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SOME COUNTEREXAMPLES CONCERNING SUFFICIENCY AND INVARIANCE

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Some conditions which are usually found in the literature on sufficiency and invariance are considered, with counterexamples given to clarify the relationship between these conditions.

Let $(\Omega, \mathscr{A}, \mathscr{P})$ be a statistical experiment [i.e., $\mathscr{P}$ is a family of probability measures on the measurable space $(\Omega, \mathscr{A})$], and let G be a group of bijective and bimeasurable maps of $(\Omega, \mathscr{A})$ onto itself leaving the family $\mathscr{P}$ invariant, that is, $gP \in \mathscr{P}$, $\forall\ P \in \mathscr{P}$, $\forall\ g \in \mathscr{G}$, where gP is the probability measure on $\mathscr{A}$ defined by $gP(A) = P(g^{-1}A)$, $A \in \mathscr{A}$. If $P \in \mathscr{P}$, two events $B, C \in \mathscr{A}$ are said to be P-equivalent (and we shall write $B \sim_P C$) if $P(B \vartriangle C) = 0$; these events are said to be equivalent (we write $B \sim C$) if they are P-equivalent for all $P \in \mathscr{P}$. The null sets are the events equivalent to $\varnothing$. Let $\mathscr{A}_I = \{A \in \mathscr{A}: gA = A,\ \forall\ g \in \mathscr{G}\}$ be the σ-field of G-invariant sets and let $\mathscr{A}_A = \{A \in \mathscr{A}: gA \sim A,\ \forall\ g \in G\}$ be the σ-field of $\mathscr{P}$-almost-G-invariant sets.

For two sub-σ-fields $\mathscr{B}, \mathscr{C}$ of $\mathscr{A}$ we shall write $\mathscr{B} \underset{\sim}{\subseteq} \mathscr{C}$ if for every $B \in \mathscr{B}$ there exists $C \in \mathscr{C}$ such that $B \sim C$; $\mathscr{B}$ and $\mathscr{C}$ will be said to be equivalent or $\mathscr{P}$-equivalent (and we shall write $\mathscr{B} \sim \mathscr{C}$) if $\mathscr{B} \underset{\sim}{\subseteq} \mathscr{C}$ and $\mathscr{C} \underset{\sim}{\subseteq} \mathscr{B}$. The sub-$\sigma$-fields $\mathscr{B}$ and $\mathscr{C}$ are said to be independent if they are P-independent for every $P \in \mathscr{P}$. A privileged dominating probability for the statistical experiment $(\Omega, \mathscr{A}, \mathscr{P})$ is a probability measure Q on $(\Omega, \mathscr{A})$ of the form $Q = \sum_{n=1}^{\infty} a_n P_n$ such that $P \ll Q$ for all $P \in \mathscr{P}$, $\{P_n: n \in \mathbb{N}\} \subset \mathscr{P}$, $\sum_n a_n = 1$ and $a_n \geq 0$, $\forall\ n$. It is well known that a privileged dominating probability exists when the experiment is dominated. $\mathscr{A}_S$ will always be a sufficient sub-σ-field of $\mathscr{A}$. The σ-fields $\mathscr{A}_{SI} = \{A \in \mathscr{A}_I: \exists\ B \in \mathscr{A}_S,\ P(A \vartriangle B) = 0,\ \forall\ P \in \mathscr{P}\}$ and $\mathscr{A}_{SA} = \{A \in \mathscr{A}_A: \exists\ B \in \mathscr{A}_S,\ P(A \vartriangle B) = 0,\ \forall\ P \in \mathscr{P}\}$ are also considered in Berk (1972).

Let $\mathscr{B}, \mathscr{C}, \mathscr{D}$ be three sub-σ-fields of $\mathscr{A}$; for $P \in \mathscr{P}$, the σ-fields $\mathscr{B}$ and C are said to be P-conditionally independent given $\mathscr{D}$, and we shall write $\mathscr{B} \perp\!\!\!\perp_P \mathscr{C} | \mathscr{D}$, if

$$E_P(I_{B \cap C} | \mathscr{D}) \sim_P E_P(I_B | \mathscr{D}) \cdot E_P(I_C | \mathscr{D}),$$

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for every $B \in \mathscr{B}$ and $C \in \mathscr{C}$. It is well known that $\mathscr{B} \perp\!\!\!\perp_P \mathscr{C} | \mathscr{D}$ if and only if

$$E_P(I_C | \mathscr{B} \vee \mathscr{D}) \sim_P E_P(I_C | \mathscr{D}) \qquad \forall\, C \in \mathscr{C},$$

where $\mathscr{B} \vee \mathscr{D}$ is the smallest σ-field containing $\mathscr{B}$ and $\mathscr{D}$. The σ-fields $\mathscr{B}$ and $\mathscr{C}$ are said to be conditionally independent given $\mathscr{D}$, and we shall write $\mathscr{B} \perp\!\!\!\perp \mathscr{C} | \mathscr{D}$, if $\mathscr{B} \perp\!\!\!\perp_P \mathscr{C} | \mathscr{D}$, $\forall\, P \in \mathscr{P}$. Other known concepts not defined here may be found in Lehmann (1986), for example.

The classical paper Hall, Wijsman and Ghosh (1965) investigates under which conditions the σ-field $\mathscr{A}_S \cap \mathscr{A}_I$ is sufficient for $\mathscr{A}_I$: it is shown that this is the case if $g\mathscr{A}_S = \mathscr{A}_S$, $\forall\, g \in G$ and $\mathscr{A}_S \cap \mathscr{A}_I \sim \mathscr{A}_S \cap \mathscr{A}_A$. The interesting analogous problem for almost-invariance is considered in Berk (1972), where it is shown that $\mathscr{A}_{SA}$ is sufficient for $\mathscr{A}_A$ if $g\mathscr{A}_S \sim \mathscr{A}_S$, $\forall\, g \in G$. A synonymous condition is that $\mathscr{A}_S$ is equivalent to the σ-field induced by an almost-equivariant statistic [see Lemma 2 of Berk (1972)] and is satisfied if $\mathscr{A}_S$ is minimal sufficient. It should be noted that the notations $\mathscr{A}_{SI}$ (resp., $\mathscr{A}_{SA}$) are used in Hall, Wijsman and Ghosh (1965) to denote the intersection of $\mathscr{A}_S$ and $\mathscr{A}_I$ (resp., $\mathscr{A}_A$).

In this paper some concepts and examples are given to clarify certain results of the papers cited above.

Let us introduce a weaker notion of equivalence between σ-fields as follows: given two sub-σ-fields $\mathscr{B}$ and $\mathscr{C}$ of $\mathscr{A}$ we will say that $\mathscr{B}$ and $\mathscr{C}$ are weakly-$\mathscr{P}$-equivalent if they are P-equivalent for all $P \in \mathscr{P}$. A σ-field will be said to be weakly-$\mathscr{P}$-trivial if it is weakly-$\mathscr{P}$-equivalent to the trivial σ-field. Using this weaker notion of triviality, a correct version of proposition (i) of Theorem 4 of Berk (1972) is as follows: The σ-fields $\mathscr{A}_S$ and $\mathscr{A}_A$ are independent if and only if they are conditionally independent given $\mathscr{A}_{SA}$ and $\mathscr{A}_{SA}$ is weakly-$\mathscr{P}$-trivial. The following counterexample shows a nontrivial group for which $\mathscr{A}_{SI}$ is not $\mathscr{P}$-equivalent to $\{\varnothing, \Omega\}$.

EXAMPLE 1. Let $\Omega = \{1, 2, 3, 4\}$, let $\mathscr{A}$ be the σ-field of all subsets of Ω and let $\mathscr{P} = \{P, Q\}$, where P is the uniform distribution on $\{2, 3, 4\}$ and Q is the probability measure concentrated at the point 1. The smallest σ-field $\mathscr{A}_S$ containing the events $\{1\}$ and $\{2\}$ is sufficient for the experiment $(\Omega, \mathscr{A}, \mathscr{P})$. Let $G = \{I, g_1, g_2\}$, where I is the identity map on Ω, g_1 is the permutation $(1, 3, 4, 2)$ and $g_2 = (1, 4, 2, 3)$. We have that $\mathscr{A}_A = \mathscr{A}_I$ is the smallest σ-field including $\{1\}$ and $\mathscr{A}_A$ and $\mathscr{A}_S$ are independent, but $\mathscr{A}_{SI} = \mathscr{A}_{SA} = \mathscr{A}_A$ is not $\mathscr{P}$-equivalent to $\{\varnothing, \Omega\}$.

REMARK 1. It is not difficult to show that, replacing the independence of $\mathscr{A}_S$ and $\mathscr{A}_A$ by the stronger condition of independence of $\mathscr{A}_S$ and $\mathscr{A}_A$ for a privileged dominating probability, $\mathscr{A}_{SA} \sim (\varnothing, \Omega\}$, and hence $\mathscr{A}_{SI} \sim \{\varnothing, \Omega\}$. We show here that independence for a privileged dominating probability implies independence when one of the σ-fields involved is sufficient, as follows. Let Q be such a privileged dominating probability. For $A \in \mathscr{A}_A$, by independence, $Q(A)$ is a version of $Q(A | \mathscr{A}_S)$, which, by sufficiency, is a common version of the conditional probabilities $P(A|\mathscr{A}_S)$, $P \in \mathscr{P}$. Hence, for $A \in \mathscr{A}_A$, $B \in \mathscr{A}_S$ and $P \in \mathscr{P}$, we have

$$\begin{aligned} P(A \cap B) &= \int_B P(A|\mathscr{A}_S)\,dP = \int_B Q(A|\mathscr{A}_S)\,dP \\ &= \int_B Q(A)\,dP = Q(A)P(B). \end{aligned} \tag{1}$$

On taking $B = \Omega$ we obtain $P(A) = Q(A)$ (this shows that $\mathscr{A}_A$ is ancillary) and then (1) shows the independence of $\mathscr{A}_S$ and $\mathscr{A}_A$. We note in passing that the preceding provides a converse to the well-known theorem of Basu, namely, any statistic independent of a sufficient statistic for a privileged dominating probability is ancillary. Example 1 also shows that this proposition is not true if we only assume independence.

We are now concerned with the relationship between the independence of $\mathscr{A}_S$ and $\mathscr{A}_A$ and the equivalence of $\mathscr{A}_{SA}$ and $\mathscr{A}_{SI}$. A correct version of an assertion of Berk (1972) states that the independence of $\mathscr{A}_S$ and $\mathscr{A}_A$ implies that $\mathscr{A}_{SA}$ is weakly-$\mathscr{P}$-equivalent to $\mathscr{A}_{SI}$. In fact, it implies the weak $\mathscr{P}$-triviality of $\mathscr{A}_{SA}$. The condition $\mathscr{A}_{SA} \sim \mathscr{A}_{SI}$ is fulfilled if $\mathscr{A}_S$ and $\mathscr{A}_A$ are independent for a privileged dominating probability. It should be noted that while $\mathscr{A}_A \sim \mathscr{A}_I$ implies that $\mathscr{A}_{SA} \sim \mathscr{A}_{SI}$, it does not imply the stronger condition that $\mathscr{A}_S \cap \mathscr{A}_A \sim \mathscr{A}_S \cap \mathscr{A}_I$ as is shown in Example 1 of Landers and Rogge (1973).

The following counterexample shows that the independence of $\mathscr{A}_S$ and $\mathscr{A}_A$ is not a sufficient condition to have $\mathscr{A}_{SA} \sim \mathscr{A}_{SI}$. For the choice of the group of transformations in the two examples below, we make use of an idea due to Berk (1970).

EXAMPLE 2. Let E_1 and E_2 be disjoint intervals of $\mathbb{R}$, $\Omega = E_1 \cup E_2$, and let $\mathscr{A}$ be the Borel σ-field of Ω. Let $\mathscr{P} = \{U_1, U_2\}$, where U_i is the uniform distribution on E_i, $i = 1, 2$. The smallest σ-field $\mathscr{A}_S$ containing E_1 and E_2 is sufficient (and complete) for the experiment considered. Let G be the group of all bijective maps of Ω onto itself moving at most a finite subset of Ω. We have that $\mathscr{A}_I = \mathscr{A}_{SI} = \{\varnothing, \Omega\}$, $\mathscr{A}_A = \mathscr{A}$ and $\mathscr{A}_{SA}$ is the smallest σ-field including $\mathscr{A}_S$ and the null sets. Hence $\mathscr{A}_{SI}$ is not equivalent to $\mathscr{A}_{SA}$. Nevertheless, $\mathscr{A}_S$ and $\mathscr{A}_A$ are independent.

A correct restatement of part (ii) of the theorem in Berk (1972) is as follows: under the assumption of weak-$\mathscr{P}$-equivalence of $\mathscr{A}_S \vee \mathscr{A}_I$ and $\mathscr{A}$, the independence of $\mathscr{A}_S$ and $\mathscr{A}_A$ implies the weak $\mathscr{P}$-equivalence of $\mathscr{A}_A$ and $\mathscr{A}_I$. The next counterexample shows that we need not have equivalence of $\mathscr{A}_A$ and $\mathscr{A}_I$, even if $\mathscr{A}_S \vee \mathscr{A}_I \sim \mathscr{A}$.

EXAMPLE 3. Let $\Omega = [0, 4] \times [0, 4]$, let $\mathscr{N}$ be the set of null Borel sets on Ω with respect to the Lebesgue measure, $A_1 = [1, 2] \times [1, 2]$, $A_2 = [2, 3] \times [2, 3]$ and $\mathscr{A}$ be the smallest σ-field containing $\mathscr{N}$, $[0, 2] \times [0, 2]$, $[2, 4] \times [2, 4]$ and $[1, 3] \times [1, 3]$. We shall write U_i, $i = 1, 2$, for the restriction to $\mathscr{A}$ of the uniform distribution on A_i and $\mathscr{P} = \{U_1, U_2\}$. Let G be the group of all transformations on Ω moving at most a finite subset of Ω and leaving the set $[1,3] \times [1,3]$ invariant. Hence $\mathscr{A}_I$ is the smallest σ-field including $[1,3] \times [1,3]$, and $\mathscr{A}_A = \mathscr{A}$. The smallest σ-field $\mathscr{A}_S$ containing $[0,2] \times [0,2]$ and $[2,4] \times [2,4]$ is sufficient for the experiment $(\Omega, \mathscr{A}, \mathscr{P})$, is independent of $\mathscr{A}_A$ and satisfies $\mathscr{A} \sim \mathscr{A}_S \vee \mathscr{A}_I$. However, $\mathscr{A}_A \nsim \mathscr{A}_I$, since the event $[2,3] \times [2,3]$ is not equivalent to any event of $\mathscr{A}_I$.

REMARK 2. It is also claimed in Berk (1972) that under the hypothesis of conditional independence of $\mathscr{A}_S$ and $\mathscr{A}_A$ given $\mathscr{A}_{SA}$ and $\mathscr{A} \sim \mathscr{A}_S \vee \mathscr{A}_I$, the propositions $\mathscr{A}_A \sim \mathscr{A}_I$ and $\mathscr{A}_{SA} \sim \mathscr{A}_{SI}$ are equivalent. The proof given there requires the not-easily-checked condition "$\mathscr{A}_I$ is sufficient for $\mathscr{A}_A$;" this condition (and, hence, $\mathscr{A}_A \sim \mathscr{A}_I$) is clearly satisfied in the dominated case. Another condition guaranteeing that $\mathscr{A}_I$ is sufficient for $\mathscr{A}_A$ is that the group acts transitively on the family $\mathscr{P}$ (this means that $\mathscr{P} = \{gP: g \in G\}$) as is shown in Lemma 2 of Berk and Bickel (1968). The condition $\mathscr{A} \sim \mathscr{A}_S \vee \mathscr{A}_I$ can be replaced by $\mathscr{A}_A \underset{\sim}{\subset} \mathscr{A}_S \vee \mathscr{A}_I$.

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