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A REPRESENTATION OF PARTIALLY ORDERED PREFERENCES[1]

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This essay considers decision-theoretic foundations for robust Bayesian statistics. We modify the approach of Ramsey, de Finetti, Savage and Anscombe and Aumann in giving axioms for a theory of *robust* preferences. We establish that preferences which satisfy axioms for robust preferences can be represented by a set of expected utilities. In the presence of two axioms relating to state-independent utility, robust preferences are represented by a *set* of probability/utility pairs, where the utilities are almost state-independent (in a sense which we make precise). Our goal is to focus on preference alone and to extract whatever probability and/or utility information is contained in the preference relation when that is merely a partial order. This is in contrast with the usual approach to Bayesian robustness that begins with a class of "priors" or "likelihoods," and a single loss function, in order to derive preferences from these probability/utility assumptions.

1. Introduction and overview.

1.1. *Robust Bayesian preferences*. This essay is about decision-theoretic foundations for robust Bayesian statistics. The fruitful tradition of Ramsey (1931), de Finetti (1937), Savage (1954) and Anscombe and Aumann (1963) seeks to ground Bayesian inference on a *normative* theory of rational choice. Rather than accept the traditional probability models and loss functions as given, Savage is explicit about the foundations. He axiomatizes a theory of preference using a binary relation over acts, $A_1 \lesssim A_2$, "act A_1 is not preferred to act A_2." Then, he shows that $\lesssim$ is represented by a unique personal probability (state-independent) utility pair according to subjective expected utility. That is, he shows there exists exactly one pair (p, U) such that, for all acts A_1 and A_2, $A_1 \lesssim A_2$ if and only if $E_{p,U}[A_1] \le E_{p,U}[A_2]$. [More precisely, in Savage's theory what is needed to justify the assertion that p is the agent's personal probability is the added assumption that each consequence has a constant value in each state. Unfortunately this is ineffa-

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ble in Savage's language of preference over acts. [See Schervish, Seidenfeld and Kadane (1990)]. We discuss this in Section 4, below.

In recent years, either under the headings of *Bayesian robustness* [Berger (1985), Section 4.7; Hartigan (1983), Chapter 12; Kadane (1984)] or *sensitivity analysis* [Rios Insua (1990)] it has become an increasingly important issue to show how to arrive at Bayesian conclusions from logically weaker assumptions than are required by the traditional Bayesian theory. Given data and a particular likelihood from a statistical model, for example, how large a class of prior probabilities leads to a class of posterior probabilities that are in agreement about some event of interest? Our work differs from the common trend in Bayesian robustness in much the same way that Savage's work differs from the traditional use of probability models and loss functions in Bayesian decision theory. Our goal is to axiomatize robust preferences directly, rather than to robustify given statistical models. Results in this theory are strikingly different from those obtained in the existing Bayesian robustness literature.

For an illustration of the difference, suppose two Bayesian agents each rank the desirability of Anscombe–Aumann ("horse lottery") acts according to his/her subjective expected utility. ("Horse lotteries" are defined in Section 2.) Let (p_1, U_1) and (p_2, U_2) be the probability/utility pairs representing these two decision makers and assume they have different beliefs and values: that is, assume $p_1 \neq p_2$ and $U_1 \neq U_2$. Denote by $\prec_1$ and $\prec_2$ their respective (strict) preference relations, each a weak order over acts. Suppose now our goal is to find those coherent (Anscombe–Aumann) preference relations $\lesssim$ corresponding to probability/utility pairs (p, U) such that the following *Pareto condition* applies:

If $E_{p_1, U_1}[A_1] < E_{p_1, U_1}[A_2]$ and $E_{p_2, U_2}[A_1] < E_{p_2, U_2}[A_2]$, then $E_{p, U}[A_1] < E_{p, U}[A_2]$. In words, when both agents strictly prefer act A_2 to act A_1, then this shared preference is robust for all efficient, cooperative Bayesian decisions that the pair make together. [We assume that though the two Bayesian agents may discuss their individual preferences, nonetheless, some differences remain in their beliefs and in their values even after such conversations. See DeGroot (1974) for a rival model.] We have the following theorem.

THEOREM 1 [Seidenfeld, Kadane and Schervish (1989)]. *Assume there exists a pair of prizes $\{r_*, r^*\}$ which the two agents rank in the same order: $r_* \prec_i r^*$ $(i = 1, 2)$. Then the set of probability/utility pairs, each of which satisfies the Anscombe–Aumann theory and each of which agrees with the strict preferences shared by these two decision makers, consists exactly of the two pairs themselves $\{(p_1, U_1), (p_2, U_2)\}$. There are no other coherent, Pareto compromises.* [*There is no coherent weak order meeting the strong Pareto condition, which requires that $A_1 \prec A_2$ if $A_1 \precsim_i A_2$ $(i = 1, 2)$ and at least one of these two preferences is strict.*]

Thus, with respect to Pareto-robust preferences, the set of probability/utility pairs for the problem of two distinct Bayesians is not connected and therefore not convex. Hence, a common method of proof—separating hyperplanes (used to develop expected utility representations)—is not available in our investigation. This is just one way in which our methods differ from the usual robust Bayesian analysis. We want the strict preferences held in common by two Bayesians to be a special case of robust preferences. Applied to a class of weak orders, the Pareto condition creates a strict partial order $\prec$: to wit, the binary relation $\prec$ is irreflexive and transitive.

Our view of robustness is that sometimes a person does not have a (strict) preference for act A_1 over act A_2 nor for A_2 over act A_1 nor are they indifferent options. Assume that strict preference is a transitive relation. Then such a person's preferences are modeled by a partial order. We ask, under what assumptions on this partial order is there a set of probability/utility pairs agreeing with it according to expected utility theory, a set which characterizes that partial order? In this we are exploring the possibility pointed out by Savage [(1954), page 21].

The general form of our inquiry is as follows. Axiomatize coherent preference $\prec$ as a partial order and establish a representation for it in terms of a set of probability/utility pairs. That is, we characterize each coherent, partially ordered preference $\prec$ in terms of the set of coherent weak orders $\{\precsim\}$ that extend it. We rely on the usual expected utility theory to depict each coherent weak order $\precsim$ by one probability/utility pair (p, U). Thus, we model $\prec$ by a set of probability/utility pairs.

In contrast with Savage's theory, which uses only personal probability, our approach is based on Anscombe–Aumann's "horse lottery" theory. Preferences over "horse lotteries" accommodate both personal and extraneous (agent-invariant) probabilities. Also, by characterizing strict preference in terms of a set of probability/utility pairs, we improve so-called one-way representations of, for example, Fishburn [(1982), Section 11], as we show more than existence of an agreeing probability/utility pair.

1.2. *Overview.* In outline, our approach is as follows: In Section 2 we introduce axioms for a partial order over Anscombe–Aumann horse lotteries (HL). Anscombe–Aumann theory contains three substantive axioms that incorporate the (von Neumann–Morgenstern) theory of cardinal utility for simple acts:

1. A postulate that preference $(\precsim)$ is a weak order—analogous to Savage's P1.
2. The independence postulate—analogous to Savage's P2, "sure thing."
3. An Archimedean condition, which plays an analogous role to Savage's P6.

Our replacement axioms for these are:

HL AXIOM 1. A postulate that strict preference $(\prec)$ is a strict partial order.

HL Axiom 2. The independence postulate.

HL Axiom 3. A modified Archimedean axiom for discrete (not just simple) lotteries.

To avoid triviality, a commonplace assumption of expected utility theory is that not all acts are indifferent, for example, there exist two acts, **W** and **B** that do *not* satisfy $\mathbf{B} \precsim \mathbf{W}$. Let $\prec$ obey our (three) preference axioms on a domain of horse-lottery acts $\mathbf{H_R}$. We show (Theorem 2) how to extend $\prec$ to a preference $\prec'$ over a larger domain that includes two new acts **B** (best) and **W** (worst) where

$$(\forall\, H_1, H_2 \in \mathbf{H_R}) \qquad \big[(\mathbf{B} \prec' H_1 \prec' \mathbf{W})\ \&\ (H_1 \prec H_2 \text{ if and only if } H_1 \prec' H_2)\big].$$

Then we establish three related theorems (Theorems 3, 4 and 5): $\prec$ is represented by a nonempty (maximal and convex) set $\mathscr{V} = \{V: \mathbf{H_R} \to (0,1)\}$ of bounded, real-valued cardinal utilities $V(\cdot)$ defined for acts. Each $V \in \mathscr{V}$ induces a weak order $\precsim_V$ that agrees with $\prec$ on the domain of simple acts and almost agrees (Definition 10b) with $\prec$ on all acts. Moreover, given a set $\mathscr{Z}$ of bounded, real-valued cardinal (so-called) "linear" utilities, $Z(\cdot)$ defined on $\mathbf{H_R}$, the partial order formed using the Pareto condition with the set $\mathscr{Z}$ satisfies our three axioms for preference.

In the light of the surprising "shape" that the family of agreeing subjective expected utilities can have (Theorem 1), we employ a modification of Szpilrajn's (1930) transfinite induction for extending a partial order. We show how to extend a partial order while preserving the other preference axioms. The proofs of all results appear in the Appendix. *Also, we number definitions, lemmas, and corollaries to coincide with their logical order in our arguments, regardless of whether they appear for the first time in the body of the text or in the Appendix.*

In Section 4 we turn our attention to the representation of $\mathscr{V}$ as a set of subjective expected utilities. We discuss when a linear utility V over acts also is a subjective expected utility for a probability/utility pair (p, U). Corollary 4.1 gives a representation of $\mathscr{V}$ in terms of sets of probability/state-dependent utility pairs $(p, \{U_j: j = 1, \dots, n\})$, where the utility $U_j(L)$ of a (von Neumann–Morgenstern) lottery L may depend upon the accompanying state s_j. (This follows up the issue raised in the first paragraph in Section 1.1.) In Section 4.3 we introduce two axioms (HL Axioms 4 and 5) that parallel the fourth Anscombe–Aumann postulate. That postulate (and our replacements for it) permits a representation of preference using a (nearly) state-independent utility: where (with high personal probability) the value of a lottery L does not depend (by more than amount $\varepsilon > 0$) upon the state in which it is awarded. In Section 4.3 we lean heavily on the proof technique of Section 3 in order to find a representation for the partial order $\prec$ in terms of a set of agreeing pairs of probabilities and (nearly) state-independent utilities, Lemma 4.3 and Theorem 6.

Section 5 is about conditional preference. Two theorems (Theorems 7 and 8) relate conditional (called-off) partially ordered preferences and Bayesian updating of the family of unconditional personal probabilities that agree with an unconditional partially ordered preference. We provide an example involving conditional probability that highlights the nonconvexity of the agreeing sets. In Section 6 we conclude with a review of several features that distinguish our results.

2. The formal theory.

2.1. *The act space*: *a domain for the preference relation*. We provide a representation for a partially ordered strict preference relation over (discrete) Anscombe–Aumann (1963) horse lotteries—acts that generalize von Neumann–Morgenstern (1947) lotteries to allow for uncertainty over states of nature.

Let $\mathbf{R}$ be a set of *rewards*. We develop our theory for countable sets $\mathbf{R}$.

DEFINITION 1. A *simple* (von Neumann–Morgenstern) *lottery* is a simple probability distribution P over $\mathbf{R}$, that is, a distribution with finite support. A *discrete lottery* is a countably additive probability over R (with a countable support). Denote a lottery by L and its distribution by P.

Horse lotteries are defined with respect to a finite partition of states. Let π be a finite partition of the sure event S into n disjoint, mutually exhaustive nonempty (sets of) states, $\pi = \{s_1, \dots, s_n\colon s_i \cap s_j = \varnothing \text{ iff } i \neq j \text{ and } \bigcup_{j \leq n}(s_j) = S\}$. Strictly speaking, elements of π are subsets of S. We take this approach rather than supposing S is finite, for example, rather than assuming $S = \{s_1, \dots, s_n\}$. Then our analysis allows for elaborations of a given preference relation in a larger domain of acts defined over (finite) refinements of the partition π. Having made this point, we allow ourselves the familiar convention of equating the set state s_j with its elements. That is, for notational convenience, often we shall use s_j when we intend "members of s_j."

DEFINITION 2. A *simple* (or *discrete*) *horse lottery* is a function from states to simple (or to discrete) lotteries. Denote a horse lottery by H and denote the space of (discrete) horse lotteries on the reward set $\mathbf{R}$ by $\mathbf{H_R}$.

In the tradition where acts are functions from states to outcomes, a horse lottery is an act with a lottery outcome. For example, the act that yields a 50–50 chance at \$10 and \$20 provided the Republicans win the next Presidential election, and which yields a 0.25 chance at \$5 and a 0.75 chance at \$10 if the Republicans do not win, is a horse lottery over a binary partition with two states: Republicans win and Republicans do not win. Thus, a *constant* horse lottery is just a von Neumann–Morgenstern lottery, and a proper subset of these are the constant von Neumann–Morgenstern lotteries, that is, the acts which yield a specific reward for certain.

Next, we define the operation of convex combination of two horse lotteries, "+", as the state-by-state mixture of their respective v.N-M lottery outcomes. Thus:

DEFINITION 3. $xH_1 + (1-x)H_2 = H_3 = \{xL_{1j} + (1-x)L_{2j}\colon\ j = 1, \dots, n;\ 0 \le x \le 1\}$.

The mixture of two lotteries is a lottery $xL_1 + (1-x)L_2 = L_3$, where $P_3(r) = xP_1(r) + (1-x)P_2(r)$. For the special cases where each H is a "constant" act, that is, if H_1 is the lottery L_1 and H_2 is L_2, Definition 3 coincides with the von Neumann–Morgenstern operation of "+" for lotteries.

2.2. *The axioms for order and independence*. The von Neumann–Morgenstern theory of preference over (simple) lotteries is encapsulated by three axioms:

1. The assumption that preference $\precsim$ is a weak order relation.
2. The independence postulate (formulated below).
3. An Archimedean condition (discussed below).

These axioms may be applied to horse lotteries also. Then the three axioms guarantee that (i) preference is represented by a (cardinal) utility V over acts with a property (ii) that utility distributes over convex combinations. To wit, given these three axioms:

(i) There exists a real-value V defined on acts, unique up to positive linear transformations, where $V(H_1) \le V(H_2)$ if and only if $H_1 \precsim H_2$; and
(ii) $V[xH_1 + (1-x)H_2] = xV(H_1) + (1-x)V(H_2)$.

DEFINITION 4. When a preference relation over acts satisfies (i) and (ii) we say it has the *expected* (or *linear*) *utility* property and we call V an agreeing expected (or, linear) utility for $\precsim$.

Anscombe–Aumann theory requires a fourth postulate ensuring the existence of a unique decomposition of V as a subjective expected, *state-independent* utility. That is, subject to a fourth postulate for preference, there exists a (unique) personal probability p defined on states and a utility U defined on lotteries (independent of states) so that:

$$\text{(iii)} \qquad V(H) = \sum_{j=1}^{n} p(s_j)U(L_j).$$

[Recall the notation $H(s_j) = L_j$.] Key, here, is that U is a state-independent utility, defined on lotteries independent of the state in which they occur. To be precise, let H_L be the constant horse lottery that yields lottery L in each state, $H_L(s_j) = L$.

DEFINITION 5. The utility $\{U_j: j = 1, \ldots, n\}$ is *state-independent* when, for each lottery L and pair of states s_j and $s_{j'}$,

$$U_j(L) = U_{j'}(L) = U(L).$$

[For our purposes, and following the usual practice, the condition of Definition 5 is required only for states s_j and $s_{j'}$ that are not "null," i.e., only when $p(s_j) \neq 0$ is it worth restricting U_j in a decomposition of a linear utility V.] If the utility is state-independent, for convenience, we drop the subscript (for states) and abbreviate it U. When (iii) obtains for a state-independent utility U, $V(H_L) = U(L)$.

DEFINITION 6. When a preference relation over acts satisfies (i)–(iii) we say it has the *subjective expected* (*state-independent*) *utility* property and we say the pair (p, U) agrees with $\precsim$.

In contrast to (iii), a decomposition of V by a subjective (possibly) state-dependent utility allows

$$\text{(iii*)} \qquad V(H) = \sum_{j=1}^{n} p(s_j) U_j(L_j).$$

We examine such state-dependent decompositions in Section 4.1.

Next, we propose versions of the first two Anscombe–Aumann axioms to accommodate our theory of preference as a partial order. We postpone our discussion of the Archimedean axiom to Section 2.4 to allow for a timely account of "indifference" in Section 2.3.

In this paper, a partial order $\prec$ identifies a *strict* preference relation.

HL AXIOM 1. $\prec$ is a strict partial order. It is a transitive and irreflexive relation on $\mathbf{H_R} \times \mathbf{H_R}$.

DEFINITION 7. When neither $H_1 \prec H_2$ nor $H_2 \prec H_1$, we say the two lotteries are *incomparable* by preference, which we denote as $H_1 \sim H_2$. When $\sim$ is transitive—corresponding to a weak order—then the relation $\precsim$ (standing for "$\prec$ or $\sim$") identifies a *weak* preference relation. Hereafter, we shall mean by "preference" the strict preference relation.

HL AXIOM 2 (Independence). $\forall$ (H_1, H_2 and H_3) and for each $0 < x \leq 1$,

$$xH_1 + (1 - x)H_3 \prec xH_2 + (1 - x)H_3 \quad \text{if and only if } H_1 \prec H_2.$$

This version of independence may be used, also, as the second axiom in the Anscombe–Aumann theory or in the von Neumann–Morgenstern theory.

2.3. *Indifference* ($\approx$). Next, we define a (transitive) relation of indifference, $\approx$, based on $\prec$, which will play a central role in our extension of $\prec$ to a weak order. [See Fishburn (1979), Exercises 9.1 and 9.4, pages 126–127, for additional discussion.]

DEFINITION 8 (Indifference). $H_1 \approx H_2$ iff $\forall\, H_3, H_4$ $(0 < x \le 1)$,

$$xH_1 + (1-x)H_3 \sim H_4 \quad \text{iff } xH_2 + (1-x)H_3 \sim H_4.$$

We establish several useful corollaries of the HL Axioms 1 and 2 about indifference.

COROLLARY 2.1. *$\forall\, H_1, H_2$, if $H_1 \approx H_2$, then $H_1 \sim H_2$.*

That is, when two acts are indifferent, neither is preferred to the other.

COROLLARY 2.2. *$\approx$ is an equivalence relation.*

COROLLARY 2.3. *$H_1 \approx H_2$ if and only if $\forall\, H_3, H_4$, and $0 < x \le 1$,*

$$xH_1 + (1-x)H_3 \prec (\succ) H_4 \quad \textit{iff } xH_2 + (1-x)H_3 \prec (\succ) H_4.$$

COROLLARY 2.4.

$$\forall\, 0 < x \le 1, H, H_1 \approx H_2 \quad \textit{iff } xH_1 + (1-x)H \approx xH_2 + (1-x)H.$$

Corollaries 2.3 and 2.4 establish important substitution properties for elements of the same indifference equivalence class.

2.4. *The Archimedean axiom*: *continuity of preference*. First, define *convergence* for acts. Let $\{H_n\}$ be a denumerable sequence of horse lotteries.

DEFINITION 9. $\{H_n\}$ *converges* to a lottery H^*, denoted by $\{H_n\} \Rightarrow H^*$, just in case the respective discrete lottery distributions $\{P_j^n(\cdot)\}$ converge (pointwise) to the lottery distribution $P_j^*(\cdot)$.

The third (Archimedean) axiom precludes infinitesimal degrees of preference. As we show in Theorem 4, it suffices for representing preferences by sets of agreeing real-valued utilities.

HL AXIOM 3. Let $\{H_n\} \Rightarrow H$ and $\{M_n\} \Rightarrow M$.

(a) If $\forall\, n(H_n \prec M_n)$ and $(M \prec N)$, then $(H \prec N)$.
(b) If $\forall\, n(H_n \prec M_n)$ and $(J \prec H)$, then $(J \prec M)$.

The familiar Archimedean condition from Anscombe–Aumann theory (also from von Neumann–Morgenstern theory), denoted here as Axiom 3*, is this:

AXIOM 3*. Whenever $H_1 \prec H_2 \prec H_3$, $\exists\, (0 < x, y < 1)$, $yH_1 + (1-y)H_3 \prec H_2 \prec xH_1 + (1-x)H_3$.

However, Axiom 3* is overly restrictive for our purposes, as a simple example shows.

EXAMPLE 2.1. Consider a set of three rewards $\mathbf{R} = \{r_w, r^*, r_b\}$ and a minimal, one element partition comprising the single sure state $\pi = \{s\}$. Then $\mathbf{H_R}$ is the set of von Neumann–Morgenstern lotteries on $\mathbf{R}$. (Denote by $\mathbf{r}$ the horse lottery with constant prize r.) Let $\mathscr{V} = \{V_x: 0 < x < 1\}$ be a (convex) set of linear utilities, where $V_x(\mathbf{r}_w) = 0$, $V_x(\mathbf{r}^*) = x$ and $V_x(\mathbf{r}_b) = 1$. Figure 1 graphs these utilities.

Let $\prec_{\mathscr{V}}$ be the partial order on lotteries generated by this set of utilities according to the weak Pareto condition. That is, $L_1 \prec_{\mathscr{V}} L_2$ iff $(\forall\ V \in \mathscr{V})$ $E_V[L_1] < E_V[L_2]$. By Theorem 4 (below), $\prec_{\mathscr{V}}$ satisfies HL Axioms 1 and 2. However, it fails Axiom 3*. Specifically, $\mathbf{r}_w \prec_{\mathscr{V}} \mathbf{r}^* \prec_{\mathscr{V}} \mathbf{r}_b$, but $\forall\ (0 < y < 1)$, $y\mathbf{r}_w + (1-y)\mathbf{r}_b \sim_{\mathscr{V}} \mathbf{r}^*$. Of course, $\prec_{\mathscr{V}}$ is represented by the (convex) set of agreeing (real-valued) utilities $\mathscr{V}$.

Next, we provide a connection between the Archimedean condition HL Axiom 3 and $\approx$-indifference.

COROLLARY 2.5. *Let* $\{H_n\}, \{H'_n\} \Rightarrow H$; $\{M_n\}, \{M'_n\} \Rightarrow M$. *If* $\forall\ n(H_n \prec M_n$ *and* $M'_n \prec H'_n)$, *then* $H \approx M$.

We conclude this discussion of HL Axiom 3 by showing that the familiar Archimedean axiom, Axiom 3*, follows from our replacement HL Axiom 3 when preference over lotteries is a weak order.

LEMMA 2.1. *If* $\precsim$ *is a weak order over discrete horse lotteries meeting conditions HL Axioms* 2 *and* 3, *then* $\precsim$ *satisfies Axiom* 3*.

2.5. *Utility for discrete lotteries.* The theory of von Neumann and Morgenstern addresses preference over simple lotteries, that is, those with finite support. These constitute the subdomain of constant, simple horse lotteries. However, there are weakly ordered preferences over lotteries which satisfy the expected utility hypothesis (Definition 4) for simple lotteries, that is, which are represented by a cardinal, linear utility V over the domain of *simple* lotteries, but which fail the expected utility hypothesis over the larger domain of discrete lotteries. [See Fishburn's (1979), Section 10, discussion;

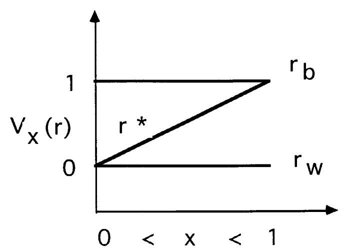

FIG. 1.

also Fishburn (1982), Section 11.3. A similar problem arises in Savage's (1954) theory, as shown by Seidenfeld and Schervish (1983). In a related matter, Aumann's (1962, 1964) argument about a utility for a partially ordered preference does not apply when the set of rewards is denumerable rather than finite, even though all lotteries are simple. Kannai (1963) showed that, and strengthened Aumann's Archimedean condition to remedy the problem.] We address this problem with an extended dominance condition.

Let $\mathbf{r}$ denote the simple horse lottery with constant prize r. Recall that H_L denotes the constant horse lottery that yields L in each state. Consider the following dominance principle:

DOMINANCE. $\forall\,(\mathbf{r}, H_L)$ if for each $r_n \in \operatorname{supp}(L)$, $\mathbf{r_n} \prec \mathbf{r}$, then it is not the case that $(\mathbf{r} \prec H_L)$ [or, alternatively, if universally, $\mathbf{r} \prec \mathbf{r_n}$, then not $(H_L \prec \mathbf{r})$].

This weak dominance condition contrasts each reward r with the lottery L through the (countably many) constant horse lotteries $\mathbf{r_n}$ taken from L's support. The condition precludes a preference for L over $\mathbf{r}$ if it occurs for $\mathbf{r}$ over each $\mathbf{r_n}$.

Our first three axioms yield dominance, as the next lemma establishes.

LEMMA 2.2. *HL Axioms* 1–3 *entail dominance*.

Based on Lemma 2.2, we may apply Fishburn's [(1979), page 139] Theorem 10.5 to argue that a weakly ordered preference $\precsim$ over *discrete* (horse) lotteries which satisfies our HL Axioms 2 and 3 has the expected utility property. (See Remark 1.) The import for our representation theorems is given in terms of *agreeing* and *almost agreeing* utilities for a partial order:

DEFINITION 10a. A utility V *agrees with* a partial order $\prec$ iff $V[H_1] < V[H_2]$ whenever $(H_1 \prec H_2)$.

DEFINITION 10b. A utility V *almost agrees* with a partial order $\prec$ iff $V[H_1] \leq V[H_2]$ whenever $(H_1 \prec H_2)$.

Thus, when our strategy for extending a partial order $\prec$ to a weak order $\precsim$ succeeds, it induces a linear utility V that agrees with $\prec$ for *discrete* horse lotteries, not just for simple ones.

Unfortunately, our argument for extending a partial order $\prec$ produces a set of expected utilities $\{V\}$ each of which agrees with $\prec$ for simple acts, and only *almost agree* with it for discrete acts. Of course, by itself the condition of "almost agreeing" is quite weak. A utility V that makes all options indifferent almost agrees with every partially ordered preference. The point, however, is that we consider an almost agreeing utility for a partial order $\prec$ only in the case in which it agrees with $\prec$ on all simple acts. [This idea parallels a similar distinction between a qualitative probability (a weak order on events) and a quantitative probability that agrees or almost agrees with it. See

Savage (1954), Section 3.3.] Through Corollary 3.2, we provide a sufficient condition for the existence of a (convex) set of utilities that agree with $\prec$ on all of $\mathbf{H_R}$.

REMARK 1. Fishburn's Theorem 10.5 uses the traditional Archimedean axiom Axiom 3*. However, by Lemma 2.2, Axiom 3* follows from the assumptions that $\precsim$ is a weak order satisfying HL Axioms 2 and 3. Also, dominance is equivalent to Fishburn's [(1979, page 138] Axiom 4c, given the other three axioms and our structural assumptions about the domain of lotteries.

2.6. *Bounded preferences*. Next in our discussion of the axioms, we settle the question whether a partial order $\prec$ satisfying HL Axioms 1–3 admits an *unbounded* utility that agrees with it or agrees with it on simple lotteries. It is well known that utilities for von Neumann–Morgenstern lotteries are finite. In light of our assumption that all discrete lotteries are acts, utilities that agree with $\prec$ are bounded as well.

COROLLARY 2.6. *Let* $\prec$ *satisfy HL Axioms* 1 *and* 2. *If* V *agrees with* $\prec$, *it is bounded*. *Hence*, *all utilities that agree with* $\prec$ *are bounded*.

As noted above, we sometimes construct a utility V that agrees with a partial order $\prec$ for all simple lotteries but (merely) almost agrees with $\prec$ for discrete lotteries. Thus, as V may fail to agree with $\prec$ on nonsimple acts, it is worthwhile to show (appealing only to simple acts) that each utility V we construct is bounded. For this purpose we formalize a condition that a partially ordered preference is bounded, and establish it as a corollary of two of our axioms, HL Axioms 1 and 3. From the fact that a partial order $\prec$ is bounded, we show that a utility V agreeing with it on simple acts also is bounded.

Call a countable (finite or denumerably infinite) sequence of lotteries $\{H_n: n = 1, \dots\}$ an *increasing* (*decreasing*) *chain* if $H_i \prec H_j$ ($H_j \prec H_i$) whenever $i < j$. The following concepts deal with chains of strict preference.

DEFINITION 11a. Say $\prec$ is *bounded above* if, for each increasing chain $\{H_n\}$,

$$\lim_{n \to \infty} \sup\{x : (H_2 \prec xH_1 + (1-x)H_n)\} < 1.$$

DEFINITION 11b. Say $\prec$ is *bounded below* if, for each decreasing chain $\{H_n\}$,

$$\lim_{n \to \infty} \sup\{x : (xH_1 + (1-x)H_n \prec H_2)\} < 1.$$

DEFINITION 11c. Call $\prec$ *bounded* if it is bounded both above and below.

LEMMA 2.3. *If* $\prec$ *satisfies HL Axioms* 1 *and* 3, *then* $\prec$ *is bounded*, *that is*, *all* $\prec$*-chains are bounded.*

Also, Lemma 2.3 yields the following claim about utilities for rewards:

COROLLARY 2.7. *Let W be a* (*real-valued*) *utility and assume that*, *in the domain of simple lotteries*, *its strict order* $\prec_W$ *satisfies HL Axioms* 1 *and* 3. *Then* $\sup_{\mathbf{R}}|W(r)| < \infty$.

Recall that a linear utility V is defined only up to a positive linear transformation. We use the facts reported by Corollaries 2.6 and 2.7 to standardize the units (0 and 1) for each V in a set of agreeing utilities $\mathscr{V}$ (agreeing with $\prec$ on simple acts, at least).

DEFINITION 11d. A set $\mathscr{V} = \{V\}$ of utilities is *bounded* if, for some standardization of its elements,

$$\sup_{\mathscr{V}, \mathbf{H_R}} |V(H)| < \infty.$$

The problem we face is this. Though each $V \in \mathscr{V}$ is bounded, there exist what are for our purposes undesirable standardizations of the V's which fail to satisfy Definition 11d. For an illustration, recall Example 2.1. There, the domain of (simple) lotteries $\mathbf{H_R}$ is generated by three rewards $\mathbf{R} = \{r_w, r^*, r_b\}$ using a partition of one (sure) state. That is, Example 2.1 is about preferences over von Neumann–Morgenstern lotteries. A partially ordered preference $\prec_{\mathscr{V}}$ over $\mathbf{H_R}$ arises (by the Pareto rule) from the convex set of utilities $\mathscr{V} = \{V_x\colon 0 < x < 1\}$, where $V_x(\mathbf{r}_w) = 0$, $V_x(\mathbf{r}^*) = x$ and $V_x(\mathbf{r}_b) = 1$. That is, $H_{L1} \prec_{\mathscr{V}} H_{L2}$ iff $V(H_{L1}) < V(H_{L2})$ for each $V \in \mathscr{V}$.

Obviously, the two constant acts (the rewards) $\mathbf{r_w}$ and $\mathbf{r_b}$ bound the partial order $\prec_{\mathscr{V}}$, that is, for each act H_L different from $\mathbf{r_w}$ and $\mathbf{r_b}$, $\mathbf{r_w} \prec_{\mathscr{V}} H \prec_{\mathscr{V}} \mathbf{r_b}$. Moreover, in this standardization of $\mathscr{V}$, $\sup_{\mathscr{V}, \mathbf{H_R}}|V(H)| = 1$. Hence, it is a bounded set of utilities. However, we may standardize the elements of $\mathscr{V}$ so that it fails the condition in question. Rewrite each V_X, instead, so that $V_X(\mathbf{r}_w) = 0$, $V_X(\mathbf{r}^*) = 1$ and $V_X(\mathbf{r}_b) = 1/X$. Then $\lim_{X \to 0} V_X(\mathbf{r}_b) = \infty$.

To ensure a simple standardization which establishes our $\mathscr{V}$s are, indeed, bounded sets of utilities, we verify that (without loss of generality) we may introduce two rewards W and B (analogous to r_w and r_b in Example 2.1) that serve to bound the preferences for all other acts: Theorem 2. Then, the sets $\mathscr{V}$ are bounded sets of utilities since we standardize all $V \in \mathscr{V}$ with $V(\mathbf{W}) = 0$ and $V(\mathbf{B}) = 1$.

2.7. *Standardizing* $\prec$*-preferences with* "*best*" *and* "*worst*" *acts.* In this section we show how to extend the domain of a partially ordered preference by bounding it with "worst" and "best" acts. First, however, we review two concepts of "null" events.

DEFINITION 12. An *event* e is the set of states in a subset T of π: $(\forall\, e)\, \exists\, (T \subset \pi)$, $e = \bigcup_{s \in T}[s]$.

DEFINITION 13. Call H_1 and H_2 a pair of *e-called-off* acts when $H_1(s) = H_2(s)$ if $s \notin e$.

Distinguish two senses of "null" events.

DEFINITION 14a. An event e is *potentially null* iff for each pair of e-called-off acts H_1 and H_2, $H_1 \sim H_2$.

DEFINITION 14b. Event e is *essentially null* iff for each pair of e-called-off acts H_1 and H_2, $H_1 \approx H_2$.

It is evident that when event e is essentially null, so too is each state that comprises it. Denote by $\mathbf{n}$ the union of the essentially null states. It follows (as is proven next) that the union of essentially null states is an essentially null event. Hence, $\mathbf{n}$ is the maximal essentially null event.

COROLLARY 2.8. *Let $N \subset \pi$ be the subset of all essentially null states $N = \{s_{j_1}, \dots, s_{j_k}\}$, with $\mathbf{n} = \bigcup_{s \in N}(s)$. Then $\mathbf{n}$ is essentially null.*

THEOREM 2. *Assume $\prec$ is a partially ordered preference (satisfying HL Axioms* 1–3) *over a set of discrete horse lotteries $\mathbf{H_R}$, defined on the partition $\pi^n = \{s_j\colon j = 1, \dots, n\}$. Let $\mathbf{R}' = \mathbf{R} \cup \{\mathbf{W}, \mathbf{B}\}$, where neither W nor B is an element of $\mathbf{R}$. Then we may extend $\prec$ to a partially ordered preference $\prec'$ over $\mathbf{H_R}$, so that*:

1. $\prec'/\mathbf{H_R} = \prec$. *That is, $\prec'$ restricted to $\mathbf{H_R}$ is just $\prec$.*
2. $\forall\, (H \in \mathbf{H_R})$, $\mathbf{W} \prec' H \prec' \mathbf{B}$.
3. $\prec'$ *satisfies HL Axioms* 1–3.

Since $\mathbf{n} = S$ iff $\prec$ is trivial, that is, iff $\forall\, (H_1, H_2)$, $H_1 \approx H_2$ [also, iff $\forall\, (H_1, H_2)$, $H_1 \sim H_2$], without loss of generality, by Theorem 2, assume preference is not trivial by including rewards W and B. (This proposition, warranted by Theorem 2, is the counterpart in our theory to Savage's P5.)

3. Extending strict partial orders: the inductive argument.

3.1. *An overview.* Let $\mathbf{R} = \{r_1, r_2, \dots\}$ be a countable (finite or denumerable) set of rewards and let $\prec$ be a preference over $\mathbf{H_R}$ satisfying HL Axioms 1–3. Based on Theorem 2, without loss of generality, assume the existence of two distinguished rewards *not* in $\mathbf{R}$: reward W, where $\mathbf{W}$ is the worst act, and reward B, where $\mathbf{B}$ is the best act. Acts $\mathbf{W}$ and $\mathbf{B}$ are to serve as the common 0 and 1 in a (convex) set $\mathscr{V}$ of bounded utility functions V that agree with $\prec$. Hence for all $H \in \mathbf{H_R}$, $\mathbf{W} \prec H \prec \mathbf{B}$.

Let us highlight the major results in this section of our essay.

Our strategy is to use a transfinite induction to extend the preference $\prec$ (a partial order) to a weak order $\precsim$ over *simple* horse lotteries in $\mathbf{H_R}$. Let $\prec$ $(= \prec_0)$ serve as the basis for the induction. The induction at the ith stage extension of $\prec$, $\prec_i$, obtains by assigning a utility v_i to act $\tilde{H}_i$, $V(\tilde{H}_i) = v_i$, so that $\tilde{H}_i \approx_i v_i\mathbf{B} + (1 - v_i)\mathbf{W}$. The quantity v_i is chosen (in accord with Definitions 20 and 25 in the Appendix) from a (convex) set of *target utilities* for $\tilde{H}_i$, $\mathscr{T}_i(\tilde{H}_i)$. The sequence $\{\tilde{H}_i\}$ is chosen (see Definition 26 in the Appendix) so that the limit stage $\prec_\omega$ is a weak order over $\mathbf{H_R}$. We use $\mathbf{W}$ and $\mathbf{B}$ as the 0 and 1 of our utility, as follows.

Assume $\{H_n\} \Rightarrow H$ and $H_n \in \mathbf{H_R}$. The general target sets $\mathscr{T}_i(H)$ are defined through endpoints that bound the candidate utilities:

DEFINITION 17. Let $v_i^*(H)$ be the lim inf of the quantities x_n for which $H_n \prec_{i-1} x_n\mathbf{B} + (1 - x_n)\mathbf{W}$.

DEFINITION 18. Let $v_{i*}(H)$ be the lim sup of the quantities x_n for which $x_n\mathbf{B} + (1 - x_n)\mathbf{W} \prec_{i-1} H_n$.

[The "utility" bounds $v_*(H)$ and $v^*(H)$ do not depend upon which sequence $\{H_n\} \Rightarrow H$ is used, as explained in the Appendix.] Next, define the (*closed*) *target set of utilities* for an act $H \in \mathbf{H_R}$:

DEFINITION 19. $\mathscr{T}(H) = \{v\colon v_*(H) \le v \le v^*(H)\}$.

We report two key properties of $\mathscr{T}(H)$ with the following lemma.

LEMMA 3.1. *Assume $\prec$ satisfies the three axioms. Then*:

(i) $v_*(H) \le v^*(H)$; *and*
(ii) $v_*(H) = v^*(H) = v_H$ *iff* $H \approx v_H B + (1 - v_H)W$.

Our plan succeeds because $\prec_i$ extends $\prec_{i-1}$, it satisfies the three HL axioms and it preserves the $\approx_{i-1}$-indifference relations. Each weak order $\precsim_V = \precsim_\omega$ (corresponding to the limit stage relation "$\prec_\omega$ or $\approx_\omega$") is defined by inequalities in expected V-utility, based on the utilities v_i for each act $\tilde{H}_i$ in a (finite or) denumerable class $\mathscr{H} \subset \mathbf{H_R}$. (As explained below, $\mathscr{H}$ is finite or denumerable depending upon whether $\mathbf{R}$ is.) We choose $\mathscr{H}$ to form a basis for $\precsim_V$, that is, each $H \in \mathbf{H_R}$ is a limit point of simple acts and each simple act has its utility fixed by some finite stage of the transfinite induction. Then, $\precsim_V$ extends $\prec$ on simple acts in $\mathbf{H_R}$. Also, the utility V *almost agrees* with $\prec$ over the discrete lotteries $\mathbf{H_R}$. That is, if $H_1 \prec H_2$, then $H_1 \precsim_V H_2$. In Corollary 3.1, we provide sufficient conditions under which $\precsim_V$ extends $\prec$ for all the acts in $\mathbf{H_R}$. (See Remark 2.)

We show in Theorem 4 that each set $\mathscr{Z}$ of (bounded, standardized) real-valued utility functions over $\mathbf{R}$ induces a partial order, $\prec_{\mathscr{Z}}$, according to the Pareto preference relation, and $\prec_{\mathscr{Z}}$ satisfies our axioms. Of course, each utility $Z \in \mathscr{Z}$ agrees with $\prec_{\mathscr{Z}}$. That is, $\mathscr{Z}$ is a subset of the set of all utilities agreeing with $\prec_{\mathscr{Z}}$. However [Seidenfeld, Schervish and Kadane (1990), E.3], distinct convex sets of bounded utilities may induce the same strict partial order. Thus, our representation of the partial order $\prec$ is in terms of the largest convex set of agreeing linear utilities—the union of all sets of utilities where each set induces $\prec$ according to the Pareto condition.

Assume $\prec$ satisfies our axioms and let $\mathscr{Z}^{\mathscr{S}}$ be the nonempty (convex) set of bounded utilities that agree with $\prec$ for simple acts. That is, $\mathscr{Z}^{\mathscr{S}}$ is the set of all bounded utilities with the property that, for simple acts H_1 and H_2, $H_1 \prec H_2$ only if for each utility Z in $\mathscr{Z}^{\mathscr{S}}$, the expected Z-utility of H_2 is greater than that of H_1. Let $\mathscr{V}$ be the nonempty (convex) set of utilities created for $\prec$ by (our method of induction in) Theorem 3. Theorem 5 asserts $\phi \neq \mathscr{V} = \mathscr{Z}^{\mathscr{S}}$. Last, when the conditions of Corollary 3.1 apply, then (Corollary 3.2) $\mathscr{V}$ is the nonempty set of all utilities that agree with $\prec$.

REMARK 2. If the Archimedean axiom is ignored and only simple lotteries are considered, the induction for extending $\prec$ to a weak order (in fact, to a total order) $\precsim$ over $\mathbf{H_R}$ is elementary and applies without a cardinality restriction on the reward set $\mathbf{R}$ and without need of the special acts $\mathbf{W}$ and $\mathbf{B}$. See the Appendix to Schervish and Kadane (1990). There, we show the following: Let κ be the cardinality of $\mathbf{R}$. Using Hausner's (1954) result, the order $\precsim$ $(= \prec_\kappa)$ is a lexicographic expected utility.

3.2. *The central theorem.*

THEOREM 3. *Let* $\prec$ *be a nontrivial partial order over* $\mathbf{H_R}$ *satisfying* HL *Axioms* 1–3. *Then*:

(i) *For simple lotteries in* $\mathbf{H_R}$, $\prec$ *can be extended to a weak order* $\precsim_\omega = \precsim$ *satisfying* HL *Axioms* 2 *and* 3. *That is,* $\precsim$ *is uniquely represented by a* (*bounded*) *real-valued utility* V *over* $\mathbf{R}$ *which agrees with* $\prec$ *for simple acts. In symbols,* $\forall$ (*simple* $H_1, H_2 \in \mathbf{H_R}$), *if* $(H_1 \prec H_2)$, *then* $E_V[H_1] < E_V[H_2]$, *and if* $(H_1 \approx H_2)$, *then* $E_V[H_1] = E_V[H_2]$.

(ii) V *almost agrees with* $\prec$. $\forall$ $(H_1, H_2 \in \mathbf{H_R})$, *if* $(H_1 \prec H_2)$, *then* $E_V[H_1] \leq E_V[H_2]$.

It is instructive to illustrate how $\precsim_\omega$ may fail to agree with $\prec$ for some nonsimple lotteries in $\mathbf{H_R}$. The example motivates a condition on $\prec$ which proves sufficient for $\prec_\omega$ to extend $\prec$.

EXAMPLE 3.1. Let $\mathscr{W} = \{W_j\colon j = 1, \dots\}$ be a countable set of utilities on $\mathbf{R} = \{r_i\colon i = 1, \dots\}$ with the two properties that $W_j(\mathbf{r}_m) = 0.25$ if $m \neq 2j$, while $W_j(\mathbf{r}_{2j}) = 0.5$. According to Theorem 4 (below), under the (weak) Pareto rule, $\mathscr{W}$ induces a partial order $\prec_{\mathscr{W}}$ which satisfies our three horse lottery axioms. Define the constant, nonsimple acts H_a and H_b by $H_a = \{P(r_i) =$

$1/2^m$ if $i = 2m - 1$, $P(r_i) = 0$ otherwise} and $H_b = \{P(r_i) = 1/2^m$ if $i = 2m$, $P(r_i) = 0$ otherwise}. Then, evidently $(H_a \prec_{\mathscr{W}} H_b)$. However, at the kth stage $\prec_k$ in the extension of $\prec_{\mathscr{W}}$, we may arrange our choices of utilities for rewards so that $V(r_k) = 0.25$ $(k = 1, \dots)$. However, then $\precsim_\omega$ does not extend $\prec_{\mathscr{W}}$ as $H_a \approx_\omega H_b$.

DEFINITION 28. Given two subsets of $\prec$-preferences $\mathscr{Q}$ and $\mathscr{R}$, say that $\mathscr{Q}$ *is a basis for* $\mathscr{R}$ if every preference, $(H_1 < H_2) \in \mathscr{R}$, is a consequence (under HL Axioms 1–3) of preferences in $\mathscr{Q}$.

COROLLARY 3.1. *If there exists a countable basis* $\mathscr{B}$ *for* $\prec$, *then there exists a (bounded) real-valued utility* V *and corresponding weak order* $\precsim$ *that agrees with* $\prec$ *on all of* $\mathbf{H}_\mathbf{R}$. (*Thus, a sufficient condition for the existence of an agreeing* $\precsim$ *is that* $\prec$ *is a separable partial order.*)

Next, we show that our axioms are not overly restrictive for representing a partial order by a (convex) set of agreeing utilities. We investigate relationships between a partial order $\prec_{\mathscr{Z}}$ (formed by the Pareto rule with a set $\mathscr{Z}$ of utilities) and the set $\mathscr{V}$ of utilities created by induction on $\prec_{\mathscr{Z}}$. Let $\mathscr{Z}$ be a set of bounded utilities on $\mathbf{R}$, standardized so that for $Z \in \mathscr{Z}$ and $H \in \mathbf{H}_\mathbf{R}$, $0 = Z(\mathbf{W}) < Z(H) < Z(\mathbf{B}) = 1$. Define the relation $\prec_{\mathscr{Z}}$ on $\mathbf{H}_\mathbf{R}$ by the Pareto condition:

DEFINITION 29. $(H_1 \prec_{\mathscr{Z}} H_2)$ iff $\forall\, Z \in \mathscr{Z}$, $Z(H_1) < Z(H_2)$.

THEOREM 4. $\prec_{\mathscr{Z}}$ *satisfies* HL *Axioms* 1–3.

Next, let $\mathscr{V}$ be the set of utilities that can be generated from the partial order $\prec$ according to our induction. Let $\mathscr{Z}^{\mathscr{S}}$ be the set of all bounded utilities Z that agree with $\prec$ on simple lotteries.

THEOREM 5. $\phi \neq \mathscr{V} = \mathscr{Z}^{\mathscr{S}}$.

Last, assume $\prec$ satisfies our axioms, let $\mathscr{Z}$ be the set of all utilities that agree with $\prec$ and let $\mathscr{V}$ be the set of utilities created by (our induction in) Theorem 3. We state three immediate corollaries of Theorem 5:

COROLLARY 3.2. *When* $\prec$ *satisfies the separability condition of Corollary* 3.1, *then* $\phi \neq \mathscr{V} = \mathscr{Z}$.

COROLLARY 3.3. *The set* $\mathscr{V}$ *does not depend upon the ordering of* $\mathscr{H}$.

COROLLARY 3.4. *The set* $\mathscr{V}$ *is convex.*

4. A representation of $\prec$ in terms of probabilities and state-dependent utilities.

4.1. *The underdetermination of personal probability by* HL *Axioms* 1–3. Let $\mathscr{V}$ be the set of utilities V, each of which (by Theorem 5) corresponds to a limit stage $\precsim_V$ in our inductive extensions of the partial order $\prec$. According to Theorem 5, $\mathscr{V}$ is the set of all and only utilities that agree with $\prec$ on simple acts. According to Corollary 3.2, when $\prec$ is separable, $\mathscr{V}$ is the set of utilities that agree with $\prec$. We examine decompositions of $V \in \mathscr{V}$ as a subjective expected (state-dependent) utility.

Let $\precsim$ be a weak order over the discrete horse lotteries $\mathbf{H_R}$. Let $p(\cdot)$ be a (personal) probability defined on states in π, with $P(\mathbf{n}) = 0$ for the set of $\precsim$-null states $\mathbf{n}$. Finally, let $U_j(\cdot)$ be a (possibly) state-dependent utility on the discrete v.N-M lotteries $\mathbf{L_R}$, defined for the $\precsim$-nonnull states s_j. That is, for each $\precsim$-nonnull state, $s_j \notin \mathbf{n}$, U_j is a v.N-M utility. (For completeness, we may take U_j to be a constant function when $s_j \in \mathbf{n}$.)

DEFINITION 30. Say that $\precsim$ represented as a *subjective expected* (*state-dependent*) *utility* by the pair $(p, \{U_j\colon j = 1, \dots, n\})$, whenever

$$H_1 \precsim H_2 \quad \text{iff} \quad \sum_j p(s_j) U_j(L_{1j}) \le \sum_j p(s_j) U_j(L_{2j}). \tag{4.1}$$

For convenience, abbreviate the probability/(state-dependent) utility pairs as (p, U_j).

We rely on a result due to Fishburn [(1979), Theorem 13.1] to show that each $\precsim_V$, $V \in \mathscr{V}$, bears the subjective expected utility property for a large class of (p, U_j) pairs. In fact, for each such $\precsim_V$, the (p, U_j) pairs range over all mutually absolutely continuous probabilities defined on the $\precsim_V$-nonnull states. Specifically:

LEMMA 4.1. *Let* $\precsim$ *be a* (*nontrivial*) *weak order on* $\mathbf{H_R}$ *satisfying* HL *Axioms* 2 *and* 3. *For each probability* $p(\cdot)$ *with support the* (*nonempty*) *set of* $\precsim$*-nonnull states*, *there is a* (*possibly*) *state-dependent utility* $U_j(\cdot)$ *on discrete lotteries for which* $\precsim$ *has property* (4.1) *under* (p, U_j).

Putting Theorem 5 and Lemma 4.1 together, we have the following corollary:

COROLLARY 4.1. *There exists a set of pairs* $\{(p(\cdot), U_j(\cdot))\}$, *with* p *a* personal probability *defined on the set of* $\precsim_V$*-nonnull states in* p *and* U_j *a* state-dependent utility *over the discrete lotteries*, *where* $\forall\ (H \in \mathbf{H_R})$, $V(H) = \sum_j p(s_j) \times U_j(L_j)$.

[Being linear utilities, the U_j have the expected utility property for lotteries. That is, $U_j(xL_1 + (1-x)L_2) = xU_j(L_1) + (1-x)U_j L_2$. Moreover, for each $V \in \mathscr{V}$, the set of personal probabilities $\{p\colon \exists\ U_j$ with $\precsim_V$ represented by $(p, U_j)]$ is closed under the relation of mutual absolute continuity.]

4.2. *State-independent utilities and a counterexample*. Anscombe and Aumann's theory of horse lotteries introduces a fourth axiom which suffices for a *unique* expected utility representation of a weak order $\precsim$ by a pair (p, U), with p a personal probability over states and U a *state-independent* utility over rewards.

Recall Definition 5, when U is state-independent, a lottery L has the same utility independent of the (nonnull) state s_j. Hence [as in Savage's (1954) theory], U assigns a constant utility across (nonnull) states to each "constant" act. In Anscombe and Aumann's theory, then (4.1) is strengthened to read:

$$H_1 \precsim H_2 \quad \textit{iff} \quad \sum_j p(s_j)U(L_{1j}) \leq \sum_j p(s_j)U(L_{2j}) \tag{4.2}$$

and each $\precsim$ is so represented by a *unique* (p, U) pair.

The existence of a state-independent utility for $\precsim$ is assured through a contrast between (unconditional) preferences over constant horse lotteries and preferences over s_j-called-off horse lotteries: pairs of acts that differ only in one state. Specifically, let H_{L_i} $(i = 1, 2)$ be two constant horse lotteries that award, respectively, the v.N-M lottery L_i in all states. Let H_i $(i = 1, 2)$ be two s_j-called-off horse lotteries with $H_i(s_j) = L_i$ [and $H_1(s) = H_2(s)$ for $s \neq s_j$]. The Anscombe–Aumann (AA) axiom for state-independent utility reads:

AA Axiom 4. Provided $s_j \notin \mathbf{n}$, for each such quadruple of acts, $H_{L_1} \precsim H_{L_2}$ iff $H_1 \precsim H_2$.

(Recall, their Axiom 1 stipulates that preferences are weakly ordered, $\precsim$; hence, in their theory there is no difference between "potentially null" and "essentially null" states.)

This axiom requires that $\precsim$-preferences over "constant" acts (such as the H_{L_i}) are reproduced by *called-off* choices (the H_i) given each nonnull s_j. The unconditional preference for v.N-M lotteries is their conditional (that is, called-off) preference, given a nonnull state. (We discuss conditional partially ordered preferences in Section 5.)

It is significant to understand that AA Axiom 4, though sufficient to create state-independent utilities when preference satisfies the usual ordering, independence and Archimedean conditions, does not preclude alternative expected utility representations by *state-dependent* utilities. Lemma 4.1 continues to apply, even in the presence of the extra axiom for state-independent utilities. Weak orderings that satisfy the independence, Archimedean and state-independent utility axioms admit a continuum of different probability/utility representations, each in accord with (4.1).

What the Anscombe–Aumann fourth axiom achieves, however, is to guarantee that precisely one probability/utility pair, among the set of all pairs $\{(p, U_j)\}$ indicated by Lemma 4.1, satisfies the more restrictive condition, (4.2). In Anscombe and Aumann's theory, as in Savage's theory, this probability/utility pair (p, U) is given priority over the others. That is, these theories select the one (and only one) expected state-independent utility representation of preference, in accordance with (4.2) and, thereby, fix a personal probability uniquely from $\precsim$-preferences.

We are not satisfied with a conventional resolution of the representation problem indicated by Lemma 4.1. If state-dependent utilities are plausible candidates for an agent's values, and we think sometimes they are, then the measurement question remains open despite the fourth axiom. What justification is there for a convention which gives priority to state-independent values? In two essays [Schervish, Seidenfeld and Kadane (1990, 1991)], we examine the case of weakly ordered preferences without the extra axiom for "state-independent" utility. Here, instead, we adopt the strategy of imposing a modified Axiom 4 and asking which probability/state-independent utility pairs agree with the partial order $\prec$. Unlike the Anscombe–Aumann or Savage theories, ours does *not* assert that these pairs of probability/(state-independent) utility functions identify the agent's degrees of beliefs and values.

We adapt Anscombe and Aumann's final axiom to our construction by restricting it to states which are not potentially null. This produces the following axiom:

HL AXIOM 4. If s_k is not $\prec$-potentially null, then for each quadruple of acts H_{L_i}, H_i $(i = 1, 2)$ as described above, $H_{L_1} \prec H_{L_2}$ iff $H_1 \prec H_2$.

Suppose, $\prec$ is a preference on horse lotteries subject to HL Axioms 1–4. Surprisingly, there may not exist a probability and *state-independent* utility agreeing with $\prec$ [according to (4.2)], even for simple acts. Moreover, the problem has nothing to do with existence of potentially null states. That is, even if no state is potentially null, the fourth axiom (HL Axiom 4) is *insufficient* for the existence of a probability/state-independent utility pair agreeing with $\prec$.

EXAMPLE 4.1. Let $\mathbf{R} = \{r_*, r, r^*\}$ be three rewards and consider the set $\mathbf{H_R}$ of horse lotteries defined on the binary partition $\{s_1, s_2\}$. Next, consider two probability/utility pairs (p^i, U^i) $(i = 1, 2)$, where $U^i(r_*) = 0$, $U^i(r^*) = 1$, $U^1(r) = 0.1$ and $U^2(r) = 0.4$; also, $p^1(s_1) = 0.1$ and $p^2(s_1) = 0.3$. Define $H_1 \prec H_2$ iff $p^i(s_1)U^i(L_{1,1}) + p^i(s_2)U^i(L_{1,2}) < p^i(s_1)U^i(L_{2,1}) + p^i(s_2)U^i(L_{2,2})$ $(i = 1, 2)$. Then, by Theorem 4, $\prec$ satisfies HL Axioms 1–3, and we claim it satisfies HL Axiom 4 as well. Moreover neither state is potentially null under $\prec$.

The proof that $\prec$ satisfies HL Axiom 4 is straightforward. We observe the following (expected utility) bounds on $\prec$-preferences for the constant horse lottery $\mathbf{r}$. $(\forall\ 0.1 > \varepsilon > 0)$, $(0.9 + \varepsilon)\mathbf{r}_* + (0.1 - \varepsilon)\mathbf{r}^* \prec \mathbf{r} \prec (0.6 - \varepsilon)\mathbf{r}_* + (0.4 + \varepsilon)\mathbf{r}^*$. However, the utilities U^i are state-independent and neither state is null for either p^i $(i = 1, 2)$. That is, using conditional preference (see Definition 34, $(\forall\ 0.1 > \varepsilon > 0)$ $(0.9 + \varepsilon)\mathbf{r}_* + (0.1 - \varepsilon)\mathbf{r}^* \prec_{\mathbf{s_j}} \mathbf{r} \prec_{\mathbf{s_j}} (0.6 -$

$\varepsilon)\mathbf{r}_* + (0.4 + \varepsilon)\mathbf{r}^*$ $(j = 1, 2)$. The utility bounds for $\mathbf{r}$ reproduce in both families of s_j-called-off acts. Hence, $\prec$ satisfies HL Axiom 4.

According to Theorem 1, the two pairs (p^i, U^i) are the sole state-independent expected utilities agreeing with $\prec$ [according to (4.2)]. Next, we assert that $\prec$ may be extended to a strict partial order $\prec''$, also satisfying HL Axioms 2–4, but where $\prec''$ narrows the expected utility bounds for r, as follows: $0.9r_* + 0.1r^* \prec'' \mathbf{r} \prec'' 0.6r_* + 0.4r^*$.

We outline a general result for extending $\prec$ by forcing a new strict preference $H_1 \prec' H_2$, when $H_1 \sim H_2$. This contrasts with the extension created through Definition 20, which, instead, forces a new indifference relation.

Suppose H_1 and H_2 are elements of $\mathbf{H_R}$ that satisfy (1) $H_1 \sim H_2$ and (2) there do not exist two sequences $\{H_{i,n}\} \Rightarrow H_i$ $(i = 1, 2)$, where $\forall$ $(n = 1, \dots)$, $H_{2,n} \prec H_{1,n}$. Create an extension $\prec'$ of $\prec$ as follows:

DEFINITION ($\prec'$). $\forall$ $(H_a, H_b \in \mathbf{H_R})$, $H_a \prec' H_b$ if and only if either:

(i) $H_a \prec H_b$ (so $\prec'$ extends $\prec$)

or

(ii) $\exists$ $\{H_{a,n}\} \Rightarrow H_a$ and $\exists$ $\{H_{b,n}\} \Rightarrow H_b$ and $\exists$ $\{x_n\}$ with $\lim_{n\to\infty}\{x_n\} \neq 1$,

$$x_n H_{a,n} + (1 - x_n) H_2 \prec x_n H_{b,n} + (1 - x_n) H_1.$$

CLAIM. *$\prec'$ satisfies* HL *Axioms* 1–3, *provided $\prec$ does. Also*, $H_1 \prec' H_2$.

We omit the proof which follows along similar lines for the demonstration of Lemma 3.3. Regarding HL Axiom 4, it suffices that $\prec'$ is formed by extending $\prec$ using a target set endpoint, for example, let $H_1 = v_* B + (1 - v_*)W$, where $\mathcal{T}(H_2) = [v_*, v^*]$ and this interval has interior, that is, $v_* < v^*$. Then $\prec'$ satisfies HL Axiom 4 too.

Last, for Example 4.1, apply the claim, twice over, first to force $0.9\mathbf{r}_* + 0.1\mathbf{r}^* \prec' \mathbf{r}$, then to force $\mathbf{r} \prec'' 0.6\mathbf{r}_* + 0.4\mathbf{r}^*$.

Consider the convex sets $\mathscr{V}$ and $\mathscr{V}''$ of *agreeing* utilities for $\prec$ and $\prec''$ provided by Corollary 3.2. (These utilities agree since $\mathbf{R}$ is finite.) Because $\prec''$ extends $\prec$, then $\mathscr{V}'' \subset \mathscr{V}$. A fortiori, each agreeing expected state-independent utility model for $\prec''$ also is one for $\prec$. However, by Theorem 1, there does not exist an agreeing expected state-independent utility model for $V'' \in \mathscr{V}''$, since $\mathscr{V}''$ excludes all (that is, both) expected state-independent utility models for $\prec$. Nonetheless, $\prec''$ satisfies HL Axioms 1–4. This ends our discussion of Example 4.1.

4.3. *Representation of $\prec$ in terms of* (*nearly*) *state-independent utilities*. The four axioms HL Axioms 1–4 are insufficient for the existence of an agreeing state-independent utility. However, with the addition of a fifth axiom to regulate state-dependence for potentially null states, the resulting theory is sufficient for an agreeing "almost" state-independent utility. First, we make precise the notion of an "almost" state-independent utility.

Consider a set of probability/state-dependent utility pairs $\{(p, U_j)\}$, each pair agreeing with the partial order $\prec$ for simple acts, according to (4.1).

DEFINITION 31. Say that $\prec$ *admits almost state-independent utilities for a set of n-rewards* $\{r_1, \dots, r_n\}$ if, for each $\varepsilon > 0$, there exists a pair (p, U_j) that agrees with $\prec$ on simple acts (and almost agrees, otherwise), where for a set of states $S^{\#} = \{s_{j1}, \dots, s_{jk}\}$, $p(S^{\#}) \geq 1 - \varepsilon$,

$$\max_{\substack{s_j, s_{j'} \in S^{\#} \\ 1 \leq i \leq n}} |U_j(r_i) - U_{j'}(r_i)| \leq \varepsilon.$$

Say $\prec$ *admits almost state-independent utilities* if it does so for each set of n-rewards, $n = 1, \dots$.

Obviously, if (p, U) agrees with $\prec$ and U is state-independent, then $\prec$ admits almost state-independent utilities.

There are two problems created by state-dependent utilities. First, given the partial order $\prec$, we would like to indicate probability bounds for an event E by $\prec$-preferences between a constant act of the form $H_x(s) = xB + (1 - x)W$ and the act $H_E(s) = B$ if $s \in E$, and $H_E(s) = W$ if $s \notin E$. That is, in general, we want the upper probability bound $p^*(E)$ to be the l.u.b.$\{x$: $H_E \prec H_x\}$ (or 1, if $H_E \sim \mathbf{B}$), and we want the lower bound, $p_*(E)$, to equal the g.l.b.$\{x$: $H_x \prec H_E\}$ (or 0, if $H_E \sim \mathbf{W}$). However, if such preferences are to indicate probability bounds, then we require that the rewards B and W carry state-independent utilities 1 and 0, respectively. Thus the first problem.

Second, if a state s_j is potentially null under $\prec$, then there are no $\prec$-preferences among pairs of acts called-off in case s_j does not obtain. Let $\mathbf{Hs_j}$ be the family of s_j-called-off acts that yield outcome W for all $s \notin s_j$. When s_j is a potentially null state, $\forall\, (H_1, H_2 \in \mathbf{Hs_j})$, $H_1 \sim H_2$. Suppose $\precsim_V$ $(V \in \mathscr{V})$ extends $\prec$ (on simple acts) and let $\{(p, U_j)\}$ be the set of probability/(possibly) state-dependent utilities which represent $\precsim_V$ according to (4.1). Then, if state s_j is potentially null under $\prec$, unfortunately, HL Axioms 1–4 impose too few restrictions on U_j (the state-dependent utility, given state s_j) even when $p(s_j) > 0$. In particular, it may be that $V(\mathbf{r}_1) > V(\mathbf{r}_2)$, yet for all the U_j, $U_1(r_1) \leq U_1(r_2)$.

To resolve both these problems, we impose a fifth axiom—a requirement of "stochastic dominance" among lotteries. For each state s_j and each v.N-M lottery L_α, define the set of acts $\{H^\alpha_{j,m}$: $H^\alpha_{j,m}(s) = (1 - 2^{-m})W + (2^{-m})L_\alpha$, if $s \notin s_j$; $H^\alpha_{j,m}(s_j) = L_\alpha$ for state $s_j\}$ $(m = 1, \dots)$. Observe that, $(\forall\ j)$ $\lim_{m \to \infty}\{H^\alpha_{j,m}\} = H_{j,\alpha} \in \mathbf{Hs_j}$. Moreover, $H_{j,\alpha}(s_j) = L_\alpha$. Then, we require the following axiom:

HL AXIOM 5. For each two "constant" acts $H_{L\alpha}(s) = L_\alpha$ and $H_{L\beta}(s) = L_\beta$,

$$\forall\,(j, m)\left[H_{L\alpha} \prec H_{L\beta} \text{ iff } H^\alpha_{j,m} \prec H^\beta_{j,m}\right] \qquad (j = 1, \dots, n;\ m = 1, \dots).$$

Thus, exactly when $\mathbf{L}_\beta$ is $\prec$-preferred to $\mathbf{L}_\alpha$ (as constant acts), HL Axiom 5 imposes a $\prec$-preference on sequences of pairs of lotteries, $(H^{\alpha}_{j,m}, H^{b}_{j,m})$ which converge to the s_j-called-off pair $(H_{j,\alpha}; H_{j,\beta})$. Thus, we obtain the constraint (Definition 21 of the Appendix) "$\neg(H_{j,\beta} \prec H_{j,\alpha})$."

Lemma 4.2. *Suppose* $\prec$ *satisfies* HL *Axioms* 1–5. *Then, for each* $V \in \mathscr{V}$ (*of Theorem* 3.1) *we may select* (*exactly*) *one pair* (p^V, U_j^V) *from the set of pairs* $\{(p, U_j)\}$ *provided by Corollary* 4.1—*where each pair represents* $\precsim_V$ *in accord with* (4.1)—*so that acts* **W** *and* **B** *have constant value and bound the state-dependent utilities of other rewards. In symbols,*

$$\forall (s_j) \forall \left(L_i, L_k \in \mathbf{L}_{\mathbf{R}\text{-}\{W,B\}}\right), \qquad H_{L_i} \prec H_{L_k}$$
$$\textit{iff} \quad 0 = U_j^V(W) \le U_j^V(L_i) \le U_j^V(L_k) \le U_j^V(B) = 1,$$

with at least one outside inequality strict for each s_j *such that* $p(s_j) > 0$, *and all inequalities strict for each* s_j *that is not* $\prec$*-potentially null.*

Definition 32. We call (p^V, U_j^V) the *standard representation* of V.

Thus, HL Axiom 5 (via HL Axiom 3) constrains state-dependent utilities of the rewards **W** and **B** in potentially null states, as desired. In the course of the proof of Theorem 6 (below), we explain how HL Axiom 5 also regulates the $\prec$-potentially null, state-dependent utilities of all v.N-M lotteries.

Of course, HL Axioms 1–5 are insufficient for guaranteeing existence of a state-independent utility agreeing with $\prec$. Counterexample 4.1 applies, that is, $\prec'$ satisfies all five axioms (since no states are $\prec'$-potentially null). However, as we show next, these axioms suffice for an almost state independent utility.

Theorem 6. *Assume that* $\prec$ *satisfies* HL *Axioms* 1–5.

(i) *Then* $\prec$ *admits almost state-independent utilities.*
(ii) *If* $\prec$ *has a countable basis* $\mathscr{B}$, *each* (p, U_j) *pair in Definition* 31 *agrees with* $\prec$.

There is a sufficient condition for the existence of a state-independent utility over the finite set $\{W, B, r_1, \ldots, r_i, \ldots, r_n\}$, using closure (at one end-point, at least) of the target sets $\mathscr{T}_i(r_i)$ defined in Definition 19.

Lemma 4.3. *If the target sets* $\mathscr{T}_i(r_i)$ $(i = 1, \ldots, n)$ *are not open intervals, there exists a subset* $\mathscr{V}' \subset \mathscr{V}$ *of expected utilities for* $\prec$ (*agreeing on simple acts*), *where each* $V' \in \mathscr{V}'$ *is standardly represented by the set of pairs* $\{(p', U_j')\}$ *according to* (4.1) *and where* $U_j'(r_i)$ *is state-independent* $(i = 1, \ldots, n)$.

Note: $\mathscr{V}'$ may fail to be convex. Also, results similar to Lemma 4.3, using different assumptions, appear in Rios Insua (1992). Related ideas appear in Nau (1992).

5. Conditional preference and conditional probabilities.

Let e be an event. (Recall, we equate the set state s_j with its elements.) Let H_1 and H_2 be a pair of e-called-off acts. Suppose $\prec$ satisfies HL Axioms 1 and 2.

LEMMA 5.1. *Let H_1' and H_2' be another pair of e-called-off acts which agree with H_1 and H_2 (respectively) on e, that is,* $\forall\ (s \notin e)$, $[H_1'(s) = H_2'(s)]$ *and* $\forall\ (s \in e)$, $[H_1(s) = H_1'(s)$ *and* $H_2(s) = H_2'(s)]$: (i) $H_1 \prec H_2$ *iff* $H_1' \prec H_2'$ *and* (ii) $H_1 \approx H_2$ *iff* $H_1' \approx H_2'$.

Therefore, a $\prec$-preference (or $\approx$-indifference) among two e-called-off acts does not depend upon how they are called-off, that is, the preference (or indifference) does not depend upon how the acts agree with each other when e fails. This replicates the core of Savage's [(1954), page 23] "sure thing" postulate, P2, as that applies to our concept of a partially ordered preference.

Consider a (maximal) subset of $\mathbf{H_R}$, denoted by $\mathbf{H_e}$, where every two elements of $\mathbf{H_e}$ form an e-called-off pair. Obviously, each such family $\mathbf{H_e}$ of e-called-off acts is closed under convex combinations.

DEFINITION 33. Define $\prec_{\mathbf{e}} = \prec/\mathbf{H_e}$, the restriction of $\prec$ to the family of e-called-off acts in $\mathbf{H_e}$. We call $\prec_{\mathbf{e}}$ the *conditional $\prec$-preference relation, given e*. (The preceding lemma insures this relation is well defined, that is, it depends on e but not on how acts are called-off.)

Note: The event e^c is essentially null with respect to the conditional preference $\prec_{\mathbf{e}}$.

DEFINITION 34. Also, for each pair of horse lotteries H_1 and H_2, say that H_2 is $\prec$-preferred to H_1 given e, provided that, for some pair H_1' and H_2' (and by Lemma 5.1, provided for all pairs) of e-called-off acts agreeing (respectively) with H_1 and H_2 on e, $H_1' \prec_{\mathbf{e}} H_2'$.

In light of Lemma 5.1(i) and because $\mathbf{H_e}$ is a subset of $\mathbf{H_R}$, the following result is immediate.

THEOREM 7. *If* $\prec$ (*over* $\mathbf{H_R}$) *satisfies* (*a subset of*) HL *Axioms* 1–5, *then* $\prec_{\mathbf{e}}$ (*over* $\mathbf{H_e}$) *also satisfies the same horse lottery axioms*, *at least*.

Theorem 7 prompts an interesting question: What is the relation between (i) the set of *conditional* probability/utility pairs $\{p(\ |e), U_{j \in e}\}$, given e, that arise from the representation of $\prec$ over the family of acts $\mathbf{H_R}$ and (ii) the set of probability/utility pairs $\{p_e, U_{e, j \in e}\}$ that represent the conditional preference $\prec_{\mathbf{e}}$ over the restricted family of acts $\mathbf{H_e}$?

The following discussion of conditional indifference tells some of the answer.

DEFINITION 35. Let $\approx_{\mathbf{e}}$ be the *conditional* $\approx$*-indifference relation*, *given* e, defined by restricting $\approx$ to acts in the family $\mathbf{H_e}$. Then, say that horse lotteries H_1 and H_2 are $\approx$-indifferent, given e, provided that for some pair H_1' and H_2' (and by Lemma 5.1, provided for all such pairs) of e-called-off acts agreeing (respectively) with H_1 and H_2 on e, $H_1' \approx_{\mathbf{e}} H_2'$.

It is important, however, to see that $\approx_{\mathbf{e}}$ is not always the same as the $\approx$-indifference relation (defined by Definition 8) induced by $\prec_{\mathbf{e}}$ over acts solely in $\mathbf{H_e}$.

DEFINITION 36. Denote by $\approx_{\mathbf{H_e}}$ the $\approx$-indifference relation over elements of $\mathbf{H_e}$, induced by $\prec_{\mathbf{e}}$.

Of course $\approx_{\mathbf{e}}$-indifference entails $\approx_{\mathbf{H_e}}$-indifference, but not conversely. H_1 and H_2 may be two e-called-off acts from a family $\mathbf{H_e}$ which satisfies $H_1 \approx_{\mathbf{H_e}} H_2$, but where, nevertheless, $H_1 \not\approx H_2$, that is, $H_1 \not\approx_{\mathbf{e}} H_2$. We illustrate this phenomenon using a potentially null state which is not essentially null.

EXAMPLE 5.1. Consider a binary partition $S = \{s_1, s_2\}$ and horse lotteries defined over a binary reward set $\mathbf{R} = \{W, B\}$. Suppose $\prec$ is created by the Pareto principle applied to expected utility inequalities from the following set of probability/(state-independent) utility pairs: $S = \{(p, U)\colon 1 \geq p(s_1) \geq 0.5;\ U(B) > U(W)\}$. Then, s_2 is potentially null: acts are $\sim$-incomparable whenever they belong to a common $\mathbf{H_{s_2}}$ family. [With $p(s_1) = 1$, all elements of $\mathbf{H_{s_2}}$ have equal expected utility.] Hence, $\prec_{\mathbf{s_2}}$ is vacuous. So, based on $\prec_{\mathbf{s_2}}$ restricted to a family $\mathbf{H_{s_2}}$, all pairs of (e-called-off) acts are $\approx_{\mathbf{H_{s2}}}$-indifferent. However, the pair of s_2-called-off acts (H_1, H_2), defined by $H_1(s_1) = H_2(s_1) = W$, $H_1(s_2) = W$ and $H_2(s_2) = B$, though $\sim$-incomparable are not $\approx$-indifferent: $H_1 \sim H_2$ and $H_1 \not\approx H_2$. This is shown as follows. Consider the act H_3 defined by $H_3(s_1) = B$ and $H_3(s_2) = W$. Observe that $0.5H_1 + 0.5H_3 \sim 0.7W + 0.3B$, whereas $0.7W + 0.3B \prec 0.5W + 0.5B = 0.5H_2 + 0.5H_3$. This shows that $H_1 \not\approx H_2$.

Returning to the question, above, we state our central result about conditional probabilities and conditional preferences.

THEOREM 8. (i) *If* (p, U_j) *belongs to the set of probability/utility pairs representing* $\prec$, *then the pair* $(p(\ |e), U_{j \in e})$ *belongs to the set that represent the conditional preference* $\prec_{\mathbf{e}}$.

(ii) *Suppose that the pair* $(p_e, U_{e, j \in e})$ *belongs to the set representing the conditional preference* $\prec_{\mathbf{e}}$ *with respect to the family* $\mathbf{H_e}$. *Then for some pair* (p, U_j) *in the set that represents* $\prec$, $p(\ |e) = p_e$ *and* $U_{j \in e} = U_{e, j \in e}$, *provided two conditions obtain*: (1) *The event* e *is not potentially null* (*with respect* *to* $\prec$) *and* (2) *the expected utility* $V_{\mathbf{e}}(\ )$ (*with arguments from* $\mathbf{H_e}$), *corresponding to the pair* $(p_e, U_{e,j\in e})$, *does not use* $\prec$*-precluded target endpoints, as regulated by Definition* 21 (*of the Appendix*).

Next, we offer an example of Theorem 8, relating Bayes' updating to conditional preferences.

EXAMPLE 5.2. Consider a partition into three states $\{s_1, s_2, s_3\}$ and acts involving the three rewards, W, r and B. Let $\mathbf{r}$ denote the constant act, with outcome r in each state. For $j = 1, 2$ and 3, define the three acts $H_j(s_j) = B$, $H_j(s_k) = W$ $(j \neq k)$ and also the three acts $H_{j,r}(s_j) = r$ and $H_{j,r}(s_k) = W$ $(j \neq k)$. Apart from the strict preferences that follow because $\mathbf{W}$ and $\mathbf{B}$ are, respectively, the "worst" and "best" acts, suppose also the agent reports these preferences:

$$0.5W + 0.5H_{3,r} \prec H_1 \prec H_{3,r} \prec H_2 \prec H_3 \prec \mathbf{r} \prec 0.5H_3 + 0.5\mathbf{r}.$$

We investigate the standardized, state-independent utility representations for these preferences. That is, with $U(W) = 0$, $U(B) = 1$, let $u = U(r)$, independent of the state s_j. If we denote by p_j the probability of state s_j, then the preferences above are modeled by each probability/utility pair $(p_1, p_2, p_3; u)$ satisfying $0 < 0.5p_3u < p_1 < p_3u < p_2 < u < 0.5p_3 + 0.5$.

For each $0 < u < 1$, it is possible to determine the set $\mathscr{P}(u)$ of all (p_1, p_2, p_3) that satisfy these inequalities. For example, the set $\mathscr{P}(0.5)$ is shown in Figure 2. The union of all sets $\mathscr{P}(u) \times \{u\}$ such that $\mathscr{P}(u) \neq \emptyset$ is the set of all probability/utility pairs that agree with the strict preferences above. From this set, one can determine other preferences not listed above which must also hold if the axioms do. For example, it is required, though not obvious from the reported preferences, that $0.4B + 0.6W \prec \mathbf{r}$. [By contrast, it is obvious from the preferences above that $(1/3)\mathbf{B} + (2/3)\mathbf{W} \prec \mathbf{r}$.]

If we were to learn that, say, the event $E = \{s_1, s_2\}$ occurred, we can determine which preferences are implied in the conditional problem. The set of all pairs (q_1, u), where q_1 is a conditional probability of s_1 given E, is shown in Figure 3. Observe that, as provided by Theorem 8, the set of conditional probabilities from Figure 2 is exactly the set represented by the vertical line (at $u = 0.5$) in Figure 3. However, the set shown in Figure 3 is not convex since it contains the points $(0.415, 0.293)$ and $(0.455, 0.379)$, but does not contain the point $(0.435, 0.336) = 0.5(0.415, 0.293) + 0.5(0.455, 0.379)$.

6. Concluding remarks. There is a burgeoning literature dealing with applications of sets of probabilities. Separate from work on robust Bayesian statistical analysis, they occur also in the following settings: as a rival account to strict Bayesian theory for representing uncertainty, such as in Levi's (1974, 1980) theory for Ellsberg's (1961) "paradox"; relating to indeterminate degrees of belief, as in Smith's (1961) theory of "medial odds" developed by Williams (1976), Giron and Rios (1980), Walley (1991) and Nau

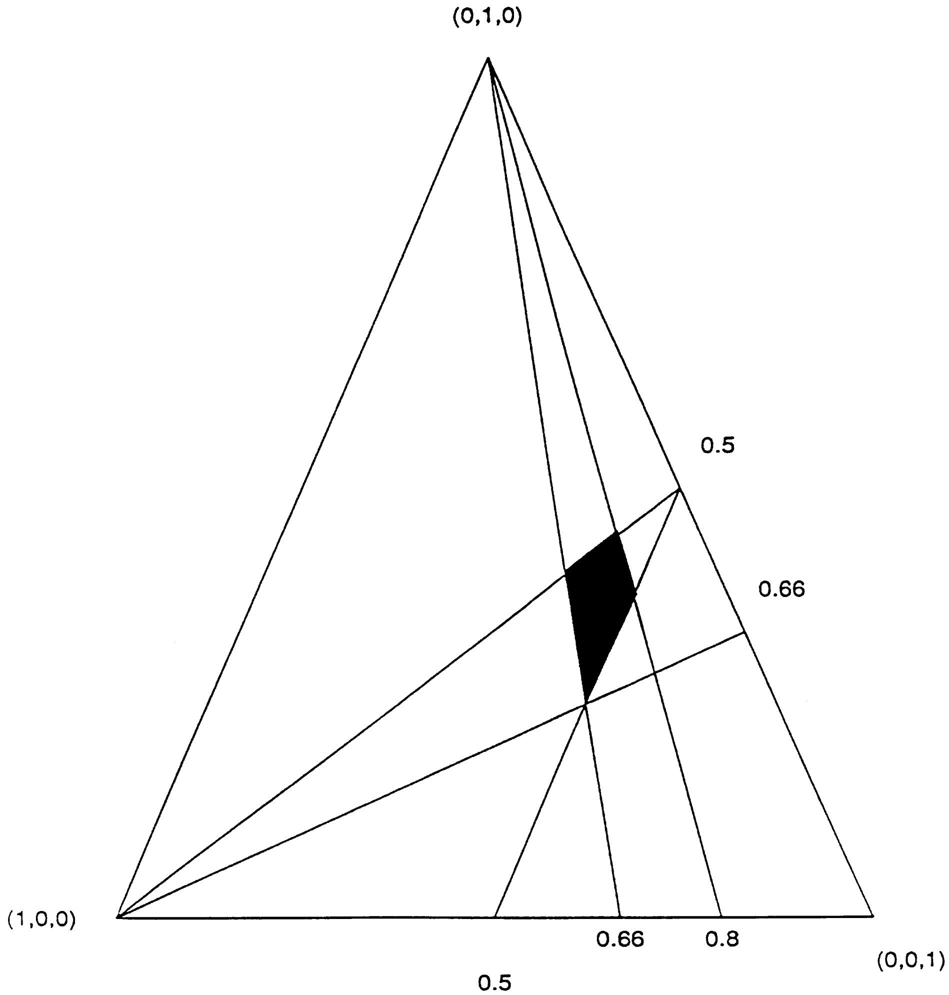

FIG. 2. *The set* $P(0.5)$ *in Example* 5.2.

(1993); and as a method for capturing multiple "expert" opinions [Kadane and Sedransk (1980); Kadane (1986)]. In addition, sets of probabilities arise from incomplete elicitations, where some but not all of an agent's opinions are formalized by inequalities in probabilities and the question is what decisions are fixed by these partially reported degrees of belief; see Moskowitz, Wong and Chu (1988) and White (1986). Dual to sets of probabilities, the articles by Aumann (1962) and Kannai (1963) explore the existence of ("linear") utilities for von Neumann–Morgenstern lotteries when probabilities are completely specified.

However, these efforts rely on convexity of the spaces of probabilities and utilities to arrive at their conclusions. From our point of view, this mathemat-

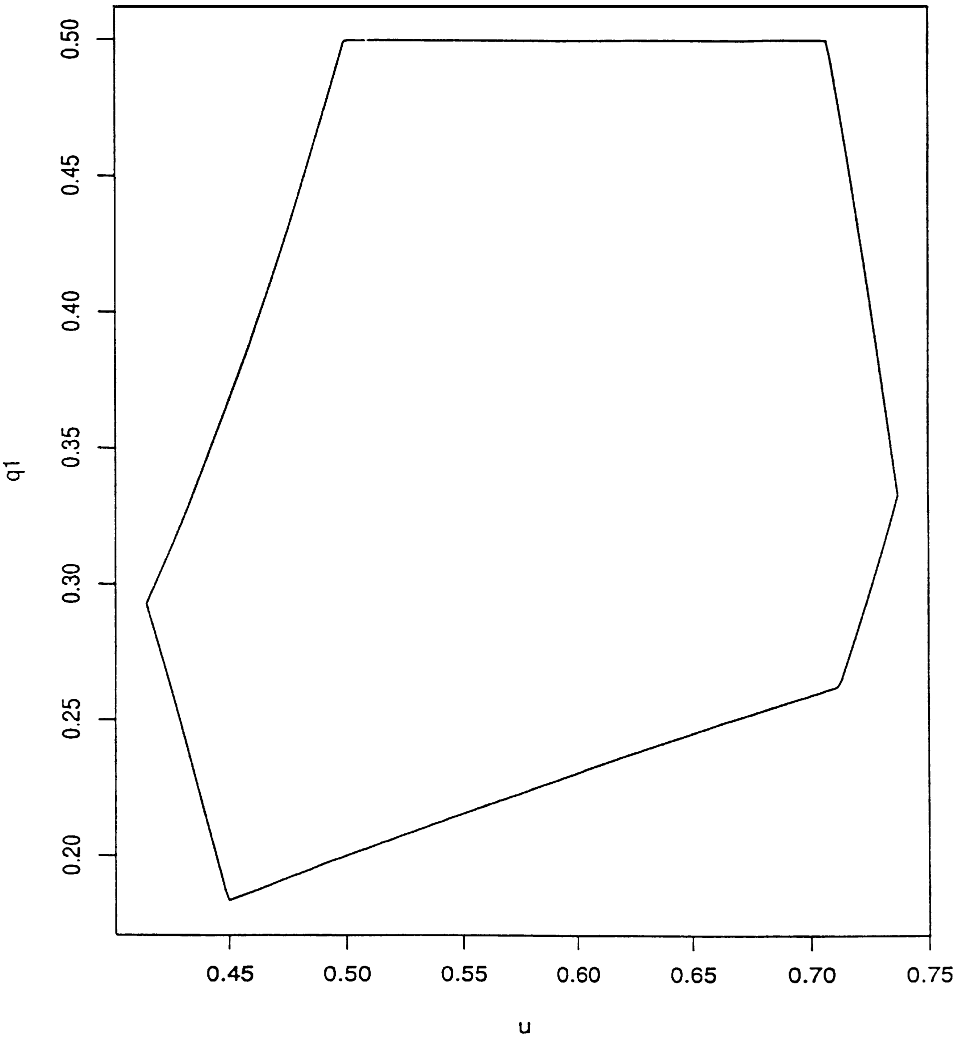

FIG. 3. *The set of conditional probabilities and utilities.*

ical convenience is justified under an assumption, for example, that at least one of the agent's probability and utility is fully determinate. For instance, in light of Corollary 3.4, convexity is appropriate for Bayesian robustness when a loss function is specified but probability is left indeterminate. Likewise, our theory endorses the use of a convex set of utilities given a determinate probability, as in Aumann's (1962) result concerning existence of a utility agreeing with a partially ordered preference over simple von Neumann–Morgenstern lotteries. However, as shown by Theorem 1, partially ordered preferences that obey a (weak) Pareto condition may not admit a convex (or even connected) set of agreeing probability/utility pairs. One way to require convexity of the agreeing sets, then, is to restrict the scope of the Pareto condition [see Levi (1990)], but that is a move we are not willing to make.

Corollary 3.4 prompts a serious question, we think, about the extent to which our proof technique for extending a partially ordered preference is useful for the representation theorems of this essay. To wit, since the set $\mathscr{V}$ of agreeing "linear" utilities in Theorem 5 is convex, why bother with the elaborate inductive argument only to arrive at what "separating hyperplanes" yields directly? The answer has two parts.

As a first reason, Theorem 5 applies without additional topological assumptions about the relation $\prec$. Specifically, to the best of our knowledge, all the existing theorems that appeal to "separating hyperplanes" in order to provide *necessary and sufficient* conditions for representing a partially ordered *strict* preference relation $\prec$ by a convex set of "linear" utilities or by a convex set of probabilities, make assumptions regarding the boundaries of $\prec$. Otherwise, for results that are based on a partially ordered *weak* preference relation $\precsim$, whether preference at the boundary of $\mathscr{V}$ is strict or not, is not determined by such an approach.

For an illustration of the former approach, Walley [(1991), Section 3.7.8, condition R8] requires that strict preference over gambles be "open" so that so-called strong separation leads to a representation in terms of sets of probabilities "closed" with respect to infimums. By avoiding "separating hyperplanes," we are able to sidestep this artifice. Surfaces of the set $\mathscr{V}$ need not have a simple topological character.

For an illustration of the latter approach, Giron and Rios (1980) use a reflexive, partial (quasi-Bayesian) preference relation, denoted in their paper by $\precsim$, which they represent with a closed, convex set of probabilities. They note [Giron and Rios (1980), footnote 3, page 20] that their method generates the same "quasi-Bayesian preorder" whether the so-called uncertainty set of probabilities (which they denote by K^*) or its closure ($\overline{K}^*$) is used. To explain our assertion about the loss of information at the boundary of $\mathscr{V}$, consider the following example involving preferences over acts using only two prizes, **W** and **B**.

EXAMPLE 6.1. Define act H_E as $H_E(s) = B$ for $s \in E$, and $H_E(s) = W$ otherwise.

Case 1. The agent reports the strict preferences $xB + (1 - x)W \prec H_E$ for $0 < x \leq 0.6$ and noncomparability $xB + (1 - x)W \sim H_E$ for $0.6 < x \leq 1$.

Case 2. The agent reports the strict preferences $xB + (1 - x)W \prec H_E$ for $0 < x < 0.6$ and noncomparability $xB + (1 - x)W \sim H_E$ for $0.6 \leq x \leq 1$.

The (closed) target set of utilities for H_E is the same in both cases: $\mathscr{T}(H_E) = [0.6, 1]$. However, in the first case the lower bound is not a "candidate utility" (Definition 24), whereas in the second case it is. Therefore, by our construction, the representation for the agent's strict preferences in Case

1 is the set $\mathscr{P} = \{P\colon 0.6 < P(E) \le 1\}$ and in the second case it is the closed set $\overline{\mathscr{P}} = \{P\colon 0.6 \le P(E) \le 1\}$.

By contrast, the Giron and Rios (1980) theory uses only a weak preference relation, $\precsim$. In both Cases 1 and 2 their theory entails

$$xB + (1-x)W \precsim H_E \quad \text{for } 0 < x \le 0.6 \quad \text{and}$$
$$xB + (1-x)W \sim H_E \quad \text{for } 0.6 < x \le 1.$$

[The weak preferences of Case 2 result from applying Giron and Rios' Axiom A5 (continuity). In particular, that axiom yields the conclusion $0.6B + 0.4W \precsim H_E$ from the premise $xB + (1-x)W \precsim H_E$ $(0 < x < 0.6)$.] In their notation, the weak-preference relation does not distinguish between these two cases: where $K^* = \{p\colon 0.6 < p(E) \le 1\})$ and $\overline{K}^* = \{p\colon 0.6 \le p(E) \le 1\}$, though our strict-preference does.

As a second reason for bypassing proof techniques using "separating hyperplanes," though the set $\mathscr{V}$ is convex, not so for the set of "linear" utilities that admit a decomposition as subjective (almost) state-independent utilities. We do not see how to show the existence of the set of agreeing probability/(almost) state-independent utility pairs, corresponding to Theorem 6, without exploring details about the surface of $\mathscr{V}$. In light of Theorem 5, we have no right to assume those surfaces are closed. By contrast, when $\mathscr{V}$ has sufficiently many closed faces, Lemma 4.3 gives a representation of $\prec$ in terms of sets of probability/state-independent utility pairs. Thus, we feel justified in our choice of an "inductive" proof technique by the increased content to the theorems reached.

APPENDIX

Proofs of selected results.

A. *Results from Section* 2. Corollaries 2.1, 2.2 and 2.3 have elementary proofs.

PROOF OF COROLLARY 2.4. From left to right, argue indirectly and apply Corollary 2.3 for a contradiction. In the other direction, assume that $xH_1 + (1-x)H \approx xH_2 + (1-x)H$ for some $1 \ge x > 0$ and some lottery H. Also, assume $yH_1 + (1-y)H_3 \prec (\succ) H_4$, with $1 \ge y > 0$. Then by HL Axiom 2, $\forall$ $(1 \ge z > 0)$, $z(yH_1 + (1-y)H_3) + (1-z)H \prec (\succ) zH_4 + (1-z)H$. Let $z = x/(x+y-xy) > 0$ and then $1 > z$ (unless $x = y = 1$, in which case we are done). Last, define the term $w = y/(x+y-xy)$ and we know that $0 < w < 1$. Thus, we have $w(xH_1 + (1-x)H) + (1-w)H_3 \prec (\succ) zH_4 + (1-z)H$. Since $(xH_1 + (1-x)H) \approx (xH_2 + (1-x)H)$, by Corollary 2.3 also we have $w(xH_2 + (1-x)H) + (1-w)H_3 \prec (\succ) zH_4 + (1-z)H$. Again by HL Axiom 2, we may cancel the common factor $(1-z)H$ from both sides, to yield $yH_2 + (1-y)H_3 \prec (\succ) H_4$. By Corollary 2.3, $H_1 \approx H_2$. $\square$

PROOF OF COROLLARY 2.5. Assume the premises and, by Corollary 2.3, show using HL Axiom 3 that $\forall\ (0 < x \le 1,\ H_a,\ H_b)$ whenever $xM + (1 - x)H_a \prec (\succ) H_b$, then $xH + (1 - x)H_a \prec (\succ) H_b$. □

Lemma 2.1 has a straightforward proof.

PROOF OF LEMMA 2.2. Without loss of generality, as L is discrete, write L as $\{P(r_n)\colon P(r_i) \ge P(r_j) \text{ for } i \le j\}$. Let

$$x_n = \sum_{i=1}^{n} P(r_i)$$

and define the simple lotteries $L_n = \{(1/x_n)P(r_i)\colon i = 1, \dots, n\}$. Then $\{H_{L_n}\} \Rightarrow H_L$. If for each $r_n \in \operatorname{supp}(L)$, $\mathbf{r_n} \prec \mathbf{r}$, then by HL Axioms 1 and 2, $H_{L_n} \prec \mathbf{r}$ (or, if $\mathbf{r} \prec \mathbf{r}_n$, then $\mathbf{r} \prec H_{L_n}$). Thus, we have the desired conclusion: not $(\mathbf{r} \prec H_L)$ [or, alternatively, not $(H_L \prec \mathbf{r})$]. For if not, by HL Axiom 3 and transitivity of $\prec$, $(H_L \prec H_L)$. □

PROOF OF COROLLARY 2.6. On the contrary, if a utility V for acts is unbounded, then there are acts with infinite utility.

Just consider the discrete horse lottery H_∞, where $H_\infty(s_j) = \sum_i (2^{-i})P_{i,j}$ for a sequence of acts H_i such that $V(H_i) \ge 2^i$ $(i = 1, \dots)$. Then, by the expected utility property, $V(H_\infty) = \infty$. The existence of such acts leads to a contradiction with the first two axioms, just as in the St. Petersburg paradox. Assume for convenience that $H_1 \prec H_2$. By axiom HL Axiom 2, $0.5H_1 + 0.5H_\infty \prec 0.5H_2 + 0.5H_\infty$. However, $V(0.5H_1 + 0.5H_\infty) = V(0.5H_2 + 0.5H_\infty) = V(H_\infty) = \infty$, which, if V agrees with $\prec$, entails the contrary result that $0.5H_1 + 0.5H_\infty \sim 0.5H_2 + 0.5H_\infty$. □

PROOF OF LEMMA 2.3. The proof is indirect. Most of the work is done by HL Axiom 3. We present the argument for the case in which $\prec$ fails to be bounded above, using Axiom 3(b). By similar reasoning using Axiom 3(a) instead, the result obtains when $\prec$ fails to be bounded below.

Let $\{H_n\colon n = 1, \dots\}$ be an *increasing chain* and suppose it is not bounded above, that is, $\lim_{n\to\infty} \sup\{x\colon (H_2 \prec xH_1 + (1 - x)H_n)\} = 1$. Choose a subsequence, also denoted by $\{H_n\}$, so that $x_n \ge 1 - 1/n$ and so that $H_2 \prec x_n H_1 + (1 - x_n)H_n$. However, $\{x_n H_1 + (1 - x_n)H_n\} \Rightarrow H_1$. Trivially, the constant sequence $\{H_2\} \Rightarrow H_2$. Also, $H_1 \prec H_2$ by assumption. Then by HL Axiom 3(b), $H_1 \prec H_1$, contradicting HL Axiom 1. □

PROOF OF COROLLARY 2.7. If not, then there is an unbounded increasing (or decreasing) $\prec_W$-chain of preferences amongst the set of rewards $\mathbf{R}$. By Lemma 2.3, $\prec_W$ does not satisfy both Axioms 1 and 3. □

PROOF OF COROLLARY 2.8. Let H_1 and H_2 be a pair of acts which are "called-off" in case $\mathbf{n}$ does not obtain, that is, $\forall\ (s \notin \mathbf{n})$, $H_1(s) = H_2(s)$.

(Properties of "called-off" acts are examined in Section 5.) Define k pairs of acts "called-off" in case s_{j_i} obtains, H_{1_i} and H_{2_i} $(i = 1, \dots, k)$ as follows: Let l be a lottery. $\forall\ (s \in s_{j_i})$, $[H_{1_i}(s) = H_1(s)$ and $H_{2_i}(s) = H_2(s)]$; $\forall\ (s \in \mathbf{n}\ \&\ s \notin s_{j_i})$, $[H_{1_i}(s) = H_{2_i}(s) = L]$; $\forall\ (s \notin \mathbf{n})$, $[H_{1_i}(s) = H_{2_i}(s) = H_1(s) = H_2(s)]$. By assumption, each $s_{j_i} \in \mathbf{n}$ is essentially null. Therefore, by iteration of Corollary 2.4 (and transitivity of $\approx$) $H_1' \approx H_2'$, where $H_1' = \sum_{i=1}^{k}(1/k)H_{1_i}$ and $H_2' = \sum_{i=1}^{k}(1/k)H_{2_i}$. However, $H_1' = (1/k)H_1 + (k - 1/k)H$ and, likewise, $H_2' = (1/k)H_2 + (k - 1/k)H$ for act H defined by $\forall\ (s \in \mathbf{n})$, $[H(s) = L]$ and $\forall\ (s \notin \mathbf{n})\ [H(s) = H_1(s) = H_2(s)]$. Then, by Corollary 2.4, the desired result obtains, $H_1 \approx H_2$. □

B. *Proof of Theorem* 2. The extension from $\prec$ to $\prec'$ is given in steps, by adding the two new rewards one at a time. First, extend $\prec$ to a partial order $\prec^*$ on $\mathbf{H}_{\mathbf{R} \cup \{W\}}$, where W is left $\sim^*$-incomparable with elements of $\mathbf{H}_{\mathbf{R}}$. The definition of $\prec^*$ is introduced by a lemma that shows the extension is minimal.

LEMMA 2.4. *Suppose* H_1, H_2, H_1' *and* $H_2' \in \mathbf{H}_{\mathbf{R}}$ *and are related as follows*: $H_1(s_j) = x_j L_j + (1 - x_j)L_{1,j}$ *and* $H_2(s_j) = x_j L_j + (1 - x_j)L_{2,j}$, *while* $H_1'(s_j) = x_j L_j' + (1 - x_j)L_{1,j}$ *and* $H_2'(s_j) = x_j L_j' + (1 - x_j)L_{2,j}$. *Then* $H_1 \prec H_2$ *iff* $H_1' \prec H_2'$.

PROOF. The lemma is immediate by HL Axiom 2 and the identity $0.5H_1 + 0.5H_2' = 0.5H_2 + 0.5H_1'$. □

Now, let $H_i \in \mathbf{H}_{\mathbf{R} \cup \{W\}}$ be written $H_i(s_j) = x_{i,j}W + (1 - x_{i,j})L_{i,j}$, where $L_{i,j} \in \mathbf{H}_{\mathbf{R}}$ is well defined if and only if $x_{i,j} < 1$. Choose a reward $r \in \mathbf{R}$ and let $H_i^{\#} \in \mathbf{H}_{\mathbf{R}}$ be the act that results by substituting r for W in H_i. Thus, $H_i^{\#}(s_j) = x_{i,j}r + (1 - x_{i,j})L_{i,j}$. Lemma 2.4 shows this choice is arbitrary and, if $\prec^*$ is to extend $\prec$, it must satisfy the following:

DEFINITION 15 ($\prec^*$). Given $H_1, H_2 \in \mathbf{H}_{\mathbf{R} \cup \{W\}}$, as expressed above, define the preference $\prec^*$ from $\prec$ by $H_1 \prec^* H_2$ iff $x_{1,j} = x_{2,j}$ (for all $s_j \notin \mathbf{n}$) and $H_1^{\#} \prec H_2^{\#}$.

LEMMA 2.5. *The order* $\prec^*$ *is identical with* $\prec$ *on* $\mathbf{H}_{\mathbf{R}}$ *and satisfies HL Axioms* 1–3.

PROOF. If $H_1, H_2 \in \mathbf{H}_{\mathbf{R}}$, then $x_{1,j} = x_{2,j} = 0$, $H_1 = H_1^{\#}$, $H_2 = H_2^{\#}$ and thus $H_1 \prec^* H_2$ iff $H_1 \prec H_2$. Next, we show that $\prec^*$ satisfies the axioms. Consider all $H_i \in \mathbf{H}_{\mathbf{R} \cup \{W\}}$.

HL Axiom 1 (*irreflexivity*). If, on the contrary, for some H_1, $H_1 \prec^* H_1$, then $H_1^{\#} \prec H_1^{\#}$, contradicting the irreflexivity of $\prec$.

Transitivity holds because if $H_1 \prec^* H_2$ and $H_2 \prec^* H_3$, then the corresponding three $H_i^{\#}$ acts $(i = 1, 2, 3)$ can be written $x_j(r) + (1 - x_j)L_{i,j}$. Since $\prec$ is transitive, $H_1^{\#} \prec H_3^{\#}$; thus, $H_1 \prec^* H_3$.

HL Axiom 2 (*independence*). $\forall\ (0 < y \le 1)$, $\forall\ H \in \mathbf{H}_{\mathbf{R}\cup\{W\}}$ $H_1 \prec^* H_2$ iff $x_{1,j} = x_{2,j}$ and $H_1^\# \prec H_2^\#$ iff $yx_{1,j} + (1-y)x_{3,j} = yx_{2,j} + (1-y)x_{3,j}$ and $yH_1^\# + (1-y)H_3^\# \prec yH_2^\# + (1-y)H_3^\#$ iff $yH_1 + (1-y)H_3 \prec^*\ yH_2 + (1-y)H_3$.

HL Axiom 3. Let $\{H_{1n} \prec^* H_{2n}\}$ be an infinite sequence of $\prec^*$-preferences where $\{H_{1n}\} \Rightarrow H_1$, $\{H_{2n}\} \Rightarrow H_2$ and assume $H_2 \prec^* H_3$. Thus $H_{1n}^\# \prec H_{2n}^\#$, where $\{H_{1n}^\#\} \Rightarrow H_1^\#$ and $\{H_{2n}^\#\} \Rightarrow H_2^\#$. Since $H_2 \prec^* H_3$, then $H_2^\# \prec H_3^\#$. By applying HL Axiom 3 to these $\prec$-preferences, we obtain $H_1^\# \prec H_3^\#$. We derive $x_{1,j} = x_{3,j}$ from the equalities $x_{1,j}^n = x_{2,j}^n$ and $x_{2,j} = x_{3,j}$. Therefore, $H_1 \prec^* H_3$. The argument for HL Axiom 3(b) is similar. □

Lemma 2.5 shows, also, that if $\prec'$ is defined on $\mathbf{R} \cup \{W\}$, extends $\prec$ and satisfies the axioms, then it extends $\prec^*$. That is, $\prec^*$ is a minimal extension of $\prec$ to the domain $\mathbf{R} \cup \{W\}$. Next, we extend $\prec^*$ to a preference $\prec_W$ in which $\mathbf{W}$ serves as a least preferred (worst) act. (The reader is alerted to the fact that, though the partial order $\prec_W$ makes $\mathbf{W}$ a least preferred act with respect to elements of $\mathbf{H}_\mathbf{R}$, it does not guarantee that W is, state by state, a least favorable reward. This feature is addressed in Section 4.1, where we consider state-dependent utilities for partial orders.)

DEFINITION 16 ($\prec_W$). $\forall\ (H_1, H_2 \in \mathbf{H}_{\mathbf{R}\cup\{W\}})$, $H_1 \prec_\mathbf{W} H_2$ iff either (a) $H_1 \prec^* H_2$ or (b) $\exists\ \{H_n \in \mathbf{H}_\mathbf{R}\}$, $\exists\ \{H_{1n}\} \Rightarrow H_1$, $\exists\ \{H_{2n}\} \Rightarrow H_2$ and $\exists\ (q_n$: $0.5 \le q_n < 1)$ with $\lim_{n\to\infty}\{q_n\} = q$, $q < 1$, such that $\forall\ n$, $q_nH_{1n} + (1 - q_n)H_n \prec^*$ (or $\approx^*$) $q_nH_{2n} + (1-q_n)\mathbf{W}$.

LEMMA 2.6. (1) *The partial order* $\prec_W$ *agrees with* $\prec$ *on* $\mathbf{H}_\mathbf{R}$.

(2) $\mathbf{W}$ *bounds* $\prec^*$ *from below*; *that is*, $\forall\ (H \in \mathbf{H}_\mathbf{R})$, $\mathbf{W} \prec_W H$.

(3) $\prec_W$ *satisfies HL Axioms* 1–3.

PROOF. (1) Let H_1 and H_2 belong to $\mathbf{H}_\mathbf{R}$. If $H_1 \prec H_2$, then by Lemma 2.4, $H_1 \prec^* H_2$, and by clause (a) in Definition 16, $H_1 \prec_W H_2$. For the converse, if $H_1 \prec_W H_2$, then it is not by clause (b), since $\forall\ (H_n \in \mathbf{H}_\mathbf{R})$ and for all sufficiently large n, as $1 > q \ge 0.5$, $q_nH_{1n} + (1-q_n)H_n \sim^*$ (and $\not\approx^*$) $q_nH_{2n} + (1-q_n)\mathbf{W}$. Hence, it must be that clause (a) obtains. So, $H_1 \prec^* H_2$ and $H_1 \prec H_2$.

(2) For each $H \in \mathbf{H}_\mathbf{R}$, recall that $0.5H + 0.5\mathbf{W} \approx^* 0.5\mathbf{W} + 0.5H$. Then, in Definition 16(b), set $H_1 = \{H_{1n}\} = \mathbf{W}$, $H_2 = \{H_{2n}\} = H$, $H_n = H$ and $q_n = 0.5$. Thus, $\mathbf{W} \prec_W H$.

(3) We verify the axioms individually:

HL Axiom 1 (*irreflexivity*). On the contrary, suppose that $H_1 \prec_W H_1$. There are two cases to consider. If this $\prec_W$-relation results by Definition 16(a), then $H_1 \prec^* H_1$, contradicting Lemma 2.4. If we hypothesize that $H_1 \prec_W H_1$ results by Definitions 16(b), then we derive a contradiction as follows. Let $H'_{1n} = q_nH_{1n} + (1-q_n)H_n$ and $H'_{2n} = q_nH_{2n} + (1-q_n)\mathbf{W}$. A necessary condition for the relation $H'_{1n} \prec^*$ (or $\approx^*)H'_{2n}$ to obtain is that $x_{H'_{1n}} = x_{H'_{2n}}$

(except on essentially null states). However, as both $\{H_{1n}\} \Rightarrow H_1$ and $\{H_{2n}\} \Rightarrow H_1$, while $\lim_{n\to\infty} q_n = q < 1$, this is impossible. That is, $\lim_{n\to\infty} x_{H'_{1n}} = \lim_{n\to\infty} q_n x_{H_{1n}} = 0$, while $x_{H'_{2n}} \geq (1 - q_n)$; hence, for all sufficiently large n, $x_{H'_{1n}} < x_{H'_{2n}}$.

HL Axiom 1 (*transitivity*). Suppose both $H_1 \prec_W H_2$ and $H_2 \prec_W H_3$ obtain. There are four cases to consider depending upon which clause in Definition 16 is used for each $\prec_W$-preference. The argument is most complicated when Definition 16(b) is used twice; hence, we give the details for this case only. Thus, assume there are two sequences of *-relations:

$$\text{(B1)} \qquad H'_{1n} \prec^* (\text{or } \approx^*) H'_{2n} \quad \text{and} \quad H''_{2n} \prec^* (\text{or } \approx^*) H'_{3n},$$

where

$$H'_{1n} = q_n H_{1n} + (1 - q_n) H_n, \qquad H'_{2n} = q_n H_{2n} + (1 - q_n)\mathbf{W},$$
$$H''_{2n} = q'_n \hat{H}_{2n} + (1 - q'_n) H'_n, \qquad H'_{3n} = q'_n H_{3n} + (1 - q'_n)\mathbf{W}$$

and where $\{H_{1n}\} \Rightarrow H_1$, $\{H_{2n}\} \Rightarrow H_2$, $\{\hat{H}_{2n}\} \Rightarrow H_2$, $\{H_{3n}\} \Rightarrow H_3$ and $\lim_{n\to\infty}\{q_n\} = q$, $\lim_{n\to\infty}\{q'_n\} = q'$, with $0.5 \leq q, q' < 1$. Then $\forall\ (0 < r_n < 1)$, $r_n H'_{1n} + (1 - r_n) H''_{2n} \prec^*$ (or $\approx^*$) $r_n H'_{2n} + (1 - r_n) H'_{3n}$. This is an $\prec^*$-preference, unless both equations of (B1) are $\approx^*$-indifferences. Choose r_n so that $r_n q_n = (1 - r_n) q'_n$. Since $\{H_{2n}\}$ and $\{\hat{H}_{2n}\}$ both converge to H_2, by HL Axiom 2, cancel the common acts in H'_{2n} and H''_{2n} (also common with acts in H_2). Then apply clause Definition 16(b) to obtain $H_1 \prec_W H_3$.

HL Axiom 2 (*independence*). $\forall\ H \in \mathbf{H}_{\mathbf{R} \cup \{\mathbf{W}\}}$, $\forall\ 0 < x \leq 1$:

Case (a). $H_1 \prec^* H_2$ iff $xH_1 + (1 - x)H \prec^* xH_2 + (1 - x)H$ iff $xH_1 + (1 - x)H \prec_W xH_2 + (1 - x)H$.

Case (b)–(i). If $q_n H_{1n} + (1 - q_n) H_n \prec^*$ (or $\approx^*$) $q_n H_{2n} + (1 - q_n)\mathbf{W}$, then $r_n[q_n H_{1n} + (1 - q_n) H_n] + (1 - r_n) H \prec^*$ (or $\approx^*$) $r_n[q_n H_{2n} + (1 - q_n)\mathbf{W}] + (1 - r_n) H$. Write $r_n = x/(q_n + (1 - q_n)x)$. Then $xH_1 + (1 - x)H \prec_W xH_2 + (1 - x)H$ by Definition 16(b).

Case (b)–(ii). Suppose $xH_1 + (1 - x)H \prec_W xH_2 + (1 - x)H$. Let $\{H_{3n}\} \Rightarrow xH_1 + (1 - x)H$ and $\{H_{4n}\} \Rightarrow xH_2 + (1 - x)H$. Assume $q_n H_{3n} + (1 - q_n) H_n \prec^*$ (or $\approx^*$) $q_n H_{4n} + (1 - q_n)\mathbf{W}$. Apply HL Axiom 2 to cancel acts in H_{3n} and H_{4n} common with H. Regroup the remainders to yield a $\prec^*$-relation of the desired form for Definition 16(b): $q'_n H_{1n} + (1 - q'_n) H_n \prec^*$ (or $\approx^*$) $q'_n H_{2n} + (1 - q'_n)\mathbf{W}$, where $\{H_{1n}\} \Rightarrow H_1$ and $\{H_{2n}\} \Rightarrow H_2$. (A simple calculation shows that $\lim_{n\to\infty}\{q'_n\} = q' \geq 0.5$.) Thus $H_1 \prec^* H_2$.

Next, we give the details for HL Axiom 3(a) [Axiom 3(b) follows similarly.]

HL Axiom 3(a) (*Archimedes*). Assume $H_n \prec_W M_n$ and $M \prec_W N$, where $\{H_n\} \Rightarrow H$ and $\{M_n\} \Rightarrow M$. We are to show that $H \prec_W N$. Again, there are four cases to consider, depending upon how (infinitely many of) the first and the second $\prec_W$-preferences arise through Definition 16. The argument is most complicated in case clause 16(b) is used throughout.

That is, assume $\exists\ \{R_{n_m}, S_n \in \mathbf{H}_\mathbf{R}\}$, $\exists\ \{H'_{n_m}\}$, $\exists\ \{M'_{n_m}\}$, $\exists\ \{M''_n\}$ and $\exists\ \{N'_n\}$ such that, $\forall\ n$, as $m \to \infty$, $\{H'_{n_m}\} \Rightarrow H_n$ and $\{M'_{n_m}\} \Rightarrow M_n$, while as $n \to \infty$,

$\{M''_n\} \Rightarrow M$ and $\{N'_n\} \Rightarrow N$. Also assume $\exists\, \{q_{n_m}, s_n \geq 0.5\}$, so $\forall\ n$, $\lim_{m\to\infty}\{q_{n_m}\} = q_n < 1$ and $\lim_{n\to\infty}\{s_n\} = s < 1$. By Definition 16,

$$\text{(B2)}\quad \begin{aligned} q_{n_m}H'_{n_m} + (1-q_{n_m})R_{n_m} &\prec^* (\text{or } \approx^*)\ q_{n_m}M'_{n_m} + (1-q_{n_m})\mathbf{W},\\ s_nM''_n + (1-s_n)S_n &\prec^* (\text{or } \approx^*)\ s_nN'_n + (1-s_n)\mathbf{W}. \end{aligned}$$

Since $\{H_n\} \Rightarrow H$ and $\{M_n\} \Rightarrow M$, $\forall\ n$, $\exists\ (m^* = m(n))$ so that, as $n \to \infty$, both $\{H'_{n_{m^*}}\} \Rightarrow H$ and $\{M'_{n_{m^*}}\} \Rightarrow M$. Moreover, we may choose (a subsequence of) these m^* so that $\lim_{n\to\infty}\{q_{n_{m^*}}\} = q$, $0.5 \leq q < 1$. Thus we have

$$\text{(B3)}\quad q_{n_{m^*}}H'_{n_{m^*}} + (1-q_{n_{m^*}})R_{n_{m^*}} \prec^* (\text{or } \approx^*)\ q_{n_{m^*}}M'_{n_{m^*}} + (1-q_{n_{m^*}})\mathbf{W}.$$

An application of the first two axioms to (B2) and (B3) yields

$$\text{(B4)}\quad \begin{aligned} &x_n[\text{left side(B2)}] + (1-x_n)[\text{left side(B3)}]\\ &\quad\prec^* (\text{or } \approx^*)\ x_n[\text{right side(B2)}] + (1-x_n)[\text{right side(B3)}]. \end{aligned}$$

Let $x_n = s_n/(s_n + q_{n_{m^*}})$. Then as both $\{M'_{n_{m^*}}\} \Rightarrow M$ and $\{M''_n\} \Rightarrow M$, we may cancel acts common to M on both sides of (B4) to yield $z_nH''_n + (1-z_n)T_n \prec^*$ (or $\approx^*$) $z_nN'_n + (1-z_n)\mathbf{W}$, where $\{H''_n\} \Rightarrow H$, $(N''_n) \Rightarrow N$, $T_n \in H_{\mathbf{R}}$ and $\lim_{n\to\infty}\{z_n\} = z = sq/(s+q-sq)$. Last, $0.5 \leq z < 1$ because $0.5 \leq s, q < 1$. Therefore, by Definition 16(b), $H \prec_W N$ as required. □

Finally, Theorem 2 is concluded by repeating this construction in a dualized form: extend the preference $\prec_W$ to $\prec'$ by introducing the act $\mathbf{B}$ and making it most preferred in $\mathbf{H}_{\mathbf{R}\cup\{\mathbf{W}\}}$. □

C. *Proof of Theorem* 3. We show (by induction) how to extend $\prec\ (=\prec_0)$ while preserving Axioms 2 and 3 over simple lotteries until the desired weak order is achieved. At stage i of the induction, the strategy is to identify a utility v_i for act $\tilde{H}_i \in \mathscr{H}$, where v_i is chosen (arbitrarily) from a (convex) set of target utilities for $\tilde{H}_i$, $\mathscr{T}_i(\tilde{H}_i)$. We create the partially ordered preference $\prec_i$ so that $\tilde{H}_i \approx_i v_iB + (1-v_i)W$.

Begin with a function $\mathscr{T}$ which provides a set of target "utilities" for all elements of $\mathbf{H}_{\mathbf{R}}$. We use $\mathbf{W}$ and $\mathbf{B}$ as the 0 and 1 of our utility. Assume $\{H_n\} \Rightarrow H$ and $H_n \in \mathbf{H}_{\mathbf{R}}$. For $i = 1$, by Definitions 17 and 18, $v_1^*(H)$ is the lim inf of the quantities x_n for which $H_n \prec x_n\mathbf{B} + (1-x_n)\mathbf{W}$ and $v_{1*}(H)$ is the lim sup of the quantities x_n for which $x_n\mathbf{B} + (1-x_n)\mathbf{W} \prec H_n$. The two "utility" bounds, $v_*(H)$ and $v^*(H)$, do not depend upon which sequence $\{H_n\} \Rightarrow H$ is used. We show this for $v_*(H)$. The argument for $v^*(H)$ is the obvious dual.

CLAIM 1. *Let $\{H_n\} \Rightarrow H$ and $\{H'_n\} \Rightarrow H$. Then $v_*(H)$ is the same for both sequences.*

PROOF. Suppose $v_*(H) = \limsup\{x_n\colon\ x_nB + (1-x_n)W \prec H_n\}$. Then we show that $v_*(H) \leq \limsup\{x_n\colon\ x_nB + (1-x_n)W \prec H'_n\}$. This suffices, since by symmetry with $\{H'_n\}$, when $v'_*(H) = \limsup\{x_n\colon\ x_nB + (1-x_n)W \prec H'_n\}$, then $v'_*(H) \le \limsup\{x_n\colon\ x_n B + (1 - x_n)W \prec H_n\}$; hence, $v_*(H) = v'_*(H)$. Since both sequences $\{H_n\}$ and $\{H'_n\}$ converge to act H, we can write each pair (H_n, H'_n) as the pair $(y_n K_n + (1 - y_n)M_n, y_n K_n + (1 - y_n)M'_n)$, where $\lim_{n\to\infty} y_n = 1$. Assume all but finitely many $y_n < 1$; else we are finished. Acts M_n and M'_n belong to $\mathbf{H_R}$ because H_n and H'_n do. Of course, neither of the two sequences of acts $\{M_n\}$ and $\{M'_n\}$ need be convergent, but $\{K_n\} \Rightarrow H$. For each n, define the act $N_n = y_n K_n + (1 - y_n)\mathbf{W}$. Clearly, $\{N_n\} \Rightarrow H$. It follows from the preference $\mathbf{W} \prec M_n$ that $N_n \prec H_n$ and from the preference $\mathbf{W} \prec M'_n$ that $N_n \prec H'_n$. By hypothesis, there exists a sequence $\{x_n\}$ such that $x_n\mathbf{B} + (1 - x_n)\mathbf{W} \prec H_n$ and $\lim_{n\to\infty}\{x_n\} = v_*(H)$. Let α_n be the maximum of 0 and $(x_n + y_n - 1)$. Since $x_n\mathbf{B} + (1 - x_n)\mathbf{W} \prec y_n K_n + (1 - y_n)\mathbf{B}$, then $\alpha_n\mathbf{B} + (1 - \alpha_n)\mathbf{W} \prec y_n K_n + (1 - y_n)\mathbf{W} = N_n$. Transitivity of $\prec$ yields $\alpha_n\mathbf{B} + (1 - \alpha_n)\mathbf{W} \prec H'_n$. However, as $\lim_{n\to\infty}\{y_n\} = 1$ and $\lim_{n\to\infty}\{x_n\} = v_*(H)$, then $\lim_{n\to\infty}\{\alpha_n\} = v_*(H)$. Thus, $v_*(H) \le \limsup\{x_n\colon\ x_n\mathbf{B} + (1 - x_n)\mathbf{W} \prec H'_n\}$. □

Observe that if $v^*(H_1) < v_*(H_2)$, then $H_1 \prec H_2$, by Axiom 3 and the fact that $\mathbf{W} \prec \mathbf{B}$. However, these "utility" bounds are merely sufficient, not necessary, for the $\prec$-preference $H_1 \prec H_2$.

Proof of Lemma 3.1. Note that $xB + (1 - x)W \prec yB + (1 - y)W$ whenever $x < y$.

(i) Suppose, on the contrary, that $v^*(H) < v_*(H)$. Then by Corollary 2.5 applied twice over, $v^*(H)B + (1 - v^*(H))W \approx H \approx v_*(H)B + (1 - v_*(H))W$. Since $v^*(H) < v_*(H)$, also $v^*(H)B + (1 - v^*(H))W \prec v_*(H)B + (1 - v_*(H))W$, contradicting the $\approx$-relation between them, as just derived.

(ii) This is immediate, by similar reasoning. □

Next, we show that $\prec$ may be extended to $\prec_H$, a strict partial order satisfying HL Axioms 2 and 3, in which $H \approx_H v_H B + (1 - v_H)W$ and where the utility v_H may be any value in the *interior* of the closed target set $\mathscr{T}(H)$. We resolve when an endpoint of the (closed) target set may be a utility afterward.

Definition 20. For $H \in \mathbf{H_R}$, let $v \in \operatorname{int}\mathscr{T}(H)$. [When $\mathscr{T}(H)$ has no interior, when $v_*(H) = v^*(H) = v$, then by Lemma 3.1(ii), $H \approx v\mathbf{B} + (1 - v)\mathbf{W}$. Thus it is appropriate that Definition 20 creates no extension of $\prec$. Then act H already has its "utility" fixed by $\prec$.] Define $\prec_H$ by

$$(H_1 \prec_H H_2) \quad \text{iff } \exists\,(0 < x < 1)\,\exists\,(G, G'),$$
$$xH_1 + (1 - x)G \prec xH_2 + (1 - x)G',$$

where G and G' are *symmetric mixtures* of H and $vB + (1 - v)W$.

Specifically, $\exists\ y$ with $G = yH + (1 - y)[vB + (1 - v)W]$ and $G' = y[vB + (1 - v)W] + (1 - y)H$.

LEMMA 3.2. $\prec_H$ *extends* $\prec$.

PROOF. Assume $H_1 \prec H_2$. Choose $y = 0.5$ in Definition 20, so $G = G'$. By Axiom 2, $xH_1 + (1-x)G \prec xH_2 + (1-x)G'$, so that $(H_1 \prec_H H_2)$. □

LEMMA 3.3. $\succ_H$ *satisfies HL Axioms* 1–3.

PROOF. We establish Axioms 1–3 separately.

HL Axiom 1 (*irreflexivity*). Assume not $(H_1 \prec_H H_1)$. Then $xH_1 + (1-x)G \prec xH_1 + (1-x)G'$, which by Axiom 2 yields $G \prec G'$. By Definition 20 and another application of Axiom 2, either $H \prec vB + (1-v)W$ or else $vB + (1-v)W \prec H$. Either contradicts the relation $H \sim vB + (1-v)W$. That follows from the assumption $v_* < v < v^*$.

HL Axiom 1 (*transitivity*). Assume $(H_1 \prec_H H_2)$ and $(H_2 \prec_H H_3)$. Then we have

$$xH_1 + (1-x)G \prec xH_2 + (1-x)G' \quad \text{and}$$
$$wH_2 + (1-w)J \prec wH_3 + (1-w)J',$$

where both pairs (G, G') and (J, J') satisfy Definition 20. These equations may be combined to create $\forall\ z$, $z(xH_1 + (1-x)G) + (1-z)(wH_2 + (1-w)J) \prec z(xH_2 + (1-x)G') + (1-z)(wH_3 + (1-w)J')$. Choose $z/(1-z) = w/x$. By HL Axiom 2, we may cancel the common term zxH_2 $[= (1-z)wH_2]$ from both sides and recombine the pairs (G, J) and (G', J') to yield $uH_1 + (1-u)K \prec uH_3 + (1-u)K'$. Thus, $H_1 \prec_H H_3$.

HL Axiom 2. Argue that $H_1 \prec_H H_2$ iff $yH_1 + (1-y)G \prec yH_2 + (1-y)G'$ iff $\forall\ (0 < z < 1)$, $z(yH_1 + (1-y)G) + (1-z)H_3 \prec z(yH_2 + (1-y)G') + (1-z)H_3$ iff $(\exists\ w)$ $w(xH_1 + (1-x)H_3) + (1-w)G \prec w(xH_2 + (1-x)H_3) + (1-w)G'$ (choose $w = zy/x$) iff $xH_1 + (1-x)H_3 \prec_H xH_2 + (1-x)H_3$.

HL Axiom 3(a). Assume $\forall\ n$ $(M_n \prec_H N_n)$, and $(N \prec_H O)$. Then show $(M \prec_H O)$.

1. Thus (a) $(x_n M_n + (1-x_n)G_n) \prec (x_n N_n + (1-x_n)G_n')$ and also (b) $(yN + (1-y)J) \prec (yO + (1-y)J')$.

As Definition 20 applies to the pairs $(G_n, G_n'), (J, J')$, we may cancel (by Axiom 2) common terms to create:

2. Either (a) $u_n M_n + (1-u_n)H \prec u_n N_n + (1-u_n)(vB + (1-v)W)$ or (b) $u_n M_n + (1-u_n)(vB + (1-v)W) \prec u_n N_n + (1-u_n)H$.
3. Also, in addition, either (a) $wN + (1-w)H \prec wO + (1-w)(vB + (1-v)W)$ or (b) $wN + (1-w)(vB + (1-v)W) \prec wO + (1-w)H$, where $u_n \geq x_n$ and $w \geq y$.

At least one of 2(a) or 2(b) occurs infinitely often. Without loss of generality, assume 2(a) does. Since $v_*(H) < v < v^*(H)$, then $\liminf\{u_n\} = u > 0$, in this infinite subsequence. [Only here do we use the fact that v is an interior point of $\mathscr{T}(H)$. See Lemma 3.5 for additional remarks.]

Thus, we are justified in considering a convergent sequence of the form 2(a), also indexed by n, with coefficients converging to $u > 0$. We argue by cases: Assume 3(a) obtains. Using Axiom 2, we mix in the act H to both sides of 2(a) and the act $(vB + (1 - v)W)$ to both sides of 3(a), yielding:

4(a) $x[u_n M_n + (1 - u_n)H] + (1 - x)H \prec x[u_n N_n + (1 - u_n)(vB + (1 - v)W)] + (1 - x)H.$

4(b) $z[wN + (1 - w)H] + (1 - z)(vB + (1 - v)W) \prec z[wO + (1 - w) \times (vB + (1 - v)W)] + (1 - z)(vB + (1 - v)W).$

Choose $xu = zw = q \neq 0$, $(1 - x) = z(1 - w)$. Note all of the following occur: the l.h.s. of 4(a) converges to the act $(qM + (1 - q)H)$; the r.h.s. of 4(b) is the act $(qO + (1 - q)(vB + (1 - v)W))$; the r.h.s. of 4(a) converges to the l.h.s. of 4(b). Then by HL Axiom 3(a), $(qM + (1 - q)H) \prec (qO + (1 - q)(vB + (1 - v)W))$, so by Definition 20, $M \prec_H O$.

In case 3(b) obtains, instead, we modify this argument by mixing the term $(vB + (1 - v)W)$ into 2(a) in case $u < w$ or into 3(b) in case $w < u$, so that (as above) the r.h.s. of 4(a) converges to the l.h.s. of 4(b) and so forth.

HL Axiom 3(b). This is verified just as HL Axiom 3(a) is.

Thus, $\prec_H$ satisfies the axioms. □

To complete our discussion of $\mathscr{T}(H)$, we explain when $\prec_H$ may be created using an endpoint of the target set. To motivate our analysis, consider when a partial order $\prec$ precludes an extension by a particular new preference or indifference.

DEFINITION 21. Say that a *preference* for act H_a over act H_b *is precluded* by the partial order $\prec$, denoted as $\neg(H_b \prec H_a)$, if there exist two convergent sequences of acts $\{H_{a,n}\} \Rightarrow H_a$ and $\{H_{b,n}\} \Rightarrow H_b$, where $(\forall\ n)\ H_{a,n} \prec H_{b,n}$. [*Note*: $H_a \approx H_b$ or $H_a \prec H_b$ implies the condition $\neg(H_b \prec H_a)$.]

DEFINITION 22. Say that *indifference* between acts H_a and H_b *is precluded* by the partial order $\prec$, denoted as $\neg(H_b \approx H_a)$, if assuming the relation $(H_a \approx H_b)$, the three axioms and the preferences $\prec$ all yield a preference precluded by $\prec$.

EXAMPLE 3.2. We illustrate $\neg(H_b \approx H_a)$. Suppose $\prec$ satisfies the axioms and the following obtain. Let $H_a \sim H_b$. However, there exist two convergent sequences of acts $\{M_n\} \Rightarrow M$ and $\{N_n\} \Rightarrow N$ and coefficients $\{x_n\colon x_n > 0, \lim_{n\to\infty} x_n = 0\}$, where $x_n M_n + (1 - x_n)H_a \prec x_n N_n + (1 - x_n)H_b$. However, for some $y > 0$, $yN + (1 - y)H_a \prec yM + (1 - y)H_b$. Thus, $(H_a \approx H_b)$ entails $M_n \prec N_n$ and $N \prec M$. By Axiom 3, then $M \prec M$, which is a $\prec$-precluded preference since $M \approx M$ obtains whenever $\prec$ satisfies the axioms.

[We sketch a model for these $\prec$-preferences. Let $H_a = v_* B + (1 - v_*)W$ and $H_b = H$. Suppose $\mathscr{V}$ is a set of utilities $\{V_d\colon 1 > d > 0;\ V_d(H) = v_* + d$ and $V_d(M) - V_d(N) = d^{0.5}\}$. Consider $\prec_{\mathscr{V}}$, when $\mathscr{T}(H) = [v_*, v^*]$ yet $H_a \prec_{\mathscr{V}}$

H_b. Let $\prec$ be as $\prec_{\mathscr{V}}$ except that $H_a \sim H_b$ is forced. $V_d(xM + (1-x)v_*) \leq V_d(xN + (1-x)H)$ entails that $x/(1-x) \leq d^{0.5}$. Since d assumes each value in (0.1), $x = 0$ is a necessary condition for the preferences of Example 3.2.]

CLAIM 2. *If both* $\neg(H_a \prec H_b)$ *and* $\neg(H_b \prec H_a)$, *then* $H_a \approx H_b$.

PROOF. When $\neg(H_a \prec H_b)$ and $\neg(H_b \prec H_a)$, then there exist pairs of convergent acts $\{H_{a,n}\}, \{H'_{a,n}\} \Rightarrow H_a$ and $\{H_{b,n}\}, \{H'_{b,n}\} \Rightarrow H_b$, where $(\forall\ n)$ $H_{a,n} \prec H_{b,n}$ and $H'_{b,n} \prec H'_{a,n}$. Then by Corollary 2.5, $H_a \approx H_b$. $\square$

Example 3.2 illustrates that our axioms are not strong enough to ensure the preference $H_a \prec H_b$ when, for example $\neg(H_b \prec H_a)$ and $\neg(H_a \approx H_b)$. It so happens that when both the conditions $\neg(H_b \prec H_a)$ and $\neg(H_a \approx H_b)$ obtain and these two acts do *not* involve the distinguished rewards W and B, then each extension $\prec^*$ of $\prec$ which fixes "utilities" for H_a and H_b (and where $\prec^*$ arises by iteration of Definition 20) has the desired relation $H_a \prec^* H_b$. Our specific problem, however, is with the case when one of these two acts is a utility endpoint of the (closed) target set $\mathscr{T}(H)$, for example, let $H_a = v_* B + (1 - v_*)\mathbf{W}$ and $H_b = H$, as in the model for the $\prec$-preferences sketched in Example 3.2. We require an extra consideration, then, to determine whether, though $v_* B + (1 - v_*)W \sim H$, a combination of $\prec$-preferences arises which prohibits an extension $\prec_H$ of $\prec$ that assigns the "utility" v_* for H.

Our solution is to show how to extend the partial order $\prec$ to a partial order $\prec^+$ that includes all the so-called missing preferences $H_a \prec H_b$.

DEFINITION 23. Define $\prec^+$ from $\prec$ by $H_a \prec^+ H_b$ iff $H_a \prec H_b$ or, both $\neg(H_b \prec H_a)$ and $\neg(H_a \approx H_b)$.

The next lemma establishes (very weak) conditions under which the $\prec^+$-closure of a partial order $\prec$ satisfies all three axioms. In particular, it is not necessary that $\prec$ satisfies HL Axiom 3. [The condition $\neg(H_b \prec H_a)$ is well defined according to Definition 23 even though $\prec$ is known only to satisfy HL Axioms 1 and 2. Specifically, the indifference relation $\approx$ is well defined and satisfies all those properties, e.g., Corollary 2.4, which depend on HL Axioms 1 and 2 alone.]

LEMMA 3.4. *The partial order* $\prec^+$ *satisfies all three axioms provided* $\prec$ *satisfies the first two axioms*, HL *Axioms* 1 *and* 2, *and provided closure of* $\prec$ *under all three axioms does not produce a* $\prec$*-precluded preference*.

PROOF. We verify the axioms separately.

HL Axiom 1 (*irreflexivity*). Since $H \approx H$ obtains and $\prec$ yields no $\prec$-precluded preference (under the three axioms), no act H satisfies $H \prec^+ H$. That is, $H \approx H$ is not $\prec$-precluded.

HL Axiom 1 (*transitivity*). Assume $H_a \prec^+ H_b$ and $H_b \prec^+ H_c$. Each of these $\prec^+$-preferences may arise two ways, according to Definition 23. We examine a general case: $\neg(H_b \prec H_a)$, $\neg(H_c \prec H_b)$, $\neg(H_a \approx H_b)$ and $\neg(H_b \approx H_c)$. We show that (i) $\neg(H_c \prec H_a)$ and (ii) $\neg(H_a \approx H_c)$.

(i) From the two assumptions $\neg(H_b \prec H_a)$ and $\neg(H_c \prec H_b)$, we conclude that there exist convergent sequences $\{H_{a,n}\} \Rightarrow H_a$, $\{H_{b,n}\}$ and $\{H'_{b,n}\} \Rightarrow H_b$ and $\{H_{c,n}\} \Rightarrow H_c$, with $(\forall\, n)\ H_{a,n} \prec H_{b,n}$ and $H'_{b,n} \prec H_{c,n}$. Thus, by Axioms 1 and 2, $0.5H_{a,n} + 0.5H'_{b,n} \prec 0.5H_{b,n} + 0.5H_{c,n}$. Using HL Axiom 2 to cancel common terms in $H_{b,n}$ and $H'_{b,n}$, we obtain $\prec$-preferences of the form $H'_{a,n} \prec H'_{c,n}$, where $\{H'_{a,n}\} \Rightarrow H_a$ and $\{H'_{c,n}\} \Rightarrow H_c$. Thus, $\neg(H_c \prec H_a)$.

(ii) Assume $H_a \approx H_c$. Because $H_{a,n} \prec H_{b,n}$ we may construct new convergent sequences $\{H'_{c,n}\} \Rightarrow H_c$ and $\{H''_{b,n}\} \Rightarrow H_b$, where $H'_{c,n} \prec H''_{b,n}$.

This exercise is done as follows. From the indifference $H_a \approx H_c$ conclude $(1/n)W + ([n-1]/n)H_c \prec (1/n)B + ([n-1]/n)H_a$. Then $0.5H_{a,n} + 0.5[(1/n)W + ([n-1]/n)H_c] \prec 0.5H_{b,n} + 0.5[(1/n)B + ([n-1]/n)H_a]$. Use Axiom 2 to cancel common terms involving act H_a.

We already have assumed $H'_{b,n} \prec H_{c,n}$. Then, since we are entitled to use HL Axiom 3 in determining the consequences of adopting $H_a \approx H_c$, by Corollary 2.5, from $H_a \approx H_c$ we derive $H_b \approx H_c$. $\neg(H_b \approx H_c)$ means that adding the $\approx$-indifference $H_b \approx H_c$ yields a $\prec$-precluded preference. Thus, adding $H_a \approx H_c$ to $\prec$ yields the same $\prec$-precluded preference. Hence, $\neg(H_a \approx H_c)$.

HL Axiom 2 (*independence*). This axiom is easy to verify, since $\prec$ satisfies HL Axioms 1 and 2. We illustrate the argument from right to left. Suppose $xH_a + (1-x)H \prec^+ xH_b + (1-x)H$. We are to show that (i) $\neg(H_b \prec H_a)$ and (ii) $\neg(H_a \approx H_b)$.

(i) We know that both $\neg(xH_b + (1-x)H \prec xH_a + (1-x)H)$ and $\neg(xH_a + (1-x)H \approx xH_b + (1-x)H)$. As in previous cases, we may assume existence of convergent sequences $\{H_{1,n}\} \Rightarrow xH_a + (1-x)H)$ and $\{H_{2,n}\} \Rightarrow xH_b + (1-x)H)$, where $H_{1,n} \prec H_{2,n}$. Use HL Axiom 2 to cancel common terms (involving act H) in each pair $H_{1,n}$ and $H_{2,n}$. The results are $\prec$-preferences of the form $H_{a,n} \prec H_{b,n}$, with $\{H_{a,n}\} \Rightarrow H_a$ and $\{H_{b,n}\} \Rightarrow H_b$. Thus, $\neg(H_b \prec H_a)$.

(ii) By Corollary 2.4, from the assumption $H_a \approx H_b$ it follows that $xH_a + (1-x)H \approx xH_b + (1-x)H$, which yields a $\prec$-precluded preference as $\neg(xH_a + (1-x)H \approx xH_b + (1-x)H)$.

HL Axiom 3(a). Assume $M_n \prec^+ N_n$ and $N \prec^+ O$, where $\{M_n\} \Rightarrow M$ and $\{N_n\} \Rightarrow N$. We need to show that (a) $\neg(O \prec M)$ and (b) $\neg(M \approx O)$.

(a) Thus $\neg(N_n \prec M_n)$, $\neg(M_n \approx N_n)$, $\neg(O \prec N)$ and $\neg(N \approx O)$. As in previous cases, assume each of these $\prec$-precluded preferences arises from corresponding sequences of $\prec$-preferences. That is, for each n there is a pair of convergent sequences $\lim_{j\to\infty}\{M_{n,j}\} \Rightarrow M_n$ and $\{N_{n,j}\} \Rightarrow N_n$, where $M_{n,j} \prec N_{n,j}$. Also, there is a pair of convergent sequences $\{N'_n\} \Rightarrow N$ and $\{O_n\} \Rightarrow O$, where $N'_n \prec O_n$. Since $\{M_n\} \Rightarrow M$ and $\{N_n\} \Rightarrow N$, for each n we may choose a value j_n so that $\lim_{n\to\infty}\{M_{n,j_n}\} \Rightarrow M$ and $\{N_{n,j_n}\} \Rightarrow N$. Of course, $M_{n,j_n} \prec N_{n,j_n}$.

Then, $0.5M_{n,j_n} + 0.5N'_n \prec 0.5N_{n,j_n} + 0.5O_n$. Use HL Axiom 2 to cancel terms common to act N, yielding $\prec$-preferences sufficient for $\neg(O \prec M)$.

(b) If we assume $M \approx O$, then (because $M_{n,j_n} \prec N_{n,j_n}$) there are sequences $\{O'_n\} \Rightarrow O$ and $\{N''_n\} \Rightarrow N$, with $O'_n \prec N''_n$. Since $N'_n \prec O_n$, using Axiom 3, by Corollary 2.5, then $N \approx O$. However, $\neg(N \approx O)$. Hence, assuming $M \approx O$ entails some $\prec$-precluded preference. Therefore, $\neg(M \approx O)$. HL Axiom 3(b) is demonstrated in the identical fashion. □

In the next definition, based on Lemma 3.4, we indicate whether either endpoint of $\mathscr{T}(H)$ is eligible as a utility for H when extending $\prec$ to form $\prec_H$.

DEFINITION 24. Say that $v_*(H)$ is a *candidate utility* for H if $v_*\mathbf{B} + (1 - v_*)\mathbf{W} \sim^+ H$. Likewise, $v^*(H)$ is a *candidate utility* for H if $H \sim^+ v^*\mathbf{B} + (1 - v^*)\mathbf{W}$.

We conclude our discussion of the extension $\prec_H$ for the special case when it is generated by a target set endpoint provided, of course, the endpoint is a candidate utility for H. The idea behind the extension, is that as it stands, Definition 20 fails with $v = v_*$ or $v = v^*$ only because the resulting partial order is incomplete with respect to Axiom 3. (See Lemma 3.5, below.) Then, in light of Lemma 3.4, the $+$-closure (using Definition 23) corrects the omissions. (See Lemma 3.6.)

When extending $\prec$ with a candidate utility, $v = v_*$ or $v = v^*$, that is, using an endpoint of $\mathscr{T}(H)$, we define the extension $\prec_H$ in two steps, as follows: Analogous with Definition 20, let G and G' be *symmetric mixtures* of H and $vB + (1 - v)W$.

DEFINITION 25. Define $H_1 \prec_{\#} H_2$ iff $\exists$ $(0 < x < 1)$ $\exists$ (G, G'), $xH_1 + (1 - x)G \prec xH_2 + (1 - x)G'$, and let $\prec_H$ result by closing $\prec_{\#}$ using Definition 23, that is, $\prec_H = \prec_{\#}^+$.

LEMMA 3.5. *The partial order $\prec_{\#}$ extends $\prec$ and satisfies axioms HL Axioms* 1 *and* 2.

PROOF. Since v is a candidate utility, $v\mathbf{B} + (1 - v)\mathbf{W} \sim H$. Then, as Definition 25 duplicates Definition 20, the proofs from Lemmas 3.2 and 3.3 apply to show that $\prec_{\#}$ extends $\prec$ and satisfies the first two axioms. □

LEMMA 3.6. *The partial order $\prec_{\#}^+$ extends $\prec$ and satisfies all three axioms.*

PROOF. In light of Lemma 3.5, the result follows by Lemma 3.4 once we show that $\prec_{\#}$ may be closed under the axioms without generating a $\prec_{\#}$-precluded preference. Note that $\prec_{\#}$ extends $\prec$ by some, but not necessarily all, preferences entailed (by the three axioms) from the $\approx$-indifference $H \approx vB + (1-v)W$. Then, since v is a candidate utility, closing $\prec_{\#}$ under the three axioms does not lead to a $\prec$-precluded preference. We claim, next, that it does not lead to a $\prec_{\#}$-precluded preference either. Suppose, on the contrary, it does. Suppose, for example, closing $\prec_{\#}$ with the axioms results in a relation $H_b \prec_{\#} H_a$, where also $\neg(H_b \prec_{\#} H_a)$. The former means that adding $H \approx vB + (1-v)W$ to the set of $\prec$-preferences entails (by the axioms) that $H_b \prec H_a$. The latter requires that, for two convergent sequences $\{H_{a,n}\}$ and $\{H_{b,n}\}$, $(H_{a,n} \prec_{\#} H_{b,n})$. Thus, adding $H \approx vB + (1-v)W$ to the set of $\prec$-preferences entails (by the axioms) that $(H_{a,n} \prec H_{b,n})$. By HL Axiom 3, these lead to a $\prec$-precluded preference $(H_b \prec H_b)$. Then v is not a candidate utility for H with respect to $\prec$, a contradiction. □

Thus, with Definition 20, we have indicated how to extend $\prec$ to $\prec_H$, where act H is assigned a utility v from the interior of its target set $\mathscr{T}(H)$, and with Definition 25, how to extend to $\prec_H$ using a candidate utility endpoint.

We interject two simple, but useful results about $\approx_H$-indifferences. The first confirms that the extension $\prec_H$ preserves $\approx$-indifferences. The second shows that the extension $\prec_H$ makes act H $\approx_H$-indifferent with its assigned utility v.

LEMMA 3.7. *If $M \approx N$, then $M \approx_H N$.*

PROOF. Suppose $M \approx N$ and that $xM + (1-x)H_3 \prec_H H_4$. We are to show that $xN + (1-x)H_3 \prec_H H_4$. By Definition 23, $[y(xM + (1-x)H_3) + (1-y)G] \prec [yH_4 + (1-y)G']$. After rearranging terms, by Corollary 23, $[y(xN + (1-x)H_3) + (1-y)G] \prec [yH_4 + (1-y)G']$, so that $xN + (1-x)H_3 \prec_H H_4$. □

LEMMA 3.8. $H \approx_H v\mathbf{B} + (1-v)\mathbf{W}$.

PROOF. Since $\mathbf{W} \prec \mathbf{B}$, we have the following:

$$\begin{aligned}&(1/n)W + [(n-1)/n][0.5H + 0.5(vB + (1-v)W)]\\ &\qquad \prec (1/n)B + [(n-1)/n][0.5H + 0.5(vB + (1-v)W)].\end{aligned}$$

This equation may be written as $x_n H_n + (1-x_n)(vB + (1-v)W) \prec x_n M_n + (1-x_n)H$, where $\{x_n\} \to 0.5$, $\{H_n\} \Rightarrow H$ and $\{M_n\} \Rightarrow (vB + (1-v)W)$. By Definition 20, $H_n \prec_H M_n$. Similarly, it may be written $x_n M'_n + (1-x_n)(H) \prec x_n H'_n + (1-x_n)(vB + (1-v)W)$, where $\{x_n\} \to 0.5$, $\{H'_n\} \Rightarrow H$ and $\{M'_n\} \Rightarrow (vB + (1-v)W)$. By Definition 20, $M'_n \prec_H H'_n$. Then, by Corollary 2.5, $H \approx_H v\mathbf{B} + (1-v)\mathbf{W}$. □

We iterate Definition 20 (or Definition 25) in a denumerable sequence of extensions of $\prec$.

DEFINITION 26. Define the set $\mathscr{H} = \{\tilde{H}_i^k\colon\ \tilde{H}_i^k(s_j) = r_1$ if $j \neq k$, and $\tilde{H}_i^k(s_k) = r_i\}$. Let $\mathbf{r}_1$ denote the constant act that yields reward r_1 in each state, so that $\mathbf{r}_1 \in \mathscr{H}$.

LEMMA 3.9. *$\mathscr{H}$ is countable and finite if* $\mathbf{R}$ *is finite.*

The proof is obvious.

[$\mathscr{H}$ remains countable even when π is a *denumerable* partition. Then it follows from HL Axiom 3 that personal probabilities over π are σ-additive. That is, HL Axiom 3 entails "continuity": $\lim_{n\to\infty} p\{\bigcap E_n\} = p\{\lim_{n\to\infty} \bigcap E_n\}$. In the light of Fishburn's (1979), page 139, result Theorem 10.5, we *conjecture* that our central theorems, e.g., Theorems 3 and 6, carry over to countably infinite partitions. However, this is not evident, e.g., our proof of Claim 1 (for Theorem 3) does not apply when π is infinite. Our use of finite partitions avoids mandating σ-additivity of personal probability.]

Hereafter, we enumerate $\mathscr{H}$ with a single subscript i. At stage i of the induction, $\prec_i$ is obtained by choosing a target utility v_i for act $\tilde{H}_i \in \mathscr{H}$, denoted $V(\tilde{H}_i) = v_i$. Here $v_i \in \mathscr{T}_i(\tilde{H}_i)$ and $\mathscr{T}_i(\cdot)$ identifies sets of target utilities, based on $\prec_{i-1}$. By Lemma 3.7, extensions preserve utilities already assigned, so that all utilities fixed by stage i are well defined over stages $j \geq i$. Next, we show that each simple act has its "utility" V determined by a finite subset of $\mathscr{H}$.

LEMMA 3.10. *If $H \in \mathbf{H_R}$ is a simple act, then there is a (finite) stage $\prec_m$ such that $\mathscr{T}_m(H)$ is a unit set, that is, by stage $\prec_m$, H is assigned a precise utility $V(H)$.*

PROOF. First we verify that V has the expected utility property over elements of $\mathscr{H}$. Consider $\tilde{H}_a, \tilde{H}_b \in \mathscr{H}$. Without loss of generality, let $b = \max\{a, b\}$. Both $\tilde{H}_a$ and $\tilde{H}_b$ have their respective utilities by stage $\prec_b$. That is, $\tilde{H}_a \approx_b v_a\mathbf{B} + (1 - v_a)\mathbf{W}$ and $\tilde{H}_b \approx_b v_b\mathbf{B} + (1 - v_b)\mathbf{W}$. By Corollary 2.4, $x\tilde{H}_a + (1 - x)\tilde{H}_b \approx_b x(v_a\mathbf{B} + (1 - v_a)\mathbf{W}) + (1 - x)(v_b\mathbf{B} + (1 - v_b)\mathbf{W})$. Hence, $V(x\tilde{H}_a + (1 - x)\tilde{H}_b) = xV(\tilde{H}_a) + (1 - x)V(\tilde{H}_b)$.

Next, write $H(s_j) = \sum_{i=1}^{k_j} P_j(r_i)$. Define the act $H_j'(s) = H(s)$ if $s = s_j$; otherwise $H'(s) = r_1$. Since H is simple, each H_j' is a finite combination of $\tilde{H}_i \in \mathscr{H}$. Specifically, $H_j' = \sum_{i=1}^{k_j} P_j(\tilde{H}_i^j)$, where $\tilde{H}_i^j(s) = r_i$ if $s = s_j$ and $\tilde{H}_i^j(s) = r_1$ otherwise. Observe that $(1/n)H + (n - 1/n)\mathbf{r}_1 = \Sigma(1/n)H_j'$. Thus the utility $V(H)$ is determined once $V(\mathbf{r}_1)$ and the n values $V(H_j')$ are fixed, all of which occurs after finitely many elements of $\mathscr{H}$ are assigned their utilities. □

We create a weak order $\precsim_\omega$ from the partial orders $\prec_i$ $(i = 1, \dots)$ using the fact that each $H \in \mathbf{H_R}$ is a limit point of simple horse lotteries. For $H \in \mathbf{H_R}$ consider a sequence $\{H_n\} \Rightarrow H$, where H_n is a simple act. Let $V(H) = \lim_{n\to\infty} V(H_n)$. Then:

LEMMA 3.11. *$V(H)$ is well defined.*

PROOF. We show that if $\{H_n\} \Rightarrow H$, then $\lim_{n\to\infty} V(H_n)$ exists and is unique. Assume $\{H_n\} \Rightarrow H$ and $\{H'_n\} \Rightarrow H$, where all these acts belong to $\mathbf{H_R}$. Without loss of generality, since the simple acts form a dense subset of $\mathbf{H_R}$ under the topology of pointwise convergence, suppose that each of H_n, H'_n is simple. Then write H_n as $y_n K_n + (1-y_n)M_n$ and H'_n as $y_n K_n + (1-y_n)M'_n$, where $\lim_{n\to\infty} y_n = 1$ and each of K_n, M_n and M'_n is a simple act in $\mathbf{H_R}$. By Lemma 3.10, $V(H_n) - V(H'_n) = (1-y_n)[V(M_n) - V(M'_n)]$. Since $\lim_{n\to\infty} y_n = 1$ and V is in the unit interval $[0,1]$, $\lim_{n\to\infty} V(H_n) - V(H'_n) = 0$. □

The next lemma establishes that V has the expected utility property for all $H \in \mathbf{H_R}$.

LEMMA 3.12. *If $H_a, H_b \in \mathbf{H_R}$, then $V(xH_a + (1-x)H_b) = xV(H_a) + (1-x)V(H_b)$.*

PROOF. Consider two sequences $\{H_{a,n}\} \Rightarrow H_a$ and $\{H_{b,n}\} \Rightarrow H_a$, where each of $H_{a,n}$ and $H_{b,n}$ is simple and belongs to $\mathbf{H_R}$. Then, for each n, the act $x(H_{a,n}) + (1-x)H_{b,n}$ is simple and belongs to $\mathbf{H_R}$. It is evident that $\{x(H_{a,n}) + (1-x)H_{b,n}\} \Rightarrow xH_a + (1-x)H_b$. By Lemma 3.10, $V(xH_{a,n} + (1-x)H_{b,n}) = xV(H_{a,n}) + (1-x)V(H_{b,n})$. Then by Lemma 3.11, $V(xH_a + (1-x)H_b) = xV(H_a) + (1-x)V(H_b)$. □

Last, define the weak order $\precsim_\omega$ for $H \in \mathbf{H_R}$ using the utilities fixed by V:

DEFINITION 27. $(H_1 \precsim_\omega H_2)$ iff $\mathbf{V}(H_1) \le \mathbf{V}(H_2)$.

We complete the proof of Theorem 3:

(i) That $\precsim_\omega$ is a weak order over elements of $\mathbf{H_R}$ follows simply by noting that V is real-valued. By Lemma 3.12, it satisfies the independence axiom. The Archimedean axiom also is a simple consequence of Lemmas 3.10 and 3.11, that is, if $\{M_n\} \Rightarrow M$, $\{N_n\} \Rightarrow N$ and $M_n \prec_\omega N_n$, then $V(M) \le V(N)$. Next, let H_a and H_b be simple, that is, each with finite support. Suppose $(H_a \prec H_b)$. According to Lemma 3.10, the utilities $V(H_a)$ and $V(H_b)$ are determined by some stage k of the induction, where k is the maximum index of the (finitely many) elements of $\mathscr{H}$ in the combined supports of H_a and H_b. Lemma 3.2 establishes that $\prec_k$ extends $\prec$. Then $(L_1 \prec_k L_2)$ and thus $V(H_a) < V(H_b)$. Therefore, $\precsim_\omega$ extends $\prec$ for simple lotteries.

(ii) We argue that V almost agrees with $\prec$, that is, if $(H_1 \prec H_2)$, then $(H_1 \precsim_V H_2)$. Here is a simple lemma about the changing endpoints of target sets which completes the theorem.

LEMMA 3.13. *For every act $H \in \mathbf{H_R}$ and stage $j = 2, \ldots,$* (i) $v_{j-1*}(H) \le v_{j*}(H) \le v_j^*(H) \le v_{j-1}^*(H)$ *and* (ii) $\lim_{j\to\infty} v_{j*}(H) = v_j^*(H) = V(H)$.

PROOF. (i) Since $\prec_j$ extends $\prec_{j-1}$, any sequence of $j-1$ stage preferences $H_n \prec_{j-1} x_n B + (1 - x_n)W$ also obtain at stage j. Thus, by Definitions 20 and 25 and Lemma 3.2(i), $v_{j-1*}(H) \le v_{j*}(H) \le v_j^*(H) \le v_{j-1}^*(H)$.

(ii) For each act $\tilde{H}_i \in \mathscr{H}$, $\forall\,(j > i)$, $v_{j*}(\tilde{H}_i) = v_j^*(\tilde{H}_i) = V(\tilde{H}_i) = v_i$. Hence, (ii) is obvious for all simple lotteries. Assume H is not simple. It is easy to find a convergent sequence of simple acts in $\mathbf{H_R}$, $\{K_n\} \Rightarrow H$, where $v_{n*}(K_n) = v_n^*(K_n) = V(K_n)$ and $H = y_n K_n + (1 - y_n)M_n$. The sequence of acts M_n, though elements of $\mathbf{H_R}$, need not converge. Since H is not simple, $y_n < 1$. Then, $y_n K_n + (1 - y_n)\mathbf{W} \prec y_n K_n + (1 - y_n)M_n \prec y_n K_n + (1 - y_n)\mathbf{B}$. As each $\prec_n$ extends $\prec$, we have $y_n V(K_n) < v_{n*}(H) \le v_n^*(H) < y_n V(K_n) + (1 - y_n)$. However, $\lim_{n\to\infty} y_n = 1$ and, by Lemma 3.8, $\lim_{n\to\infty} V(K_n) = V(H)$. Thus, $\lim_{j\to\infty} v_{j*}(H) = \lim_{j\to\infty} v_j^*(H) = V(H)$. □

Finally, if $(H_1 \prec H_2)$, since for each n, $\prec_n$ extends $\prec$, we have that $v_{n*}(H_1) \le v_n^*(H_2)$. Then by Lemma 3.13, $V(H_1) \le V(H_2)$. □

D. *Other results from Section* 3.

PROOF OF COROLLARY 3.1. The extensions $\prec_i$ created in Theorem 3 rely on the existence at stage $i-1$ of a nonempty target set $\mathscr{T}_i(\tilde{H}_i)$, only for the acts $\tilde{H}_i \in \mathscr{H}$. However, $\mathscr{T}_i(\cdot)$ is defined on all of $\mathbf{H_R}$, including the nonsimple acts. Hence, we can amend the sequence of extensions of $\prec$ to fix utilities for any countable set of acts, in addition to fixing utilities for each element of $\mathscr{H}$. Just modify the argument of Theorem 3 to assign utilities to the countable set $\mathscr{H} \cup \mathscr{B}$. □

In connection with Example 3.1, for instance, we can introduce acts H_a and H_b into a well ordering of $\mathscr{H}$, for example, $\{\tilde{H}_1, H_a, \tilde{H}_2, H_b, \tilde{H}_3, \dots\}$, so that by stage 4 of the sequence of extensions, $k_1 = V(H_a) < V(H_b) = k_2$, which precludes the undesired limit stage in which $V(r_i) = 0.25$ $(i = 1, \dots)$.

Theorem 4 is easily demonstrated.

PROOF OF THEOREM 5. That $\phi \ne \mathscr{V} \subseteq \mathscr{Z}^{\mathscr{S}}$ is part (i) of Theorem 4. For the converse, argue indirectly. If $Z \in \mathscr{Z}^{\mathscr{S}}/\mathscr{V}$ then let $\tilde{H}_k$ be the first element of $\mathscr{H}$ (that is, let k be the least integer) for which $Z(\tilde{H}_k) \notin \mathscr{T}_k(\tilde{H}_k)$, even though $v_1 = Z(\tilde{H}_1), \dots, v_{k-1} = Z(\tilde{H}_{k-1})$ for acts $\tilde{H}_1, \dots, \tilde{H}_{k-1}$. Then Z agrees with $\prec_{k-1}$, since $\prec_{k-1}$ is the result of extending $\prec$ by the conditions $\tilde{H}_i \approx v_i B + (1 - v_i)W$ $(i = 1, \dots, k-1)$. That is, expand each $\prec_{k-1}$-preference into a $\prec$-preference. The former follows from the latter by adding a set of $k-1$ assumptions $\{\tilde{H}_i \approx v_i B + (1 - v_i)W\colon (i = 1, \dots, k-1)\}$ to $\prec$. But these $k-1$ conditions are satisfied under Z, and Z agrees with $\prec$ on simple acts. Hence, it must be that either $Z(\tilde{H}_k) = v_k = v_{k*}(\tilde{H}_k)$ and $\mathscr{T}_k(\tilde{H}_k)$ is open at the lower end or else $Z(\tilde{H}_k) = v_k = v_k^*(\tilde{H}_k)$ and $\mathscr{T}_k(\tilde{H}_k)$ is open at the upper end. However, if the target set is open and if an endpoint v_k of $\mathscr{T}_k(\tilde{H}_k)$ is not a candidate utility for $\tilde{H}_k$, then adding $\tilde{H}_k \approx v_k B + (1 - v_k)W$ to $\prec_{k-1}$ produces a $\prec_{k-1}$-precluded preference. Since Z agrees with $\prec_{k-1}$, Z does not agree with any $\prec_{k-1}$-precluded preference. Thus, Z cannot assign act $\tilde{H}_k$ the utility v_k, which contradicts the assumption $Z(\tilde{H}_k) = v_k$. □

E. *Results from Section* 4. The proof of Lemma 4.1 is immediate after Theorem 13.1 of Fishburn (1979).

PROOF OF LEMMA 4.2. Recall the strict preferences $\mathbf{W} \prec H \prec \mathbf{B}$, whenever $W, B \notin \text{supp}(H)$. Hence, for each V, we may standardize the (expected) utility of act $\mathbf{W}$ as 0 and the (expected) utility of act $\mathbf{B}$ as 1, where all other acts (not involving W and B) have (expected) cardinal utilities in the open interval $(0, 1)$. Next, define a set of simple, called-off acts $\{H_{i,j} \in \mathbf{Hs_j}\}$, which yield the lottery outcome $L_i \in \mathbf{L}_{\mathbf{R}\text{-}\{W,B\}}$ in state s_j and outcome W in all other states. In keeping with this notation, let $H_{W,j} = W$ and let $H_{B,j}$ be the $\mathbf{Hs_j}$ act with outcome B in state s_j. Recall, for each j, $\lim_{m\to\infty}\{H^i_{m,j}\} \Rightarrow H_{i,j}$ and $H^W_{m,j} = H_{W,j} = \mathbf{W}$. Then, whenever $H_{L_i} < H_{L_k}$ (by HL Axiom 5), $\mathbf{W} \prec H^i_{m,j} \prec H^k_{m,j} \prec H^B_{m,j}$. Hence, by the Archimedean HL Axiom 3 (as in Lemma 2.3), we have the restriction $\neg(H_{B,j} \prec H_{k,j} \prec H_{i,j} \prec \mathbf{W})$. Moreover, this constraint obtains also for each extension of $\prec$, including all the limit extensions $\precsim_V$ since these $\prec$-preferences involve simple acts. Then, for each $V, \mathbf{W} \precsim_V H_{k,j} \precsim_V H_{i,j} \precsim_V H_{B,j}$. Trivially, either $\mathbf{W} \approx_V H_{B,j}$ or else $\mathbf{W} \prec_V H_{B,j}$. The upshot is that, for each V, one of two circumstances obtains:

Case 1. If $\mathbf{W} \approx_V H_{B,j}$, $H_{\alpha,j} \approx_V H_{\beta,j}$ and s_j is null under $\precsim_V$, so $p(s_j) = 0$.

Case 2. If $\mathbf{W} \prec_V H_{B,j}$, then s_j is V-nonnull and for each representation of V as an expected, state-dependent utility [in accord with condition (4.1)], $U_j(W) \le U_j(L_i) \le U_j(L_k) \le U_j(B)$, with at least one of the outside inequalities strict. However, since the U_j are defined only up to a similarity transformation, without loss of generality choose $U_j(W) = 0$ and $U_j(B) = 1$ and rescale p accordingly. □

PROOF OF LEMMA 4.3. Without loss of generality (Corollary 3.3), let the denumerable sequence $\mathscr{H} = \{\tilde{H}_i\}$ of simple horse lotteries, used to create the set $\mathscr{V}$ of extensions for $\prec$, take $\{H_{r_1}, \dots, H_{r_n}\}$ as its initial segment: the constant acts that award r_i in each state. Suppose the interval $\mathscr{T}_1(r_1)$ is *not* open, for example, $\mathscr{T}_1(r_1) = [v_{1*}, v_1^*)$. Then $0 < v_{1*}$. Extend $\prec$ according to the condition $H_{r_1} \approx_1 v_{1*}\mathbf{B} + (1 - v_{1*})\mathbf{W}$. That is (by Definition 2.3), $H_1 \prec_1 H_2$ iff $xH_1 + (1-x)G_1 \prec xH_2 + (1-x)G_1'$, where G_1 and G_1' are constant acts, symmetric mixtures of outcomes r_1 and $v_{1*}B + (1 - v_{1*})W$.

We show that each $V \in \mathscr{V}$ which extends $\prec_1$ (where V is standardly represented by the set of pairs $\{(p, U_j)\}$ according to condition (4.1)) carries only state-independent utilities for r_1. That is, for each such U_j, $U_j(r_1) = v_{1*}$ if s_j is p-nonnull. To verify this claim, define act H_j as follows:

$$H_{-\varepsilon,j}(s_j) = (v_{1*} - \varepsilon)B + (1 - [v_{1*} - \varepsilon])W \quad \text{and} \quad H_{-\varepsilon,j}(s) = W \quad \text{for } s \notin s_j.$$

If state s_j is not $\prec$-potentially null then, since v_{1*} is the lower bound of $\mathscr{T}_1(r_1)$, by HL Axiom 4, we have $\forall\ (v_{1*} > \varepsilon > 0)$, $H_{-\varepsilon,j} \prec H_{1,j}$. [Recall,

$H_{1,j}(s_j) = r_1$ and $H_{1,j}(s) = W$ when $s \notin s_j$.] By the Archimedean condition HL Axiom 3, letting $\varepsilon \to 0$, we find that these $\prec$-preferences create the constraint $\neg(H_{1,j} \prec H_{\varepsilon=0,j})$, which applies also to each extension of $\prec$. Thus, if s_j is not $\prec$-potentially null, each V which extends $\prec_1$ has v_{1*} as a lower bound on the state dependent utility $U_j(r_1)$. Likewise, by appeal to HL Axiom 5 in case s_j is $\prec$-potentially null, it follows that $\neg(H_{1,j} \prec H_{\varepsilon=0,j})$ and this applies also to all extensions of $\prec$. So, again, v_{1*} is a lower bound on the state dependent utility $U_j(r_1)$ for cases where s_j is $\prec$-potentially null but V-nonnull and $\precsim_V$ extends $\prec$ (on simple acts). (*Note*: Here we use axiom HL Axiom 5 to regulate the state-dependent utility of lotteries in $\prec$-potentially null states.) Because $V(r_1) = v_{1*}$ for each $\precsim_V$ that extends $\prec_1$ on simple acts, v_{1*} also is an upper bound on all such V-nonnull state-dependent utilities $U_j(r_1)$. This is so because $v_{1*} = V(r_1)$ is the p-expectation of $U_j(r_1)$. Hence, each $\precsim_V$ that so extends $\prec_1$ assigns to reward r_1 the state-independent utility v_{1*}.

Next, assume that $\mathscr{T}_2(r_2)$ is not an open interval, for example, let $\mathscr{T}_2(r_2) = (v_{2*}, v_2^*]$, and we know $v_2^* < 1$. Thus, $H_{r_2} \prec_1 (v_2^* + \varepsilon)\mathbf{B} + (1 - [v_2^* + \varepsilon])\mathbf{W}$. Extend $\prec_1$ to $\prec_2$ by introducing the $\approx_2$-condition $H_{r_2} \approx_2 v_2^*\mathbf{B} + (1 - v_2^*)\mathbf{W}$. That is, define $\prec_2$ by $H_1 \prec_2 H_2$ iff $xH_1 + (1-x)G_2 \prec_1 xH_2 + (1-x)G_2'$, where G_2 and G_2' are constant horse lotteries, which are symmetric mixtures of acts H_{r_2} and $v_2^*\mathbf{B} + (1 - v_2^*)\mathbf{W}$.

To see that all $\precsim_V$-extensions of $\prec_2$ impose a state-*independent* utility on r_2, that is, to show $U_j(r_2) = v_2^*$, it suffices to demonstrate that $v_2^* B + (1 - v_2^*)W$ serves as an upper utility bound for r_2 over all $\prec_1$, s_j-called-off preferences, called-off if s_j fails. In other words, we are to establish that, for each state s_j, the constraint $\neg(H_{v_2^*+\varepsilon,j} \prec_1 H_{2,j})$ applies to $\prec_1$ and its extensions. Then, by the reasoning we used above, since $V(r_2) = v_2^*$ for all $\precsim_V$ which extend $\prec_2$ (on simple acts), v_2^* also is a lower utility bound for each state-dependent utility $U_j(r_2)$, and thus $U_j(r_2)$ is state-independent. That is, since $V(r_2) = v_2^*$ is the p-expectation of quantities, none of which is greater than v_2^*, then $U_j(r_2) = v_2^*$ if $P(s_j) > 0$.

To establish that v_2^* is such a state-independent upper bound, expand each of the relevant $\prec_1$-preferences, to wit, $\forall\ (1 - v_2^* > \varepsilon > 0)$ expand $H_{r_2} \prec_1 (v_2^* + \varepsilon)\mathbf{B} + (1 - [v_2^* + \varepsilon])\mathbf{W}$, into its respective $\prec$-preference: $\exists\ x_\varepsilon > 0$, $\exists\ (G_{1\varepsilon}, G_{1\varepsilon}')$,

$$x_\varepsilon H_{r_2} + (1 - x_\varepsilon)G_{1\varepsilon} \prec x_\varepsilon\big[(v_2^* + \varepsilon)\mathbf{B} + (1 - [v_2^* + \varepsilon])\mathbf{W}\big] + (1 - x_\varepsilon)G_{1\varepsilon}'.$$

Each pair $(G_{1\varepsilon}, G_{1\varepsilon}')$ is a symmetric mixture of acts H_{r_1} and $v_{1*}\mathbf{B} + (1 - v_{1*})\mathbf{W}$. These $\prec$-preferences are between constant horse lottery acts. By appeal to HL Axiom 4 in case s_j is not $\prec$-potentially null, or by appeal to HL Axiom 5 in case s_j is $\prec$-potentially null, we arrive at a constraint for called-off acts involving the two lottery outcomes $x_\varepsilon r_2 + (1 - x_\varepsilon)G_{1\varepsilon}$ and $x_\varepsilon[(v_2^* + \varepsilon)B + (1 - [v_2^* + \varepsilon])W] + (1 - x_\varepsilon)G_{1\varepsilon}'$. Specifically, we obtain the restriction $\neg(H_{x+\varepsilon,j} \prec H_{x_2,j})$—a constraint on all extensions of $\prec$—where

$$H_{x+\varepsilon,j}(s_j) = x_\varepsilon\big[(v_2^* + \varepsilon)B + (1 - [v_2^* + \varepsilon])W\big] + (1 - x_\varepsilon)G_{1\varepsilon}' \quad \text{and}$$

$$H_{x+\varepsilon, j}(s) = W \quad \text{if } s \notin s_j,$$

$$H_{x_2, j}(s_j) = x_\varepsilon r_2 + (1 - x_\varepsilon)G_{1\varepsilon} \quad \text{and} \quad H_{x_2, j}(s) = W \quad \text{if } s \notin s_j.$$

However, each $\precsim_V$ extension of $\prec_1$ (on simple acts) assigns to r_1 the state-independent utility v_{1*}. Thus, each extension assigns $G_{1\varepsilon}$ and $G'_{1\varepsilon}$ this same state-independent utility v_{1*}. Then, as the constraint $\neg(H_{x+\varepsilon, j} \prec_1 H_{x_2, j})$ obtains, so too does the constraint which results at the limit, when $\varepsilon = 0$, and terms $G_{1\varepsilon}$ and $G'_{1\varepsilon}$ are canceled according to HL Axiom 2. Hence, each $\precsim_V$ extension of $\prec_1$ has the quantity v_2^* as an upper bound on the state-dependent utility $U_j(r_2)$ of r_2, provided s_j is not null under $\precsim_V$. Therefore, since V is a weighted average of U_j values, $U_j(r_2) = v_2^*$ for each $\precsim_V$ that extends $\prec_2$ on simple acts.

Proceed, this way, through the first n stages in the extension of $\prec$ (using Theorem 3), by choosing for the ith stage either the condition $H_{r_i} \approx_i v_{i*}\mathbf{B} + (1 - v_{i*})\mathbf{W}$ or the condition $H_{r_i} \approx_i v_i^*\mathbf{B} + (1 - v_i^*)\mathbf{W}$, as $\mathscr{T}_i(r_i)$ is closed below or above (respectively). Then the set $\mathscr{V}'$ of extensions for $\prec_n$ provides the requisite subset of $\mathscr{V}$. [Note: $\mathscr{V}'$ may fail to be convex when $\mathscr{T}_i(r_i)$ is a closed interval, as in the example for Theorem 1. Then either endpoint may be chosen, but not values in between.] □

F. *Proof of Theorem* 6. The proof of Theorem 6(i) is based on the idea of the proof of Lemma 4.3. The argument is by induction on the number of rewards, that is, on the length of the initial segment of $\{r_1, r_2, \dots\}$. The method is a straightforward epsilon–delta technique of fixing the degree of state-dependence to be tolerated and then choosing target set values sufficiently close to a boundary of the target sets to force agreement with the allowed tolerance for state-dependent utilities.

The proof of Theorem 6(ii) follows the argument of Corollary 3.1; that is, use the countable set $\{\mathscr{R} \cup \mathscr{B}\}$ in forming the extensions of $\prec$, subject to the following modification in the ordering of $\{\mathscr{R} \cup \mathscr{B}\}$: Fix k, which determines the initial segment of $\mathscr{R}$, $\{r_1, \dots, r_k\}$, over which the almost state-independent utilities are to be provided. Given a nonsimple act $H \in \mathscr{B}$, insert it into the sequence of extensions based on $\mathscr{H}$ only after these k-many rewards have been assigned their utilities. This method ensures that assigning utilities to the nonsimple acts in $\mathscr{B}$ does not interfere with using the boundary regions of the target sets of the k-many rewards, $\{r_1, \dots, r_k\}$, to locate their almost state-independent utilities. For interesting discussion of this point, see Section 5 of Nau (1993). □

Two remarks help to explain the content of Theorem 6. First, in light of Example 4.1, it may be that for each $\varepsilon > 0$, $\prec$ admits an almost state-independent utility, but (corresponding to $\varepsilon = 0$) there is no agreeing probability/state-independent utility pair in the limit. That is, the limit (as $\varepsilon \to 0$) of the (nested) sets of agreeing, almost state-independent utilities is empty. Second, Definition 31 requires only that $\prec$ admit almost state-independent utilities for each finite set of n-many rewards. Obviously, by increasing n, we can form sequences of (nested) sets of probability/utility pairs. However, Definition 31 does not provide for an almost state-independent utility covering infinitely many rewards simultaneously. We do not yet know whether, given our five axioms, there exists a nonempty limit (as $n \to \infty$) to these nested sets.

G. *Results from Section* 5.

PROOF OF LEMMA 5.1. (i) By HL Axiom 2, $H_1 \prec H_2$ iff $0.5H_1 + 0.5H_1' \prec 0.5H_2 + 0.5H_1'$. Regrouping terms on the r.h.s. of the second $\prec$ relation, we obtain $H_1 \prec H_2$ iff $0.5H_1 + 0.5H_1' \prec 0.5H_1 + 0.5H_2'$. Another application of HL Axiom 2 yields the desired result: $H_1 \prec H_2$ iff $H_1' \prec H_2'$.

(ii) Suppose $H_1 \approx H_2$. By Corollary 2.3, it suffices to show that $xH_1' + (1-x)H_3 \prec H_4$ iff $xH_2' + (1-x)H_3 \prec H_4$. By HL Axiom 2, $xH_1' + (1-x)H_3 \prec H_4$ iff $z[xH_1' + (1-x)H_3] + (1-z)H_1 \prec zH_4 + (1-z)H_1$ $(0 < z \le 1)$. Since $H_1 \approx H_2$, by Corollary 2.3, substituting H_2 for H_1 on the l.h.s., the biconditional reads: iff $z[xH_1' + (1-x)H_3] + (1-z)H_2 \prec zH_4 + (1-z)H_1$. Let $zx = 1 - z$, that is, $z = (1+x)^{-1}$. Then regrouping terms in H_1' and H_2, the biconditional reads: iff $z[xH_2' + (1-x)H_3] + (1-z)H_1 \prec zH_4 + (1-z)H_1$. Another application of HL Axiom 2 yields the desired result. □

PROOF OF THEOREM 8. Part (i) is immediate as $\mathbf{H_e}$ is a subset of $\mathbf{H_R}$. Specifically, if a weak order $\precsim_V$ (of Theorem 3) agrees with $\prec$, it agrees with $\prec_{\mathbf{e}}$. That is, consider the e-called-off family $\mathbf{H_e}$, where $H(s) = W$ if $s \notin e$ and $\prec_{\mathbf{e}}$ is the restriction of $\prec$ to $\mathbf{H_e}$. Let H_1 and H_2 be simple acts that belong to $\mathbf{H_e}$. If $H_1 \prec_{\mathbf{e}} H_2$, then $H_1 \prec H_2$ and therefore $V(H_1) \prec V(H_2)$. Let the expected utility V be given by the probability/(state-dependent) utility pair (p, U_j). As $U_j(W) = 0$ and $H_i(s) = W$ for $s \notin e$ $(i = 1, 2)$, then $\sum_{s_j \in e} p(s_j)U_j(L_{1j}) < \sum_{s_j \in e} p(s_j)U_j(L_{2j})$. Hence, $(p_e, U_{j \in e})$ agrees with $\prec_{\mathbf{e}}$.

For part (ii), without loss of generality (Lemma 5.1), continue with the e-called-off family $\mathbf{H_e}$ determined by fixing $H(s) = W$ if $s \notin e$. Define the act $\mathbf{B_e} \in \mathbf{H_e}$ by $\mathbf{B_e}(s) = B$ if $s \in e$. With respect to $\prec_{\mathbf{e}}$, $\mathbf{B_e}$ serves as the "best" act and $\mathbf{W}$ serves as the "worst." Thus, for $H \in \mathbf{H_e}$, $V(H|e)V(\mathbf{B_e}) = V(H)$. Let $V_{\mathbf{e}}(\cdot)$ agree with $\prec_{\mathbf{e}}$ over the set $\mathbf{H_e}$. Assume $V_{\mathbf{e}}(\cdot)$ differs from each conditional expected utility $V(\cdot|e)$ $(V \in \mathscr{V})$. In particular, with $\mathscr{H}_{\mathbf{e}}$ ordered for applying Theorem 3 to $\prec_{\mathbf{e}}$, let $H_z \in \mathscr{H}_{\mathbf{e}}$ satisfy the following condition: For each $V \in \mathscr{V}$ such that $V_{\mathbf{e}}(H_i) = V(H_i|e)$ $(i = 1, \dots, z-1)$, $V_{\mathbf{e}}(H_z) \ne V(H_z|e)$. That is, H_z is the first e-called-off act, where $V_{\mathbf{e}}$ differs from each $V(\cdot|e)$, $V \in \mathscr{V}$. Without loss of generality, according to Corollary 3.3, put the first z-elements of $\mathscr{H}_{\mathbf{e}}$ as the initial segment of $\mathscr{H}$. Thus, H_z is the zth element in this reordering of $\mathscr{H}$.

By hypothesis, for some $V \in \mathscr{V}$, $V_{\mathbf{e}}(H_i) = V(H_i|e)$ $(i = 1, \dots, z-1)$. Then mimic the first $z-1$ extensions of $\prec_e$ in the first $z-1$ extensions of $\prec$. That is, provided e is not potentially null so that $\mathbf{W} \prec \mathbf{B_e}$, use Definition 20 to extend $\prec$ to $\prec_{z-1}$ with symmetric mixtures of the $z-1$ act pairs: H_i and $V_{\mathbf{e}}(H_i)\mathbf{B_e} + (1 - V_{\mathbf{e}}(H_i))\mathbf{W}$. Also by hypothesis, $\prec_{z-1}$ cannot be extended to $\prec_z$ using Definition 20 with symmetric mixtures of H_z and $V_{\mathbf{e}}(H_z)\mathbf{B_e} + (1 - V_{\mathbf{e}}(H_z))\mathbf{W}$.

Next, we show that $V_{\mathbf{e}}(H_z)$ is an endpoint of the conditional target set $\mathscr{T}_z(H_z)$, defined using mixtures of $\mathbf{B_e}$ and $\mathbf{W}$. Argue indirectly: either $H_z \prec_{z-1} V_{\mathbf{e}}(H_z)\mathbf{B_e} + (1 - V_{\mathbf{e}}(H_z))\mathbf{W}$ or else $V_{\mathbf{e}}(H_z)\mathbf{B_e} + (1 - V_{\mathbf{e}}(H_z))\mathbf{W} \prec_{z-1} H_z$. We give the analysis for the former case. (The reasoning for the latter case is parallel.) Expand the $\prec_{z-1}$-preference into its equivalent $\prec$-preference. Thus, for $i = 1, \dots, z-1$, there exist $x_i \geq 0$, $x_z > 0$, $\sum_i x_i + x_z = 1$, such that

$$x_1 G_1 + \cdots + x_{z-1} G_{z-1} + x_z H_z \prec x_1 G_1' + \cdots + x_{z-1} G_{z-1}' + x_z\left[V_{\mathbf{e}}(H_z)\mathbf{B_e} + (1 - V_{\mathbf{e}}(H_z))\mathbf{W}\right],$$

where the pairs (G_i, G_i') are symmetric mixtures of H_i and $V_{\mathbf{e}}(H_i)\mathbf{B_e} + (1 - V_{\mathbf{e}}(H_i))\mathbf{W}$. However, as this $\prec$-preference involves elements of $\mathscr{H}_{\mathbf{e}}$ only, then $x_1 G_1 + \cdots + x_{z-1} G_{z-1} + x_z H_z \prec_{\mathbf{e}} x_1 G_1' + \cdots + x_{z-1} G_{z-1}' + x_z[V_{\mathbf{e}}(H_z) \times \mathbf{B_e} + (1 - V_{\mathbf{e}}(H_z))\mathbf{W}]$. Thus $V_{\mathbf{e}}(H_z) \notin \mathscr{T}_{\mathbf{e},z}(H_z)$—a contraction with the assumption that $V_{\mathbf{e}}(\cdot)$ agrees with $\prec_{\mathbf{e}}$. Hence, $V_{\mathbf{e}}(H_z)$ is a *precluded* endpoint of the conditional $\mathscr{T}_z(H_z)$ according to preferences $\prec_{z-1}$, but it is not precluded from $\mathscr{T}_{\mathbf{e},z}(H_z)$ according to the subset of preferences in $\prec_{\mathbf{e},z-1}$. $\square$

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