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ASYMPTOTIC PROPERTIES OF ESTIMATORS FOR AUTOREGRESSIVE MODELS WITH ERRORS IN VARIABLES

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Let $\{X_t,\ t \in \mathbb{Z}\}$ be an observable strictly stationary sequence of random variables and let $X_t = U_t + \varepsilon_t$, where $\{U_t\}$ is an AR (p) and $\{\varepsilon_t\}$ is a strictly stationary sequence representing errors of measurement in $\{X_t\}$, with $E\{\varepsilon_1\} = 0$. Under some broad assumptions on $\{\varepsilon_t\}$ we establish the consistency properties as well as the rates of convergence for the standard estimators for the autoregressive parameters computed from a set of modified Yule–Walker equations.

1. Introduction. Data analysis of models with errors in variables started historically with the investigation of relations between statistical variables when all or some of these variables are subject to errors of measurement. A somewhat exhaustive bibliography on this subject appears in Anderson (1984). The treatment of time series models with errors in variables, however, is of recent origin. Regressions for time series models with errors in measurement have been discussed by Hannan (1963), Moran (1971) and Robinson (1986). Identifiability problems for such models appear in Anderson and Deistler (1984), Maravall (1979), Nowak (1985) and Solo (1986). Estimation of an error in variable autoregressive model is due to Trognon (1989). However, so far, to the best of this author's knowledge, no systematic attempt has been made to analyze data which are derived from general finite parameter linear or nonlinear models and are, themselves, subject to errors in measurement.

The present paper deals with autoregressive (AR) models which have errors in variables and is an extension of the results in Chanda (1994) in the sense that the probability models for the errors of measurement are treated here in a nonparametric manner.

2. Estimation of parameters in errors-in-variable AR models. Let $\{X_t,\ t \in \mathbb{Z}\}$ be an observable strictly stationary sequence of random variables (rv) and let

$$X_t = U_t + \varepsilon_t, \qquad \phi(B)(U_t - \mu) = \eta_t, \tag{2.1}$$

where $\{U_t;\ t \in \mathbb{Z}\}$ is an AR sequence of unobservables, μ is a finite constant, $\{\varepsilon_t\}$ is a strictly stationary sequence representing errors in measurement in $\{X_t\}$, $\{\eta_t\}$ is a sequence of independent and identically distributed (iid) rv's

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with $E(\eta_1) = 0$ and $E(\eta_1^2) = \sigma_\eta^2$ and $\{\varepsilon_t\}$, $\{\eta_t\}$ are mutually independent sequences. We assume that the roots of $\phi(\xi) = 1 - \sum_{j=1}^{p} \phi_j \xi^j$ all lie outside the unit circle. We assume that $E(\varepsilon_1) = 0$ and from now on we use X_t in place of $X_t - \mu$ throughout this paper. Our main objective is to estimate the parameter $\phi_1, \dots, \phi_p$ from a given set of data $\{X_t, 1 \le t \le n\}$. If μ is unknown, we estimate it by the sample mean $\overline{X}$. Write $\gamma_\nu = E(X_t X_{t+\nu})$, $\gamma_\nu(U) = E(U_t U_{t+\nu})$ and $\gamma_\nu(\varepsilon) = E(\varepsilon_t \varepsilon_{t+\nu})$, $\nu \ge 0$.

2.1. *The* AR(1) *model*. Consider at first the case $p = 1$. We write $\phi_1 = \phi$ $(0 < |\phi| < 1)$. Assume that $\{\varepsilon_t\}$ is a linear process such that $|\gamma_\nu(\varepsilon)| \le M_j \delta^\nu$ $(\nu \ge 0)$, for some finite constants $M > 0$ and $0 < \delta < |\phi| < 1$. (In particular, this condition holds if $\{\varepsilon_t\}$ is, itself, a certain type of ARMA process.) Now choose $k = [g \log n / \log|\phi|]$, where $g \in (-1/2, 0)$ and $|g| > \log|\phi| / 2 \log \delta$. Set $\hat{\gamma}_k = \sum_{t=1}^{n-k} X_t X_{t+k} / n$ and $C_k = \sum_{t=1}^{n-k} (X_t \zeta_{t+k} - a_k)/n$, where $\zeta_t = \eta_t + \varepsilon_t - \phi \varepsilon_{t-1}$ and $a_k = E(X_t \zeta_{t+k}) = \gamma_k(\varepsilon) - \phi \gamma_{k-1}(\varepsilon)$, and define

$$\hat{\phi}_k = \hat{\gamma}_{k+1} / \hat{\gamma}_k. \tag{2.2}$$

Now note that $n^{1/2}|\gamma_k| \to \infty$, $k \to \infty$, but $k/n \to 0$ as $n \to \infty$. Therefore, we can write

$$n^{1/2}\gamma_k(\hat{\phi} - \phi) \approx (n^{1/2} C_{k+1} + n^{1/2} a_{k+1}) / (T_k + 1), \tag{2.3}$$

where $T_k = (\hat{\gamma}_k - \gamma_k)/\gamma_k$. Corollary 2.1 that follows will imply that $\mathscr{L}(n^{1/2} C_{k+1}) \to \mathscr{N}(0, \sigma^2)$, where $\sigma^2 = \gamma_0 \sigma_\eta^2 + \sum_{s=-\infty}^{\infty} \gamma_s \bar{\gamma}_s$ and

$$\begin{aligned} \bar{\gamma}_s &= E(\varepsilon_t - \phi \varepsilon_{t-1})(\varepsilon_{t+s} - \phi \varepsilon_{t+s-1}) \\ &= \gamma_s(\varepsilon)(1 + \phi^2) - \phi(\gamma_{s+1}(\varepsilon) + \gamma_{s-1}(\varepsilon)). \end{aligned}$$

By Theorem 2.1 (below) we have that $\mathscr{L}(n^{1/2}(\hat{\gamma}_k - \gamma_k)) \to \mathscr{N}(0, \sum_{s=-\infty}^{\infty} \gamma_s^2)$. Also, it is easy to see that $n^{1/2} a_{k+1} \to 0$ as $n \to \infty$. Therefore, as $n \to \infty$,

$$\mathscr{L}\left(n^{1/2}|\gamma_k|(\hat{\phi}_k - \phi)\right) \to \mathscr{N}(0, \sigma^2) \tag{2.4}$$

and hence

$$\mathscr{L}\left(n^{1/2}|\phi|^k (1 - \phi^2)^{-1} (\hat{\phi}_k - \phi)\right) \to \mathscr{N}(0, \sigma^2). \tag{2.4$'$}$$

Note that $n^{1/2}|\gamma_k| \sim n^{(1+2g)/2}$. Naturally we should choose $|g|$ to be as small as possible. For example, if $\phi = 0.5$ and $\delta = 0.25$, then $-0.5 < g < -0.25$. If we choose $g = -0.251$, then $n^{1/2}|\gamma_k| \sim n^{0.249}$.

One other possibility exists. Suppose k is chosen such that $k \to \infty$, $k/n \to 0$, $n^{1/2}|a_{k+1}| \to \infty$, but $\gamma_{k+1}(\varepsilon)/\gamma_k(U) \to 0$ and $\gamma_k(\varepsilon)/\gamma_k(U) \to 0$ as $n \to \infty$. Then one can show quite easily from (2.3) that

$$|\gamma_k / a_{k+1}| |\hat{\phi}_k - \phi| \to 1 \tag{2.5}$$

in probability as $n \to \infty$. For example, if $\{\varepsilon_t\}$ is, itself, an AR(1) process, $\varepsilon_t - \psi \varepsilon_{t-1} = \nu_t$, where $\{\nu_t\}$ is a sequence of iid rv's with $E(\nu_1) = 0$, $0 < E(\nu_1^2) < \infty$, and $0 < |\psi| < |\phi| < 1$, then (2.5) holds if we choose $k = [g \log n / \log|\psi|]$

with $g \in (-1/2, 0)$. In fact, one can then show that $D\tau^k|\hat{\phi}_k - \phi| \to 1$ in probability, for some finite positive constant D and $\tau = |\phi/\psi| > 1$.

2.2. *The* AR(p) *model* $p \geq 2$. For the general case $p \geq 2$, the estimators $\hat{\phi}_{k,1}, \dots, \hat{\phi}_{k,p}$ of the AR parameters $\phi_1, \dots, \phi_p$ are defined by the equations

$$\sum_{j=1}^{p} \hat{\phi}_{k,j}\hat{\gamma}_{\nu-j} = \gamma_\nu, \qquad k+1 \leq \nu \leq k+p, \tag{2.6}$$

where we choose k such that $k \to \infty$, but $k/n \to 0$ as $n \to \infty$. If, now, we set $\hat{\Delta}_j = n^{1/2}(\hat{\phi}_{k,j} - \phi_j)$, $1 \leq j \leq p$, then (2.6) reduces, in matrix form, to

$$\hat{\mathbf{\Gamma}}_{k,p}\hat{\mathbf{\Delta}} = \overline{\mathbf{C}}_k, \tag{2.6$'$}$$

where

$$\begin{aligned}
\hat{\mathbf{\Delta}}^T &= \left[\hat{\Delta}_1, \dots \hat{\Delta}_p\right],\\
\hat{\mathbf{\Gamma}}_{k,p} &= \left[\hat{\gamma}_{k+i-j}\right], \qquad 1 \leq i, j \leq p,\\
\overline{\mathbf{C}}_k^T &= \left[\overline{C}_{k+1}, \dots, \overline{C}_{k+p}\right],\\
\overline{C}_\nu &= n^{1/2}C_\nu + n^{1/2}a_\nu(n-\nu)/n + R_{\nu,n},\\
C_\nu &= \sum_{t=1}^{n-\nu}(X_t\zeta_{t+\nu} - a_\nu)/n,\\
a_\nu &= E(X_t\zeta_{t+\nu}), \qquad k+1 \leq \nu \leq k+p,\\
\zeta_t &= \eta_t + \varepsilon_t - \sum_{j=1}^{p}\phi_j\varepsilon_{t-j}
\end{aligned}$$

and

$$R_{(\nu,n} = O_p(n^{-1/2}).$$

We now assume that the following conditions hold.

1. The roots $\xi_1, \dots, \xi_p$ of $\xi^p\phi(\xi^{-1})$ are all distinct and if we denote the root with the smallest modulus by ξ_p, then ξ_p is real and $0 < |\xi_p| < |\xi_j|$, $1 \leq j \leq p-1$.
2. $\{\varepsilon_t\}$ is a linear process, $\varepsilon_t = \sum_{j=-\infty}^{\infty}\psi_j\delta_{t-j}$, where $\{\delta_t\}$ is a sequence of iid rv's with $E(\delta_1) = 0$ and $0 < E(\delta_1^2) < \infty$.
3. $\{\psi_j\}$ is such that $|\gamma_{\nu+j}(\varepsilon)/\xi_p^\nu| \to 0$, for every finite j, whenever $\nu \to \infty$.

REMARK 1. The asymptotic normality result of Theorem 2.2 below will hold (although in substantially different forms) even when the roots of $\xi^p\phi(\xi^{-1})$ are not all distinct. However, the mathematical details are somewhat complicated and hence are withheld from this report.

Note that by conditions 1–3 we can choose k such that $k \to \infty$, $n^{1/2}|\gamma_\nu(U)| \to \infty$ and $n^{1/2}|\gamma_\nu(\varepsilon)| \to 0$ as $n \to 0$, for $k+1 \le \nu \le k+p$ (the arguments are similar to those for the case $p = 1$). Since $a_\nu = \gamma_\nu(\varepsilon) - \sum_{j=1}^{p} \phi_j \gamma_{\nu-j}(\varepsilon)$, it follows that $n^{1/2} a_\nu \to 0$ as $n \to \infty$, $k+1 \le \nu \le k+p$.

We now prove the following theorem.

THEOREM 2.1. *Let $U_t = \sum_{j=-\infty}^{\infty} \beta_j \eta_{t-j}$ and $\varepsilon_t = \sum_{j=-\infty}^{\infty} \psi_j \nu_{t-j}$, where $\{\eta_t\}$ and $\{\nu_t\}$ are independent sequences of iid rv's with $E(\eta_1) = 0$, $E(\eta_1^2) = \sigma_\eta^2$, $E(\nu_1) = 0$, $E(\nu_1^2) = \sigma_\nu^2$, $0 < \sigma_\eta, \sigma_\nu < \infty$, $E(\eta_1^4) < \infty$, $E(\nu_1^4) < \infty$, $\sum_{j=-\infty}^{\infty} |j\beta_j| < \infty$ and $\sum_{j=-\infty}^{\infty} |j\psi_j| < \infty$. Assume that $k \to \infty$, but $k/n \to 0$ as $n \to \infty$. Then for any finite integer $l \ge 1$, $n^{1/2}(\hat{\gamma}_{k+j} - \gamma_{k+j})$, $1 \le j \le l$ have, asymptotically, as $n \to \infty$, a $(l+1)$-variate normal distribution with mean vector zero and covariance matrix $\mathbf{\Lambda} = [\lambda_{ij}]$, where*

$$\lambda_{ij} = \sum_{s=-\infty}^{\infty} \gamma_s \gamma_{s+i-j}, \qquad 1 \le j \le l. \tag{2.7}$$

PROOF. Theorem 2.1. in Chanda (1993) proves the result for the case when $\varepsilon_t = 0$. Routine extension of arguments similar to those in the theorem in Chanda (1993) will establish the result of Theorem 2.1 in this paper. □

COROLLARY 2.1. *Let the conditions of Theorem 2.1 hold and let C_ν be as defined in (2.6′), with $\mathbf{C}_k^T = [C_{k+1}, \dots, C_{k+p}]$. Then*

$$\mathscr{L}\left(n^{1/2}\mathbf{C}_k\right) \to \mathscr{N}\left(0, \mathbf{\Lambda}_p^*\right), \tag{2.8}$$

where $\mathbf{\Lambda}_p^* = [\lambda_{ij}^*]$, $\lambda_{ij}^* = \sigma_\eta^2 \gamma_{i-j} + \sum_{s=-\infty}^{\infty} \gamma_s \bar{\gamma}_{s+j-i}$ and $\bar{\gamma}_l = \sum_{\tau,s=0}^{p} \phi_\tau^* \phi_s^* \gamma_{l+\tau-s}(\varepsilon)$, $l = 0, \pm 1, \dots$.

PROOF. Note that as $n \to \infty$,

$$\begin{aligned} n^{1/2} C_\nu &= \sum_{t=1}^{n-\nu} (X_t \zeta_{t+\nu} - a_\nu)/n^{1/2} = \sum_{t=1}^{n-\nu} (X_t \phi(B) X_{t+\nu} - a_\nu)/n^{1/2} \\ &\simeq n^{1/2} \sum_{\tau=0}^{p} \phi_\tau^* (\hat{\gamma}_{\nu-\tau} - \gamma_{\nu-\tau}), \ \phi_j^* = -\phi_j, 1 \le j \le p, \phi_0 = 1, \end{aligned}$$

and

$$\begin{aligned} n \operatorname{Cov}\left(C_{k+i}, C_{k+j}\right) &= n \sum_{\tau,s=0}^{p} \phi_\tau^* \phi_s^* \operatorname{Cov}\left(\hat{\gamma}_{k+i-\tau}, \hat{\gamma}_{k+j-s}\right) \to \sum_{\tau,s=0}^{p} \phi_\tau^* \phi_s^* \lambda_{i-\tau, j-s} \\ &= \sum \phi_\tau^* \phi_s^* \sum_{m=-\infty}^{\infty} \gamma_m \gamma_{m+i-\tau-j+s} \quad \text{as } n \to \infty, \end{aligned}$$

which together with Theorem 2.1 proves Corollary 2.1. □

We conclude from Corollary 2.1 and the relation (2.6′) that

$$\mathscr{L}\left(\overline{C}_k\right) \to \mathscr{N}\left(0, \boldsymbol{\Lambda}_p^*\right) \tag{2.9}$$

as $n \to \infty$.

Now choose k such that $k \to \infty$ and $n^{1/2}|\xi_p|^{k-p+1} \to \infty$ as $n \to \infty$. (Since $0 < |\xi_p| < 1$, such a choice is always possible.) Define $\tilde{\boldsymbol{\Gamma}}_{k,p} = \hat{\boldsymbol{\Gamma}}_{k,p} - \boldsymbol{\Gamma}_{k,p}$ and $\boldsymbol{\Gamma}_{k,p} = [\gamma_{k+i-j}]$, $1 \le i,\ j \le p$, and note that since condition 1 holds, we can write $\gamma_\nu = \sum_{j=1}^{k} A_j \xi_j^\nu$, where $A_1, \dots, A_p$ are nonzero constants and A_p is real. Also by conditions 2 and 3, Theorem 2.1 and the choice of k above, it follows that $\tilde{\boldsymbol{\Gamma}}_{k,p}/\xi_p^{k-p+1} \to_P 0$ as $n \to \infty$. Therefore,

$$p \lim_{n\to\infty} A_p \xi_p^{k-p+1} \hat{\boldsymbol{\Gamma}}_{k,p}^{-1} = \lim_{n\to\infty} A_p \xi_p^{k-p+1} \boldsymbol{\Gamma}_{k,p}^{-1}, \tag{2.10}$$

provided that the limit on the right side of (2.10) exists. We can now prove the following theorem.

THEOREM 2.2. *Assume that conditions* 1–3 *above hold. Choose k such that $k \to \infty$ and $n^{1/2}|\xi_p^{k-p+1}| \to \infty$ as $n \to \infty$. Then*

$$\mathscr{L}\left(n^{1/2}|A_p \xi_p^{k-p+1}|\left(\hat{\phi}_k - \phi_k\right)\right) \to \mathscr{N}\left(0, \mathbf{Q}_p \boldsymbol{\Lambda}_p^* \mathbf{Q}_p^T\right) \tag{2.11}$$

as $n \to \infty$, where

$$\mathbf{Q}_p = d_p^{-2}\mathbf{H}_p,$$

$$\mathbf{H}_j = \boldsymbol{\pi}_j \boldsymbol{\pi}_j^T \mathbf{J}_p, \qquad 1 \le j \le p,$$

$$\boldsymbol{\pi}_j^T = [c_{0,j}, c_{1,j}, \dots, c_{p-1,j}], \quad \sum_{m=1}^{p} \xi^{p-m} c_{m-1,j} = \prod_{\substack{u=1\\u\ne j}}^{p} (\xi - \xi_u),$$

$$d_j = \prod_{\substack{u=1\\u\ne j}}^{p} (\xi_j - \xi_u),$$

$$\mathbf{J}_p = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & & & & \\ \vdots & & & & \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and $\boldsymbol{\Lambda}_p^*$ is as defined in (2.8).

PROOF. Note that $\boldsymbol{\Gamma}_{k,p} = \boldsymbol{\Gamma}_{k,p}(U) + \boldsymbol{\Gamma}_{k,p}(\varepsilon)$, where $\boldsymbol{\Gamma}_{k,p}(U) = [\gamma_{k+i-j}(U)]$ and $\boldsymbol{\Gamma}_{k,p}(\varepsilon) = [\gamma_{k+i-j}(\varepsilon)]$. Set $\mathbf{G}_j = [\xi_j^{l-m+p-1}]$, $\beta_j = (A_j/A_p)(\xi_j/\xi_p)^{k-p+1}$

and $\boldsymbol{\theta}_j^T = [1, \xi_j, \dots, \xi_j^{p-1}]$, $1 \le j \le p$. Then it is easy to see that $\mathbf{G}_j = \boldsymbol{\theta}_j \boldsymbol{\theta}_j^T \mathbf{J}_p$ and $\boldsymbol{\Gamma}_{k,p}(U) = A_p \xi_p^{k-p+1} \sum_{j=1}^p \beta_j \mathbf{G}_j$ and since condition 3 holds,

$$A_p \xi_p^{k-p+1} \boldsymbol{\Gamma}_{k,p}^{-1} = \left(\sum_{j=1}^p \beta_j \mathbf{G}_j + \boldsymbol{\Gamma}_{k,p}(\varepsilon)/A_p \xi_p^{k-p+1} \right)^{-1} \approx \left(\sum_{j=1}^p \beta_j \mathbf{G}_j \right)^{-1} \tag{2.12}$$

as $n \to \infty$, provided, of course, $\sum_{j=1}^p \beta_j \mathbf{G}_j$ is nonsingular. Again, since

$$\boldsymbol{\theta}_i^T \mathbf{J}_p \pi_j = \sum_{m=1}^p \xi_i^{p-m} c_{m-1,j} = \prod_{\substack{u=1 \\ u \neq j}}^p (\xi_i - \xi_u) = 0$$

whenever $i \neq j$ and $= d_i$, if $i = j$, it follows easily that $\mathbf{G}_i \mathbf{H}_j = \mathbf{0}$, if $1 \le i \neq j \le p$, and $\mathbf{G}_i \mathbf{H}_i = d_i \mathbf{C}_i$, where $\mathbf{C}_i = \boldsymbol{\theta}_i \boldsymbol{\pi}_i^T \mathbf{J}_p$, $1 \le i \le p$. Hence

$$\sum_{j=1}^p \beta_j \mathbf{G}_j \sum_{j=1}^p \beta_j^{-1} \mathbf{H}_j = \mathbf{M}_p, \tag{2.13}$$

where $\mathbf{M}_p = \sum_{i=1}^p d_i \mathbf{C}_i$. Now note that $\mathbf{M}_p \boldsymbol{\alpha} = \mathbf{0}$ implies that $\mathbf{C}_j \sum_{i=1}^p d_i \mathbf{C}_i \boldsymbol{\alpha} = \mathbf{0}$, $1 \le j \le p$. Also $\mathbf{C}_i \mathbf{C}_j = \mathbf{0}$ whenever $1 \le i \neq j \le p$ and $\mathbf{C}_i^2 = d_i \mathbf{C}_i$, $1 \le i \le p$. Since $d_i \neq 0$, $1 \le i \le p$, it follows immediately that $\mathbf{C}_i \alpha = \mathbf{0}$, $1 \le i \le p$. Therefore, $\boldsymbol{\alpha} = \mathbf{0}$ and $\mathbf{M}_p$ and hence $\sum_{i=1}^p \beta_i \mathbf{G}_i$ are nonsingular.

Now set $\mathbf{Q}_p = \lim_{n \to \infty} (\sum_{i=1}^p \beta_i \mathbf{G}_i)^{-1}$. Then since $\beta_i^{-1} \to 0$ as $n \to \infty$, $1 \le i \le p - 1$, and $\beta_p = 1$, we conclude that

$$\mathbf{Q}_p = \mathbf{H}_p \mathbf{M}_p^{-1}. \tag{2.14}$$

Write $\mathbf{L}_p = [\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_p]$ and $\mathbf{L}_p^* = [\mathbf{J}_p \boldsymbol{\pi}_1, \dots, \mathbf{J}_p \boldsymbol{\pi}_p]$. Note the $\mathbf{H}_j \boldsymbol{\theta}_i = \boldsymbol{\pi}_j \boldsymbol{\pi}_u^T \mathbf{J}_p \boldsymbol{\theta}_i = \mathbf{0}$ whenever $1 \le i \neq j \le p$. Therefore, $\mathbf{M}_p \boldsymbol{\theta}_i = \mathbf{G}_i \mathbf{H}_i \boldsymbol{\theta}_i = d_i^2 \boldsymbol{\theta}_i$ and

$$\mathbf{M}_p \mathbf{L}_p = \mathbf{L}_p \mathbf{D}_p, \tag{2.15}$$

where $\mathbf{D}_p = \operatorname{diag}[d_1^2, \dots, d_p^2]$. Similar arguments lead to

$$\mathbf{M}_p^T \mathbf{L}_p^* = \mathbf{L}_p^* \mathbf{D}_p. \tag{2.16}$$

Relations (2.15) and (2.16) imply that

$$\mathbf{M}_p = \sum_{j=1}^p g_j \boldsymbol{\theta}_j \boldsymbol{\pi}_j^T \mathbf{J}_p, \tag{2.17}$$

where $g_j = d_j^2/d_j = d_j$ [see Chapter 1c, 3.14 in Rao (1973) and note that $\boldsymbol{\theta}_i^T \mathbf{J}_p \boldsymbol{\pi}_j = 0$, if $i \neq j$, and $= d_i$, if $i = j$, $1 \le i, j \le p$]. From (2.17) we have that $\mathbf{H}_p \mathbf{M}_p = \boldsymbol{\pi}_p \boldsymbol{\pi}_p^T \mathbf{J}_p \sum_{j=1}^p d_j \boldsymbol{\theta}_j \boldsymbol{\pi}_j^T \mathbf{J}_p = d_p^2 \mathbf{H}_p$ and hence

$$\mathbf{Q}_p = \mathbf{H}_p \mathbf{M}_p^{-1} = d_p^{-2} \mathbf{H}_p. \tag{2.18}$$

Now set $\mathbf{\Delta}^* = A_p \xi_p^{k-p+1}\hat{\mathbf{\Delta}}$. Then we conclude from (2.6′) and the comments that follow (2.6′) that $\mathbf{\Delta}^* \approx A_p \xi_p^{k-p+1}\hat{\mathbf{\Gamma}}_{k,p}^{-1}\overline{\mathbf{C}}_k$ as $n \to \infty$. If, now, we use (2.9), (2.10), (2.12), (2.18) and Corollary 2.1, we immediately obtain the result (2.11) of Theorem 2.2. It is also interesting to note that $\text{rank}(\mathbf{Q}_p \mathbf{\Lambda}_p^* \mathbf{Q}_p^T) = \text{rank}(\mathbf{Q}_p) = 1$. □

REMARK 2. If the roots of $\xi^p \phi(\xi^{-1})$ do not satisfy condition 1 the result of Theorem 2.2 will not hold. In order to understand the kind of result that may be obtained, in general, we consider the case $p = 2$ and discuss two situations: (a) the roots ξ_1, ξ_2 are both complex and (b) the two roots are real and equal to ξ_1. For (a) we can show by direct calculation that

$$|A_1 \xi_1^k| \hat{\Delta}_1 \approx \big(\cos(\alpha + k\theta)/2\sin^2\theta\big)\overline{C}_{k+1} - \big(\cos(\alpha + (k-1)\theta)/2\tau\sin^2\theta\big)\overline{C}_{k+2}$$

and

$$|A_1 \xi_1^k| \hat{\Delta}_2 \approx \big(\cos(\alpha + k\theta)/2\sin^2\theta\big)\overline{C}_{k+2} - \big(\tau\cos(\alpha + (k+1)\theta)/2\sin^2\theta\big)\overline{C}_{k+1}, \tau = |\xi_1|.$$

which clearly indicates that no asymptotic distribution is possible for $\hat{\phi}_{k,1}$ and $\hat{\phi}_{k,2}$. On the other hand, for (b) routine and direct methods show that if condition 2 holds, if we replace condition 3 by 3′: $|\nu\gamma_{\nu+j}/\xi_1^\nu| \to 0$, for every finite j as $\nu \to \infty$ and choose k such that $k \to \infty$ and $n^{1/2}|\xi_1|^k/k \to \infty$ as $n \to \infty$, then

$$\mathscr{L}\Big(n^{1/2}|A_1 \xi_1^{k+1}/k|\big(\hat{\phi}_k - \phi_k\big)\Big) \to \mathscr{N}(\mathbf{0}, \mathbf{\Sigma}) \tag{2.19}$$

as $n \to \infty$, where $\mathbf{\Sigma} = \mathbf{Q}_2^* \mathbf{\Lambda}_2^* \mathbf{Q}_2^{*T}$, $\mathbf{\Lambda}_2^*$ is as defined in (2.8) for $p = 2$, $\mathbf{Q}_2^* = \begin{bmatrix} \xi_1 & -1 \\ -\xi_1^2 & \xi_1 \end{bmatrix}$, $A_1 = \sigma_\eta^2/(1 + \phi_2)^2$ and $\phi_2 = -\phi_1^2/4$.

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