad Eh(\mathbf{X}) \leq Eh(\mathbf{Y}),$$

whenever expectations in (1.4) exist.

Now let $\mathbf{U}^{k \times 1} \sim \mathscr{M}(n, k, \mathbf{p})$. Let $\mathscr{K}^*$ be a contrast cone. That is, the generators of $\mathscr{K}^*$ are contrasts. Since we study functions that are COM $[\mathscr{K}^*]$, the only type of cone which makes sense to consider is a contrast cone. More specifically, note that if $\mathscr{K}^*$ has a generator which is not a contrast, then no two points within the multinomial sample space are ordered with respect to that generator. To see this, note that if $\mathbf{x}$ lies in the sample space and $\mathbf{b}$ is a generator of $\mathscr{K}^*$, then $\mathbf{x}'\mathbf{1} = n$ and so $(\mathbf{x} + \lambda\mathbf{b})'\mathbf{1}$ must equal n for $\lambda > 0$.

We have the following preservation theorem.

THEOREM 1.3. *The inequality*

$$(1.5) \qquad\qquad E_{\mathbf{p}'}h(\mathbf{U}) \geq E_{\mathbf{p}}h(\mathbf{U})$$

holds for all $\mathbf{p}' \geq [\mathscr{K}^*]\mathbf{p}$ *and all* $h(\mathbf{U})$ *that are COM* $[\mathscr{K}^*]$ *if and only if* $\mathscr{K}^*$ *is a pairwise contrast cone.*

When (1.5) holds, we say that the family of random vectors $\mathbf{U}$ is SCO with respect to the cone $\mathscr{K}^*$.

The preservation theorem (Theorem 1.3) is particularly interesting when contrasted with a comparable preservation theorem for the normal or translation parameter case. In the case $\mathbf{U} \sim N(\mu, \Sigma)$, $\mathscr{K}^*$ can be any convex cone and the analogous preservation theorem is true.

Another result concerning the notion of SCO extends a result of Proschan and Sethuraman (1977). Let X_i, $i = 1, 2, \ldots, k$, be independent random variables with densities that belong to the same one-parameter family. The densities are with respect to μ, Lebesgue measure in the continuous case and counting measure on the integers in the integer value case. Let $\mathbf{X} = (X_1, \ldots, X_k)'$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)'$ and let $f(x; \theta)$ represent the family of densities for each X_i. The following is another preservation theorem.

THEOREM 1.4. *Suppose* $f(x; \theta)$ *satisfies*

$$(1.6) \qquad\qquad f(x; \theta) = 0 \ \textit{if} \ x < 0;$$

$$(1.7) \qquad\qquad f(x; \theta) \ \textit{is totally positive of order 2 } (TP_2);$$

f satisfies the semigroup property, that is,

$$(1.8) \qquad f(y, \theta_1 + \theta_2) = \int_{\mathscr{X}} f(x; \theta_1) f(y - x, \theta_2) \, d\nu(x)$$

for some measure ν *on* $\mathscr{X}$.

Suppose $\mathscr{K}^*$ is a pairwise contrast cone. Then the random vector $\mathbf{X}$ is SCO w.r.t. $\mathscr{K}^*$. That is, the function $\psi(\boldsymbol{\theta}) = E_{\boldsymbol{\theta}} h(\mathbf{X})$ is COM $[\mathscr{K}^*]$.

Theorems 1.1–1.4 have important applications to hypothesis-testing problems concerned with order-restricted parameters. For example, Theorem 1.2 can be used to identify a class of unbiased tests of the homogeneity of Poisson parameters versus an alternative that such parameters satisfy a matrix order. That is, in a two-way table of parameters λ_{ij}, λ_{ij} satisfy $\lambda_{(i+1)j} \geq \lambda_{ij}$ and $\lambda_{i(j+1)} \geq \lambda_{ij}$. See, for example, Robertson, Wright and Dykstra (1988) for discussions of this alternative. This is a particularly interesting application from two points of view. First, it is a practical situation of interest and, second, the number of generators of the dual cone is too large to enable previous methods to establish the unbiasedness property. This makes the results of this paper more meaningful.

In Section 2 we will discuss the problem of testing the homogeneity of Poisson parameters against the matrix order alternative. We will indicate how each of the theorems of Section 1 can be utilized or applied to this problem. The connection between the Poisson distribution and the theorems concerning the multinomial distribution will become clear.

The theorems concerning the multinomial distribution apply to other hypothesis-testing problems dealing with Poisson parameters or multinomial parameters. Alternatives other than the matrix order alternative can be considered including the simple order alternative.

Theorems 1.1 and 1.4 can apply to problems involving distributions other than the multinomial and Poisson.

In Section 3 we give extensions and further discussion. Section 4 contains the proofs of the results concerned with COA, while Section 5 contains the proofs of the results concerned with SCO.

2. Matrix order alternative for Poisson parameters. Let Z_{ij} be independent Poisson variables with parameters λ_{ij}, $i = 1, \ldots, R$; $j = 1, \ldots, C$. Test H_0: $\lambda_{11} = \lambda_{12} = \cdots = \lambda_{RC}$ (all λ_{ij} are equal) vs. $H_1 - H_0$, where H_1: $\lambda_{ij} \leq \lambda_{(i+1)j}$, $\lambda_{ij} \leq \lambda_{i(j+1)}$, all (i, j). Let $\mathbf{Z}' = (Z_{11}, Z_{12}, \ldots, Z_{RC})$ be an $RC \times 1$ vector and let $\boldsymbol{\lambda}' = (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{RC})$. The alternative H_1 may be represented as a closed convex cone

$$(2.1) \qquad \mathscr{K}_B = \{\boldsymbol{\lambda} : B\boldsymbol{\lambda} \geq 0\},$$

with $\mathscr{K}_B^*$ a pairwise contrast cone. The matrix B of rank $(RC - 1)$ is determined in Cohen, Sackrowitz and Samuel-Cahn (1995b), Section 5.2, where it is also established that $\mathscr{K}_B^* \oplus H \supset \mathscr{K}_B$. [Note that the alternative can also be represented as $\{\boldsymbol{\omega} : B\boldsymbol{\omega} \geq 0\}$, where $\boldsymbol{\omega} = (\log \lambda_{ij})$, i.e., ω_{ij} is the natural parameter when the distribution of $\mathbf{Z}$ is expressed in exponential family form, and B is the same matrix as in (2.1).]

Next recognize that, under H_0, $T = \sum_{j=1}^C \sum_{i=1}^R Z_{ij}$ is a sufficient complete statistic. Therefore, any similar test of size α must have Neyman structure [see Lehmann (1986)]. This means any size α test must have conditional size

α for every given $T = t$. Since power functions of tests in this model must be continuous, any unbiased test must be similar. We seek unbiased tests so we consider tests performed conditionally for a given $T = t$. Any test that is conditionally unbiased will be unconditionally unbiased. Since the distribution of T in this Poisson model depends only on the sum of the parameters, it follows that a test that has a conditional monotone nondecreasing power function will have an unconditional monotone nondecreasing power function.

Next note that the conditional distribution of $\mathbf{Z} | T = t$ is multinomial $M(t, RC, \mathbf{p})$, where RC is the number of cells and $\mathbf{p}$ has ijth element $(\lambda_{ij} / \Sigma_{i=1}^{R} \Sigma_{j=1}^{C} \lambda_{ij})$. Our plan is to find conditionally unbiased tests of size α based on $\mathbf{Z}$ given $T = t$. In fact, we want to establish that test functions $\phi(\mathbf{Z})$ that are COM $[\mathscr{H}_B^*]$ are unbiased. Toward this end Theorem 4.1 of Cohen, Sackrowitz and Samuel-Cahn (1995a) states that such tests are unbiased provided (1) $\mathbf{Z}$ is COA $[\mathscr{H}_B^*]$ and (2) the ratio, consisting of the density of $\mathbf{Z} | T = t$ under H_1 divided by the density of $\mathbf{Z} | T = t$ under H_0, is COM $[\mathscr{H}_B^*]$. That condition (2) is true can be seen by writing the densities in the ratio in exponential family form. The ratio for fixed $T = t$ reduces to

$$(2.2) \qquad r(\mathbf{z}; \boldsymbol{\omega}, \boldsymbol{\omega}_0) = \beta(\boldsymbol{\omega}, \boldsymbol{\omega}_0, t) e^{\mathbf{z}' \boldsymbol{\omega}},$$

where $\boldsymbol{\omega}$ is an alternative point and $\boldsymbol{\omega}_0$ is a null point. However,

$$(2.3) \qquad \begin{aligned} r(\mathbf{z} + \mathbf{b}; \boldsymbol{\omega}, \boldsymbol{\omega}_0) &= \beta(\boldsymbol{\omega}, \boldsymbol{\omega}_0, t) e^{(\mathbf{z} + \mathbf{b})' \boldsymbol{\omega}} \\ &\geq \beta(\boldsymbol{\omega}, \boldsymbol{\omega}_0) e^{\mathbf{z}' \boldsymbol{\omega}} = r(z; \boldsymbol{\omega}, \boldsymbol{\omega}_0) \end{aligned}$$

for $\boldsymbol{\omega} \in \mathscr{H}_B$ and $\mathbf{b}$ a generator of $\mathscr{H}_B^*$ (i.e., $\mathbf{b}' \boldsymbol{\omega} \geq 0$). Hence (2.3) establishes condition (2). At this point we invoke Theorem 1.2 to establish condition (1) and thereby conclude that tests that are COM $[\mathscr{H}_B^*]$ of size α are unbiased. We must keep in mind that these are the conditional tests given $T = t$ but, as mentioned before, a test that is conditionally unbiased for $T = t$ a.e. is unconditionally unbiased.

Theorem 1.3 gives a stronger result for the very same problem. First, we will apply Theorem 1.3 with $\mathbf{Z} | T = t$ playing the role of $\mathbf{U}$, ϕ, a test function, playing the role of h and noting that $\mathbf{p}' \geq [\mathscr{H}_B^*] \mathbf{p}$ if and only if $\boldsymbol{\lambda}' \geq [\mathscr{H}_B^*] \boldsymbol{\lambda}$. Theorem 1.3 implies that, for the problem at hand, tests that are COM $[\mathscr{H}_B^*]$ of size α are not only unbiased but have monotone nondecreasing power functions on lines that are determined as follows. Let $\boldsymbol{\lambda}$ be any alternative point. Since $\mathscr{H}_B \subset \mathscr{H}_B^* \oplus H$, $\boldsymbol{\lambda}$ can be expressed as $\boldsymbol{\lambda}^* + \gamma \mathbf{1}$, with $\boldsymbol{\lambda}^* \in \mathscr{H}_B^*$. The power function is monotone nondecreasing on the line from $\gamma \mathbf{1}$ that passes through $\boldsymbol{\lambda}$. This is a consequence of Theorem 1.3 because suppose we consider alternatives $\boldsymbol{\lambda}_2 = a_2 \boldsymbol{\lambda}^* + \gamma \mathbf{1}$ and $\boldsymbol{\lambda}_1 = a_1 \boldsymbol{\lambda}^* + \gamma \mathbf{1}$, with $a_2 > a_1$. Then $\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1 = (a_2 - a_1) \boldsymbol{\lambda}^*$, which implies $\boldsymbol{\lambda}_2 \geq [\mathscr{H}_B^*] \boldsymbol{\lambda}_1$. The relation (1.5) means that the power function of a COM $[\mathscr{H}_B^*]$ test under $\boldsymbol{\lambda}_2$ is greater than or equal to the power function under $\boldsymbol{\lambda}_1$.

The fact that Theorem 1.3 yields a better result for this problem than Theorem 1.2 does not mean Theorem 1.2 does not have value. When $\mathbf{U}$ is COA it lends itself to other applications such as probability inequalities.

Theorem 1.4 is also applicable for this hypothesis-testing problem and yields the same conclusion as Theorem 1.3. The advantage of Theorem 1.3 is that it has a converse and so we know this stochastic cone ordering result holds only if $\mathscr{K}^*$ is a pairwise contrast cone.

Finally, Theorem 1.1 is applicable to this testing problem. In fact, Theorem 1.1 is valuable in actually determining unbiased tests. A method of constructing unbiased tests as noted in Cohen, Sackrowitz and Samuel-Cahn (1995b), Section 4, is to find test functions that are monotone nondecreasing functions of the Y_ν's of Theorem 1.1. The unbiasedness property ensues when the Y_ν's are associated.

We note that the conditional likelihood ratio test for this problem is COM $[\mathscr{K}_B^*]$ and so this test is unbiased with a monotone power function. (The likelihood ratio test performed unconditionally would not be similar and could not be unbiased.) As a consequence of Theorem 1.1, tests based on statistics that are nonnegative linear combinations of the Y_ν, or tests based on statistics equal to the $\max_{1 \le \nu \le p}[\rho_\nu Y_\nu]$, where ρ_ν are nonnegative weights and p is the number of generators in $\mathscr{K}$, have monotone power functions. The determination of $\mathbf{a}_\nu$ used to define Y_ν is outlined in Cohen, Sackrowitz and Samuel-Cahn (1995b), Section 5.2.

Tests based on the statistic $M = \max_{1 \le \nu \le p}[\rho_\nu Y_\nu]$ have considerable versatility. If one were interested in a special type of alternative (say differences in the rows of the matrix of parameters), this would suggest giving greater weight (through the ρ_ν's) to particular Y_ν's. [See Cohen, Sackrowitz and Samuel-Cahn (1995b) where this is accomplished in a normal testing problem.] If the test required rejection when $M > C_\alpha(t)$, one could determine the $C_\alpha(t)$, the critical value, by simulation.

3. Extensions and discussion. A theorem related to Theorem 1.2, which amounts to an improvement in the sufficiency part of that theorem, can be given. Before stating the new theorem, we let $\mathscr{K}$ be a closed convex cone and let $\mathscr{K}^*$ denote its dual. Let $S_{\mathscr{K}}^*$ denote the cone generated by the totality of all pairwise contrast vectors that lie in $\mathscr{K}^*$. Clearly, then, $\mathscr{K}^* \supset S_{\mathscr{K}}^*$. Let $S_{\mathscr{K}}$ be the dual of $S_{\mathscr{K}}^*$.

Suppose $\mathscr{C}$ is a cone and $\mathscr{D}$ is a cone such that $\mathscr{C} \subset \mathscr{D}$.

THEOREM 3.1. *Let $\mathbf{U}$ be a $k \times 1$ random vector. If $\mathbf{U}$ is COA $[\mathscr{C}]$, then $\mathbf{U}$ is COA $[\mathscr{D}]$. In particular, if $\mathbf{U}$ is COA $[S_{\mathscr{D}}^*]$, then $\mathbf{U}$ is COA $[\mathscr{K}^*]$.*

PROOF. Let $\mathscr{F}_{\mathscr{C}}$ be the class of COM $[\mathscr{C}]$ functions. Then $\mathscr{F}_{\mathscr{D}} \subset \mathscr{F}_{\mathscr{C}}$. Hence, for any pair of functions $h, g \in \mathscr{F}_{\mathscr{D}}$, we have

$$(3.1) \qquad Eh(\mathbf{U})g(\mathbf{U}) \ge Eh(\mathbf{U})Eg(\mathbf{U})$$

since $\mathbf{U}$ is COA $[\mathscr{C}]$. $\square$

In Section 2 we showed how A and COA can be used to establish the unbiasedness of constant size tests for the Poisson parameter matrix order

problem. These notions as well as SCO can be used to obtain similar types of results for other cone order alternatives and for other distributions. In Cohen, Sackrowitz and Samuel-Cahn (1995a, b) the focus is on the normal distribution. For the Poisson and multinomial distributions the results in all the theorems apply to any pairwise contrast cone $\mathscr{K}^*$ such that $\mathscr{K}^* \oplus H \supset \mathscr{K}$, which includes the simple order cone. Theorem 1.1 is not limited to a polyhedral cone and so it can apply, for example, to a problem discussed in Pincus (1975), which deals with a circular cone. See also Cohen, Sackrowitz and Samuel-Cahn (1995a) where Pincus is also referenced.

A remark is in order regarding the set of distributions satisfying the conditions of Theorem 1.4. Among exponential family distributions we name the Poisson, the binomial with different n's but the same p and the gamma with the same scale parameter but different degrees of freedom. Furthermore, if Theorem 1.4 is to be applied to a hypothesis-testing problem where the hypotheses are unchanged when monotone transforms of the X_i are made, other distributions qualify.

Theorem 1.3 is applicable to a hypothesis-testing problem considered by Kochar and El Barmi (1994). The problem is concerned with bivariate symmetry against ordered alternatives in a contingency table. Again, unbiased tests with monotone power functions can be identified as those which are COM $[\mathscr{K}^*]$, where $\mathscr{K}^*$ is the dual of the cone representing the alternative space for that problem.

The necessity parts of Theorems 1.2 and 1.3 are valuable in that they demonstrate that it is only in these situations that one can achieve the strong properties of COA and SCO. It is interesting and worthwhile to contrast the result of Theorem 1.4 with other preservation theorems. For example, Theorem 2 of Boland, Proschan and Tong (1992) is a special case of Theorem 1.4 where $\mathscr{K}$ is a simple order cone, that is,

$$(3.2) \qquad \mathscr{K} = \left\{ \boldsymbol{\theta} : \boldsymbol{\theta} \in \mathbb{R}^k, \ \theta_1 \leq \theta_2 \leq \cdots \leq \theta_k \right\}.$$

The SCO property of Theorem 1.4 is equivalent to the SO property of the vector of partial sums. Another type of preservation theorem of Boland, Proschan and Tong (1992), namely their Theorem 1, is also concerned with the simple order cone. This theorem is a special case of Theorem 2.1 of Cohen and Sackrowitz (1993), where the same theorem as in Boland, Proschan and Tong (1992) applies to a larger class of distributions. In contrast to our Theorem 1.4 here, the Cohen and Sackrowitz (1993) result is limited to the simple order cone, even though the class of distributions is enlarged.

Theorem 1.4 extends Theorem 1.1 of Proschan and Sethuraman (1977). In that theorem, Schur convexity is preserved. Schur convex functions can be thought of as COM functions where the ordering is majorization, which is, in a sense, a cone ordering but defined on vectors with ordered components, that is, $(x_{(1)}, x_{(2)}, \ldots, x_{(k)})$, where $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(k)}$.

We remark that the notions of COM, COA and SCO can be extended to any partial ordering of points in $\mathbb{R}^k$. They need not be limited to cone ordering of points. However, cone ordering of points has applications of interest.

4. Proofs of theorems concerned with COA. Let $\mathscr{K}$ be a closed convex cone and let $\mathscr{K}^*$ be its dual. Let $\{\mathbf{a}_\nu: \nu \in \Gamma\}$, Γ an index set, be a collection of generators of $\mathscr{K}$ and let $y_\nu = \mathbf{a}'_\nu \mathbf{x}$ for all $\nu \in \Gamma$. For ease of exposition we write $\mathbf{y} = \{y_\nu: \nu \in \Gamma\}$.

LEMMA 4.1. *Let $h(\mathbf{x})$ be any COM $[\mathscr{K}^*]$ function. Then there exists a nondecreasing function $h^*(\mathbf{y})$ such that $h^*(\mathbf{y}) = h(\mathbf{x})$.*

PROOF. Let Q denote the linear set in $\mathbf{y}$-space representing the range space of the mapping for which

(4.1) $\qquad \mathbf{x} \in \mathbb{R}^k$ exists with $\mathbf{a}'_\nu \mathbf{x} = y_\nu$ for all $\nu \in \Gamma$.

Clearly, $\dim Q \leq k$.

Now suppose $\mathbf{x}$ and $\mathbf{x}^*$ are such that $\mathbf{a}'_\nu \mathbf{x} = \mathbf{a}'_\nu \mathbf{x}^*$ for all ν. Then $h(\mathbf{x}) = h(\mathbf{x}^*)$. To see this, observe that $\mathbf{a}'_\nu(\mathbf{x} - \mathbf{x}^*) = 0$ for all $\nu \in \Gamma$ so that $(\mathbf{x} - \mathbf{x}^*) \in \mathscr{K}^*$ and $(\mathbf{x}^* - \mathbf{x}) \in \mathscr{K}^*$, that is, $\mathbf{x} \leq [\mathscr{K}^*]\mathbf{x}^*$ and $\mathbf{x}^* \leq [\mathscr{K}^*]\mathbf{x}$, implying $h(\mathbf{x}) = h(\mathbf{x}^*)$ since h is COM $[\mathscr{K}^*]$. In light of this we can define $h^*(\mathbf{y})$ as follows. For $\mathbf{y} \in Q$ take any $\mathbf{x}$ such that $y_\nu = \mathbf{a}'_\nu \mathbf{x}$, all ν, and define

(4.2) $\qquad h^*(\mathbf{y}) = h^*(\mathbf{a}'_\nu \mathbf{x}: \nu \in \Gamma) = h(\mathbf{x})$.

It is easily verified that h^* is a nondecreasing function of $\mathbf{y}$.

The function h^* is defined only on Q. It can be extended to be a nondecreasing function on all of $\mathbf{y}$-space by letting

(4.3) $\qquad h_1^*(\mathbf{y}) = \sup\{h^*(\mathbf{u}): \mathbf{u} \in Q, \mathbf{u} \leq \mathbf{y}\}$.

If there does not exist any $\mathbf{u} \in Q$ with $\mathbf{u} \leq \mathbf{y}$, $h_1^*(\mathbf{y}) = -\infty$. □

PROOF OF THEOREM 1.1. (i) Suppose $\mathbf{Y}$ is A. Consider COM $[\mathscr{K}^*]$ functions h_1 and h_2 of $\mathbf{X}$ and note

(4.4) $\qquad Eh_1(\mathbf{X})h_2(\mathbf{X}) = Eh_1^*(\mathbf{Y})h_2^*(\mathbf{Y})$,

where h_1^* and h_2^* are nondecreasing functions of Y that are determined from Lemma 4.1. Now use the fact that $\mathbf{Y}$ is A along with (4.4) to conclude that $\mathbf{X}$ is COA $[\mathscr{K}^*]$.

(ii) Suppose $\mathbf{X}$ is COA $[\mathscr{K}^*]$. If g_1 and g_2 are nondecreasing functions of $\mathbf{Y}$, then

(4.5) $\qquad Eg_1(\mathbf{Y})g_2(\mathbf{Y}) = Eg_1^*(\mathbf{X})g_2^*(\mathbf{X})$,

where $g_i^*(\mathbf{X}) = g_i(\mathbf{a}'_1\mathbf{X}, \mathbf{a}'_2\mathbf{X}, \ldots)$, $i = 1, 2$. Furthermore, note that if $\mathbf{b} \in \mathscr{K}^*$, then $\mathbf{a}'_\nu \mathbf{b} \geq 0$, which means that $\mathbf{a}'_\nu(\mathbf{X} + \mathbf{b}) \geq \mathbf{a}'_\nu \mathbf{X}$ which along with monotonicity of g_i implies that $g_i^*(\mathbf{X})$ is COM $[\mathscr{K}^*]$. Use this fact, (4.5) and the fact that $\mathbf{X}$ is COA $[\mathscr{K}^*]$ to conclude that $\mathbf{Y}$ is A. □

To prove Theorem 1.2, we need some lemmas.

LEMMA 4.2. *Suppose $\mathbf{Z} \sim \mathscr{M}(1, k, (1/k)\mathbf{1})$ is COA $[\mathscr{K}^*]$. Then $\mathbf{U} \sim \mathscr{M}(n, k, (1/k)\mathbf{1})$ is also COA $[\mathscr{K}^*]$.*

PROOF. Note $\mathbf{U} = \sum_{j=1}^{n}\mathbf{Z}_j$, where $\mathbf{Z}_j$ are iid and $\mathbf{Z}_j \sim \mathcal{M}(1, k, (1/k)\mathbf{1})$. We use induction and assume the lemma is true for $1, 2, \ldots, n-1$ after the hypothesis assures us it is true for 1. Now let h_1 and h_2 be COM $[\mathcal{K}^*]$ functions and consider

$$Eh_1(\mathbf{U})h_2(\mathbf{U})$$

$$(4.6) \qquad = E\left\{E\left\{h_1\left(\mathbf{Z}_n + \sum_{j=1}^{n-1}\mathbf{Z}_j\right)h_2\left(\mathbf{Z}_n + \sum_{j=1}^{n-1}\mathbf{Z}_j\right)\bigg|\sum_{j=1}^{n-1}\mathbf{Z}_j\right\}\right\}$$

$$\geq E\left\{E\left\{h_1\left(\mathbf{Z}_n + \sum_{j=1}^{n-1}\mathbf{Z}_j\right)\bigg|\sum_{j=1}^{n-1}\mathbf{Z}_j\right\}E\left\{h_2\left(\mathbf{Z}_n + \sum_{j=1}^{n-1}\mathbf{Z}_j\right)\bigg|\sum_{j=1}^{n-1}\mathbf{Z}_j\right\}\right\}$$

by using the hypothesis of the lemma. Furthermore, let

$$(4.7) \qquad h_i^*\left(\sum_{j=1}^{n-1}\mathbf{Z}_j\right) = E\left\{h_i\left(\mathbf{Z}_n + \sum_{j=1}^{n-1}\mathbf{Z}_j\right)\bigg|\sum_{j=1}^{n-1}\mathbf{Z}_j\right\}, \qquad i = 1, 2.$$

Note h_i^* are COM $[\mathcal{K}^*]$. This is true since if $\mathbf{b} \in \mathcal{K}^*$ and $\sum_{j=1}^{n-1}\mathbf{Z}_j = \mathbf{z}$, then $h_i(\mathbf{Z}_n + \mathbf{z} + \mathbf{b}) \geq h_i(\mathbf{Z}_n + \mathbf{z})$ because h_i are COM $[\mathcal{K}^*]$. Now use the induction hypotheses and (4.6) to complete the proof of Lemma 4.2. $\square$

Now let $\mathcal{K}_B$ be as in (1.3), where $\mathcal{K}_B^*$ is a pairwise contrast cone. Recall that H is the equiangular line.

LEMMA 4.3.

$$(4.8) \qquad\qquad\qquad \mathcal{K}_B^* \oplus H \supset \mathcal{K}_B$$

if and only if

$$(4.9) \qquad\qquad\qquad k\boldsymbol{\theta}_1'\boldsymbol{\theta}_2 \geq (\mathbf{1}'\boldsymbol{\theta}_1)(\mathbf{1}'\boldsymbol{\theta}_2)$$

for all $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathcal{K}_B$.

PROOF. (i) Necessity. Let $\bar{\boldsymbol{\theta}}_i = (1/k)[\sum_j \theta_{ij}, \sum_j \theta_{ij}, \ldots, \sum_j \theta_{ij}]'$, $i = 1, 2$. Note that if $\boldsymbol{\theta}_i \in \mathcal{K}_B$, then $(\boldsymbol{\theta}_i - \bar{\boldsymbol{\theta}}_i) \in \mathcal{K}_B$. This is true since $B(\boldsymbol{\theta}_i - \bar{\boldsymbol{\theta}}_i) = B\boldsymbol{\theta}_i \geq \mathbf{0}$. Also $(\boldsymbol{\theta}_i - \bar{\boldsymbol{\theta}}_i) \in \mathcal{K}_B^* \oplus H$ by hypothesis, and since $\boldsymbol{\theta}_i - \bar{\boldsymbol{\theta}}_i$ is orthogonal to H it follows that $(\boldsymbol{\theta}_i - \bar{\boldsymbol{\theta}}_i) \in \mathcal{K}_B^*$. Hence $(\boldsymbol{\theta}_i - \bar{\boldsymbol{\theta}}_i)$ lies in both $\mathcal{K}_B$ and $\mathcal{K}_B^*$, which implies (4.9).

(ii) Sufficiency. Assume (4.9) holds but $\boldsymbol{\theta}_1$ is such that $\boldsymbol{\theta}_1 \in \mathcal{K}_B$, but $\boldsymbol{\theta}_1 \notin \mathcal{K}_B^* \oplus H$. Also, if $\boldsymbol{\theta}_1 \in \mathcal{K}_B$, that implies $\boldsymbol{\theta}_1 - \bar{\boldsymbol{\theta}}_1 \in \mathcal{K}_B$ but $\boldsymbol{\theta}_1 - \bar{\boldsymbol{\theta}}_1 \notin \mathcal{K}_B^*$. This implies there exists a $\boldsymbol{\theta}_2 \in \mathcal{K}_B$ such that

$$(4.10) \qquad \left(\boldsymbol{\theta}_1 - \bar{\boldsymbol{\theta}}_1\right)'\boldsymbol{\theta}_2 = \left(\boldsymbol{\theta}_1 - \bar{\boldsymbol{\theta}}_1\right)'\left(\boldsymbol{\theta}_2 - \bar{\boldsymbol{\theta}}_2\right) < 0.$$

However, (4.10) contradicts (4.9) and so (4.8) is true. $\square$

At this point we let $\mathbf{e}_i = (0, \ldots, 1, 0, \ldots, 0)'$ be a $(k \times 1)$ vector with all 0's except a 1 for the ith coordinate, $i = 1, 2, \ldots, k$.

LEMMA 4.4. *Let* $\mathbf{U} \sim \mathcal{M}(1, k, (1/k)\mathbf{1})$. *Let* $\mathcal{K}_B$ *be as in* (1.3), *where* $\mathcal{K}_B^*$ *is a pairwise contrast cone and B has rank* $(k-1)$. *Then U is COA* $[\mathcal{K}_B^*]$ *if and only if* $\mathcal{K}_B^* \oplus H \supset \mathcal{K}_B$.

PROOF. Let h_1 and h_2 be functions which are COM $[\mathcal{K}_B^*]$. Since U is uniformly distributed on $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$, we may write

$$(4.11) \qquad Eh_1(\mathbf{U})h_2(\mathbf{U}) = \mathbf{h}_1'\mathbf{h}_2/k,$$

where $\mathbf{h}_i$ is the vector $[h_i(\mathbf{e}_1), h_i(\mathbf{e}_2), \dots, h_i(\mathbf{e}_k)]'$, $i = 1, 2$. Similarly,

$$(4.12) \qquad Eh_i(\mathbf{U}) = \overline{\mathbf{h}}_i, \qquad i = 1, 2.$$

Now let $\mathbf{b}$ be any row of B with a 1 in the βth position and a -1 in the γth position. Then $\mathbf{b} = \mathbf{e}_\beta - \mathbf{e}_\gamma$ and $\mathbf{b}'\mathbf{h}_1 = h_1(\mathbf{e}_\beta) - h_1(\mathbf{e}_\gamma)$. Similarly, for $\mathbf{b}'\mathbf{h}_2$. Since h_i, $i = 1, 2$, are COM $[\mathcal{K}_B^*]$, we have $h_i(\mathbf{e}_\beta) \geq h_i(\mathbf{e}_\gamma)$, $i = 1, 2$. Thus $\mathbf{b}'\mathbf{h}_i \geq \mathbf{0}$, which means $\mathbf{h}_i \in \mathcal{K}_B$ for $i = 1, 2$. Now invoke Lemma 4.3 and (4.11) and (4.12) to conclude the proof. $\square$

Now we can prove the following result.

THEOREM 1.2. *Let* $\mathbf{U} \sim \mathcal{M}(n, k, (1/k)\mathbf{1})$. *Let* $\mathcal{K}_B^*$ *be as in Lemma* 4.4. *Then* $\mathbf{U}$ *is COA* $[\mathcal{K}_B^*]$ *if and only if* $\mathcal{K}_B^* \oplus H \supset \mathcal{K}_B$.

PROOF. (i) *Sufficiency.* If $\mathcal{K}_B^* \oplus H \supset \mathcal{K}_B$, then Lemma 4.4 implies $\mathbf{U} \sim \mathcal{M}(1, k, (1/k)\mathbf{1})$ is COA $[\mathcal{K}_B^*]$. Lemma 4.2 in turn implies that $\mathbf{U} \sim \mathcal{M}(n, k, (1/k)\mathbf{1})$ is COA $[\mathcal{K}_B^*]$.

(ii) *Necessity.* Suppose $\mathbf{U} \sim \mathcal{M}(n, k, (1/k)\mathbf{1})$ is COA $[\mathcal{K}_B^*]$ but $\mathcal{K}_B^* \oplus H \not\supset \mathcal{K}_B$. Then by Lemma 4.3 there exist $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathcal{K}_B$ such that $(\boldsymbol{\theta}_1 - \overline{\boldsymbol{\theta}}_1)'(\boldsymbol{\theta}_2 - \overline{\boldsymbol{\theta}}_2) < 0$. Note $\boldsymbol{\theta}_i'\mathbf{U}$, $i = 1, 2$, is COM $[\mathcal{K}_B^*]$. However, $\mathrm{cov}(\theta_1'\mathbf{U}, \theta_2'\mathbf{U}) = (1/k)(\boldsymbol{\theta}_1 - \overline{\boldsymbol{\theta}}_1)'(\boldsymbol{\theta}_2 - \overline{\boldsymbol{\theta}}_2) < 0$. This is a contradiction to the fact that $\mathbf{U}$ is COA $[\mathcal{K}_B^*]$. $\square$

5. Proofs of theorems concerned with SCO.

PROOF OF SUFFICIENCY OF THEOREM 1.3. Recall $\mathbf{U} \sim \mathcal{M}(n, k, \mathbf{p})$ and $\mathcal{K}^*$ is a pairwise contrast cone. Let $\mathbf{p}^* \geq [\mathcal{K}^*]\mathbf{p}$ and let $h(\mathbf{u})$ be a COM $[\mathcal{K}^*]$ function. First, let $\mathbf{p}^*$ differ from $\mathbf{p}$ in only two coordinates and, without loss of generality, we let $\mathbf{p}^* = (p_1^*, p_2^*, \dots, p_k^*)'$, $\mathbf{p} = (p_1, p_2, \dots, p_k)'$ be such that $p_1^* \geq p_1$, and $p_2^* \leq p_2$, while $p_j^* = p_j$, $j = 3, \dots, k$. In other words, we assume, without loss of generality, that $(1, -1, 0, \dots, 0)$ is a generator of $\mathcal{K}^*$. Consider

$$(5.1) \qquad E_{\mathbf{p}^*} h(\mathbf{U}) = E_{\mathbf{p}^*}\{E_{\mathbf{p}^*}[h(\mathbf{U})|U_3, \dots, U_k]\}.$$

Since $\mathbf{U} \sim \mathcal{M}(n, k, \mathbf{p})$, when $n, U_3, \dots, U_k$ are fixed,

$$U_1 | n, U_3, \dots, U_k \sim B\left(n - \sum_{j=3}^k U_j, \frac{p_1}{1 - \sum_{j=3}^k p_j}\right).$$

Therefore, by the monotone likelihood ratio property of a binomial variable and the fact that $1 - \sum_{j=3}^{k} p_j = 1 - \sum_{j=3}^{k} p_j^*$, we have that (5.1) is greater than or equal to

$$(5.2) \qquad\qquad E_{\mathbf{p}^*}\{E_{\mathbf{p}}\{h(\mathbf{U})|U_3,\ldots,U_k\}\}.$$

Now, however, the marginal distribution of $(U_3,\ldots,U_k)$ under $\mathbf{p}^*$ is the same as under $\mathbf{p}$. Hence (5.2) equals

$$(5.3) \qquad\qquad E_{\mathbf{p}} h(\mathbf{U}),$$

and we have $(5.1) \geq (5.3)$, which proves sufficiency when $(\mathbf{p}^* - \mathbf{p})$ is a multiple of $(1, -1, 0, \ldots, 0)'$. When $\mathbf{p}^* \geq [\mathscr{K}^*]\mathbf{p}$ and $\mathbf{p}^* - \mathbf{p} = \sum_{i=1}^{m} \gamma_i \mathbf{b}_i$, $\gamma_i > 0$, m an integer, where $\mathbf{b}_i$ are generators of $\mathscr{K}^*$, we proceed step by step as before and have a chain of inequalities like $(5.1) \geq (5.3)$ to complete the proof. $\square$

To prove necessity, we will use the following simple lemma, which is very useful in constructing COM $[\mathscr{K}^*]$ functions.

LEMMA 5.1.   *Let V be any set and let $\hat{V} = V + \mathscr{K}^*$. Then*

$$(5.4) \qquad\qquad H(\mathbf{u}) = \begin{cases} 1, & \text{if } \mathbf{u} \in \hat{V}, \\ 0, & \text{otherwise} \end{cases}$$

is COM $[\mathscr{K}^*]$.

PROOF OF NECESSITY OF THEOREM 1.3.   Assume (1.5) holds and suppose $\mathscr{K}^*$ is not a pairwise contrast cone. Our plan is to exhibit a function $H(\mathbf{u})$ which is COM $[\mathscr{K}^*]$ such that $E_{\mathbf{p}} H(\mathbf{U})$ is not COM $[\mathscr{K}^*]$, that is, a function $H(\mathbf{u})$ for which (1.5) does *not* hold. This will provide the contradiction to establish necessity.

Recall $\mathbf{e}_i$ is the vector whose ith component is 1 and whose other components are 0. Let $\boldsymbol{\delta}_{ij} = \mathbf{e}_i - \mathbf{e}_j$ so that any pairwise contrast vector is some $\boldsymbol{\delta}_{ij}$. Since we assumed $\mathscr{K}^*$ is not a pairwise contrast cone, let $\mathbf{w}$ be a generator of $\mathscr{K}^*$, where $\mathbf{w} \neq \boldsymbol{\delta}_{ij}$ for any (i, j). Since $\mathbf{w}$ is a generator and $\mathscr{K}^*$ is convex, by the supporting hyperplane theorem, there exists r_0 such that $r_0'\mathbf{w} = 0$ and $r_0'\mathbf{u} \geq 0$ for all $\mathbf{u} \in \mathscr{K}^*$. Let S^* be the cone generated by all δ_{ij} such that $r_0'\delta_{ij} > 0$ or $\delta_{ij} \in \mathscr{K}^*$. We claim $\mathbf{w} \notin S^*$. Since if $\mathbf{w} \in S^*$, then $\mathbf{w} = \sum_{\delta_{ij} \in S^*} \lambda_{ij}\delta_{ij}$, $\lambda_{ij} \geq 0$. However, $0 = r_0'\mathbf{w} = \sum_{\delta_{ij} \in S^*} \lambda_{ij}r_0'\delta_{ij}$, which implies $\mathbf{w} = \sum_{\delta_{ij} \in \mathscr{K}^*} \lambda_{ij}\delta_{ij}$, contradicting the fact that $\mathbf{w}$ is a generator of $\mathscr{K}^*$.

Since $\mathbf{w} \notin S^*$, it follows that there exists $\mathbf{r}$ such that $\mathbf{r}'\mathbf{w} = 0$, $r'\delta_{ij} > 0$ for all $\delta_{ij} \in S^*$. Then there exists a generator of S, the dual of S^*, say $\mathbf{s}$, such that $\mathbf{w}'\mathbf{s} < 0$. If not, $\mathbf{w}'\mathbf{s} \geq 0$ for all generators of S, which would imply that $\mathbf{w} \in S^*$.

By the upper sets algorithm of Berk and Marcus (1996), it follows that $\mathbf{s}$ is of the form $(1, 1, \ldots, 1, 0, \ldots, 0)' - (m/k)\mathbf{1}$, where there are m 1's in $(1, 1, \ldots, 1, 0, \ldots, 0)'$.

Now $\boldsymbol{\delta}_{ij} \in S^*$ if and only if $\mathbf{r}'\boldsymbol{\delta}_{ij} > 0$ if and only if $r_i > r_j$. This implies that $r_{m+1}, \ldots, r_k$ must all be less than or equal to the $\min(r_1, \ldots, r_m)$. To see this, suppose, for example, that $r_{m+1} > r_m$. Then $\boldsymbol{\delta}_{m+1, m} \in S^*$ and by the upper

sets algorithm $s_{m+1} \geq s_m$, which contradicts the form of $\mathbf{s}$. Hence we may take, without loss of generality, $r_1 \geq r_2 \geq \cdots \geq r_m \geq r_{m+1} \geq \cdots \geq r_k$.

We are finally ready to construct $H(\mathbf{u})$. Let

$$(5.5) \qquad V = \{\mathbf{y} \colon \mathbf{y} = (n-1)\mathbf{e}_1 + \mathbf{e}_j, j = 1, \ldots, m\},$$

and let $\hat{V} = V + \mathscr{K}^*$. Then Lemma 5.1 implies $H(\mathbf{u})$, defined in (5.4) is COM $[\mathscr{K}^*]$.

Before computing $E_{\mathbf{p}}H(\mathbf{U})$, we show that $\mathbf{e}_q + (n-1)\mathbf{e}_1 \notin \hat{V}$ for $q > m$. Suppose $\mathbf{e}_q + (n-1)\mathbf{e}_1 \in \hat{V}$. Then there exists $\mathbf{y} \in V$ such that $(n-1)\mathbf{e}_1 + \mathbf{e}_q - \mathbf{y} \in \mathscr{K}^*$. This occurs if and only if $(n-1)\mathbf{e}_1 + \mathbf{e}_q - ((n-1)\mathbf{e}_1 + \mathbf{e}_i) \in \mathscr{K}^*$ for some $i = 1, 2, \ldots, m$. However, this would imply $\boldsymbol{\delta}_{qi} \in \mathscr{K}^*$ which in turn would imply $r_q > r_1$, which is a contradiction.

Recall $\mathbf{U} \sim \mathscr{M}(n, k, \mathbf{p})$ so

$$
\begin{aligned}
W(\mathbf{p}) = E_{\mathbf{p}}H(\mathbf{U}) &= p_1^n + np_1^{n-1}(p_2 + \cdots + p_m) \\
(5.6) \qquad &+ \sum_{\mathbf{u} \in \hat{V} - V} R(\mathbf{u}) \prod_{i=1}^{k} p_i^{\gamma_i(\mathbf{u})},
\end{aligned}
$$

where $R(\mathbf{u})$ is a positive function of $\mathbf{u}$, $\gamma_1(\mathbf{u}) \leq n - 2$ and $\sum_{i=2}^{k} \gamma_i(\mathbf{u}) \geq 2$. It follows from Lemma 3.1 of Cohen, Sackrowitz and Samuel-Cahn (1995a) that $W(\mathbf{p})$ is COM $[\mathscr{K}^*]$ if and only if $\mathbf{b}' \nabla W(\mathbf{p}) \geq 0$ for all $\mathbf{b} \in \mathscr{K}^*$ and all $\mathbf{p}$, where $\nabla W(\mathbf{p})$ is the gradient of W. Now choose $\mathbf{b} = \mathbf{w}$, the generator of $\mathscr{K}^*$ that is not a $\boldsymbol{\delta}_{ij}$. We compute

$\mathbf{w}' \nabla W(\mathbf{p})$

$$
(5.7) \qquad = \mathbf{w}' \begin{pmatrix} np_1^{n-1} & + & n(n-1)p_1^{n-2}(p_2 + \cdots + p_m) & + & P_1(\mathbf{p}) \\ np_1^{n-1} & & + & & P_2(\mathbf{p}) \\ \vdots & & & & \vdots \\ np_1^{n-1} & & + & & P_m(\mathbf{p}) \\ & & P_{m+1}(\mathbf{p}) & & \\ & & \vdots & & \\ & & P_k(\mathbf{p}) & & \end{pmatrix}
$$

$$
= np_1^{n-1}\mathbf{w}'\mathbf{s} + \mathbf{w}' \begin{pmatrix} P_1^*(\mathbf{p}) \\ P_2(\mathbf{p}) \\ \vdots \\ P_k(\mathbf{p}) \end{pmatrix},
$$

where $P_1^*(\mathbf{p}) = n(n-1)p_1^{n-2}(p_2 + \cdots + p_m) + P_1(\mathbf{p})$ and $P_j(\mathbf{p})$, $j = 1, 2, \ldots, k$, are functions of $\mathbf{p} = (p_1, \mathbf{p}^{(2)})'$, $\mathbf{p}^{(2)} = (p_2, \ldots, p_k)'$ such that $P_j(\mathbf{p}) \to 0$ as $\mathbf{p}^{(2)} \to \mathbf{0}$. However, since $\mathbf{w}'\mathbf{s} < 0$, it follows that (5.7) is negative for some $\mathbf{p}$, which implies that $W(\mathbf{p})$ is not COM $[\mathscr{K}^*]$. $\square$

PROOF OF THEOREM 1.4.   The proof of this theorem uses the ideas of the sufficiency proof of Theorem 1.3 and uses the steps in the proof of Theorem J.2 of Chapter 3 of Marshall and Olkin (1979). We omit the details.

REFERENCES

AHMED, A.-H. N., LEÓN, R. and PROSCHAN, F. (1981). Generalized association, with applications in multivariate statistics. *Ann. Statist.* **9** 168–176.

BERK, R. and MARCUS, R. (1996). Dual cones, dual norms and simultaneous inference for partially ordered means. *J. Amer. Statist. Assoc.* **91** 318–328.

BOLAND, P. J., PROSCHAN, F. and TONG, Y. L. (1992). A stochastic ordering of partial sums of independent random variables and of some random processes. *J. Appl. Probab.* **29** 645–654.

COHEN, A. and SACKROWITZ, H. B. (1993). On stochastic ordering of partial sums with application to reliability theory. In *Statistics and Probability* (J. K. Ghosh, S. K. Mitra, K. R. Parthasarathy and B. L. S. Prakasa Rao, eds.) 165–170. Wiley Eastern, New Delhi.

COHEN, A. and SACKROWITZ, H. B. (1994). Association and unbiased tests in statistics. In *Stochastic Orders and Their Application* (M. Shaked and J. G. Shanthikumar, eds.). Academic Press, San Diego.

COHEN, A., SACKROWITZ, H. B. and SAMUEL-CAHN, E. (1995a). Cone order association. *J. Multivariate Anal.* **55** 320–330.

COHEN, A., SACKROWITZ, H. B. and SAMUEL-CAHN, E. (1995b). Constructing tests for normal order restricted inference. *Bernoulli* **1** 321–333.

ESARY, J. D., PROSCHAN, F. and WALKUP, D. W. (1967). Association random variables with applications. *Ann. Math. Statist.* **38** 1466–1474.

KOCHAR, S. C. and EL BARMI, H. (1994). Likelihood ratio tests for bivariate symmetry against ordered alternatives in a square contingency table (abstract 94t-15). *IMS Bulletin* **38**.

LEHMANN, E. L. (1986). *Testing Statistical Hypotheses*. Wiley, New York.

MARSHALL, A. W. and OLKIN, I. (1979). *Inequalities*: *Theory of Majorization and Its Applications*. Academic Press, New York.

PINCUS, (1975). Testing linear hypotheses under restricted alternatives. *Math. Oper. Statist.* **6** 733–751.

PROSCHAN, F. and SETHURAMAN, J. (1977). Schur functions in statistics. I. The preservation theorem. *Ann. Statist.* **5** 256–262.

ROBERTSON, T., WRIGHT, F. T. and DYKSTRA, R. L. (1988). *Order Restricted Statistical Inference*. Wiley, New York.

DEPARTMENT OF STATISTICS
RUTGERS UNIVERSITY
HILL CENTER FOR THE MATHEMATICAL SCIENCES
BUSCH CAMPUS
PISCATAWAY, NEW JERSEY 08855
E-MAIL: artcohen@rci.rutgers.edu
        hsack@rci.rutgers.edu