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CONSERVATIVE CONFIDENCE REGIONS IN MULTIVARIATE CALIBRATION[1]

BY THOMAS MATHEW AND WENXING ZHA

University of Maryland Baltimore County

In the multivariate calibration problem using a multivariate linear model, some conservative confidence regions are constructed. The regions are nonempty and invariant under nonsingular transformations. Situations where the explanatory variable occurs nonlinearly in the model are also considered. Computational aspects of the confidence region and its practical implementation are discussed. The results are illustrated using two examples. The examples show that our confidence regions are much more satisfactory compared to those based on some of the existing procedures. Furthermore, simulation results for the examples reveal that the coverage probability of the conservative confidence regions are very close to the assumed confidence level.

1. Introduction. The multivariate calibration problem deals with using the statistical relationship between a $p \times 1$ response variable $\mathbf{y}$ and an $m \times 1$ explanatory variable $\mathbf{x}$ for statistical inference concerning an unknown value of $\mathbf{x}$, corresponding to a future value of $\mathbf{y}$, using available data on $\mathbf{x}$ and $\mathbf{y}$. The focus of this article is the construction of a confidence region for the unknown value of $\mathbf{x}$, when the explanatory variable $\mathbf{x}$ is assumed to be fixed (i.e., not random). We shall deal only with the situation when the relationship between $\mathbf{y}$ and $\mathbf{x}$ is a multivariate linear model and $\mathbf{y}$ follows the multivariate normal distribution. Thus $\mathbf{y} \sim N(B\mathbf{x}, \Sigma)$, where B is an unknown $p \times m$ parameter matrix and Σ is an unknown $p \times p$ positive definite matrix. If $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N$ are N independent observations corresponding to N known values $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ of the explanatory variable $\mathbf{x}$, writing $Y = (\mathbf{y}_1, \ldots, \mathbf{y}_N)$ and $X = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$, the columns of Y are independent multivariate normal random variables with

$$E(Y) = BX \quad \text{and} \quad \operatorname{Cov}(\operatorname{Vec}(Y)) = I_N \otimes \Sigma, \tag{1.1}$$

where the $m \times N$ matrix X is assumed to be of rank m. Now consider another $p \times 1$ normally distributed random vector $\mathbf{y}$ corresponding to an unknown value θ of $\mathbf{x}$ and independent of Y in (1.1). Assuming that the same multivariate linear model [as in (1.1)] holds, we get

$$E(\mathbf{y}) = B\theta \quad \text{and} \quad \operatorname{Cov}(\mathbf{y}) = \Sigma. \tag{1.2}$$

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Our problem is the construction of a confidence region for θ. Note that since B is a $p \times m$ matrix, for the identifiability of θ in (1.2), we need the condition $p \geq m$. This will be assumed throughout this article. As a generalization of the model (1.2), we also consider models where θ in (1.2) is a nonlinear function of fewer unknown parameters, denoted by an $r \times 1$ vector ξ ($r \leq m$). When this is the case, instead of (1.2), we have the model

$$E(\mathbf{y}) = B\mathbf{h}(\xi) \quad \text{and} \quad \operatorname{Cov}(\mathbf{y}) = \Sigma, \tag{1.3}$$

where $\mathbf{h}(\xi)$ is an $m \times 1$ vector-valued function of ξ. The problem how is the construction of a confidence region for ξ. An example where (1.3) is applicable is polynomial regression. The model (1.3) was considered by Oman (1988) and a specific application appears in Oman and Wax (1984). Note that the models (1.1)–(1.3) do not have an intercept term even though in many applications the means of the $\mathbf{y}_i$'s and $\mathbf{y}$ will involve a common intercept term. Models with such an intercept term can actually be reduced to those without the intercept; see Remark 2.2 in Mathew and Kasala (1994). It is also natural to work with such reduced models, since, when a common intercept term is present, inference concerning θ in (1.2), or ξ in (1.3), is invariant under a common translation of the $\mathbf{y}_i$'s and $\mathbf{y}$.

Several authors have addressed the problem of constructing exact or conservative confidence regions for θ in the model (1.2). In this context, asymptotic results are due to Fujikoshi and Nishii (1984), Davis and Hayakawa (1987) and Brown and Sundberg (1987), while finite sample results have been obtained by Brown (1982), Oman (1988) and Mathew and Kasala (1994). A review of some of these results is given in the articles by Osborne (1991) and Sundberg (1994) and in the recent book by Brown (1993). The confidence regions that are available for the finite sample case are known to have some drawbacks. Brown's (1982) region can be empty and Oman's (1988) region lacks a natural invariance property, namely, invariance under the action of the group of $p \times p$ nonsingular matrices acting on Y and $\mathbf{y}$ as $Y \to AY$ and $\mathbf{y} \to A\mathbf{y}$, where A is a $p \times p$ nonsingular matrix. The region due to Mathew and Kasala (1994), though nonempty and invariant, is based on a pivotal statistic that is a fairly complicated function of θ and, hence, it is difficult to study the shape of the region. It should also be pointed out that except Oman (1988), the other authors have actually considered the model (1.2), whereas, Oman (1988) considered the more general model (1.3).

Our concern in this article is to obtain a region that avoids some of the above drawbacks. However, our region will be only conservative, that is, its coverage probability is more than the assumed confidence level. The construction of our region is explained in the next section. We have used several ideas from Oman's (1988) article for the derivation of our region and its practical implementation. We first describe the procedure for the models (1.1) and (1.2). We then extend our procedure to the model (1.3). Some computational aspects are discussed in Section 3, and two examples are presented in Section 4. Both examples are taken from published literature. The first example is based on the paint finish data analyzed in Brown (1982) and Brown and Sundberg

(1987). For this example, our procedure gave confidence regions that were slightly wider compared to the likelihood-based confidence region due to Brown and Sundberg (1987). However, it turns out that the coverage probability of the likelihood-based region can be substantially lower than the assumed confidence level. Our second example is based on the problem considered in Oman and Wax (1984) dealing with the estimation of gestational age (i.e., week of pregnancy) using ultrasound measurements of two fetal bone lengths (some details are given in Section 4). Oman and Wax (1984) modeled the relation between the bone lengths and gestational age using a model that is quadratic in the gestational age. The problem is to predict the unknown gestational age corresponding to a measurement of the bone lengths. For this problem, our results turned out to be very similar to those in Oman (1988) with the additional property that our region is invariant. Furthermore, for both the examples, the coverage probabilities of our conservative regions turned out to be very close to the assumed confidence level for a variety of parameter values. Some concluding remarks are mentioned in Section 5.

2. The confidence regions. We shall first consider the models (1.1) and (1.2) and work with the following canonical form derived in Mathew and Kasala (1994). Let

$$\begin{aligned} Y_1 &= YX'(XX')^{-1/2}, \qquad & S &= Y\left(I - X'(XX')^{-1}X\right)Y', \\ B_1 &= B(XX')^{1/2}, & \theta_1 &= (XX')^{-1/2}\theta. \end{aligned} \tag{2.1}$$

Then

$$Y_1 \sim N(B_1, I_m \otimes \Sigma), \qquad \mathbf{y} \sim N(B_1\theta_1, \Sigma), \qquad S \sim W_p(\Sigma, N-m), \tag{2.2}$$

where we assume $N - m \geq p$. Furthermore, Y_1, $\mathbf{y}$ and S are independently distributed. (2.2) is the canonical form that we shall work with. Once we obtain a confidence region for θ_1, the transformation in (2.1) can be used to obtain a confidence region for θ.

The confidence region that we shall construct will be based on the statistic

$$\begin{aligned} &(\mathbf{y} - Y_1\theta_1)'S^{-1}Y_1\left(Y_1'S^{-1}Y_1\right)^{-1}Y_1'S^{-1}(\mathbf{y} - Y_1\theta_1) \\ &\qquad = \left(\hat{\theta}_1 - \theta_1\right)'Y_1'S^{-1}Y_1\left(\hat{\theta}_1 - \theta_1\right), \end{aligned} \tag{2.3}$$

where $\hat{\theta}_1 = (Y_1'S^{-1}Y_1)^{-1}Y_1'S^{-1}\mathbf{y}$. Clearly, it is reasonable to use (2.3) to obtain a confidence region for θ_1. The possibility of using (2.3) is mentioned in Williams (1959) and Wood (1982). The asymptotic results in Fujikoshi and Nishii (1984) and Davis and Hayakawa (1987) are based on (2.3). These authors have derived the conditional distribution of the statistic in (2.3), conditionally given Y_1 and $\mathbf{y}'[S^{-1} - S^{-1}Y_1(Y_1'S^{-1}Y_1)^{-1}Y_1'S^{-1}]\mathbf{y}$ [see

Fujikoshi and Nishii (1984), Theorem 3, and Davis and Hayakawa (1987), page 145]. They have shown that if

$$U(\theta_1) = \frac{N-p-m+1}{m} \tag{2.4}$$
$$\times \frac{(\mathbf{y} - Y_1\theta_1)'S^{-1}Y_1\left(Y_1'S^{-1}Y_1\right)^{-1}Y_1'S^{-1}(\mathbf{y} - Y_1\theta_1)}{1 + \mathbf{y}'\left[S^{-1} - S^{-1}Y_1\left(Y_1'S^{-1}Y_1\right)^{-1}Y_1'S^{-1}\right]\mathbf{y}},$$

then, conditionally given Y_1 and $\mathbf{y}'[S^{-1} - S^{-1}Y_1(Y_1'S^{-1}Y_1)^{-1}Y_1'S^{-1}]\mathbf{y}$,

$$U(\theta_1) \sim F_{m, N-p-m+1}(\Lambda(B_1, \theta_1)), \tag{2.5}$$

the noncentral F distribution with degrees of freedom $(m, N-p-m+1)$ and noncentrality parameter $\Lambda(B_1, \theta_1)$ given by

$$\Lambda(B_1, \theta_1) = \frac{\theta_1'(B_1 - Y_1)'\Sigma^{-1}Y_1\left(Y_1'\Sigma^{-1}Y_1\right)^{-1}Y_1'\Sigma^{-1}(B_1 - Y_1)\theta_1}{1 + \mathbf{y}'\left[S^{-1} - S^{-1}Y_1\left(Y_1'S^{-1}Y_1\right)^{-1}Y_1'S^{-1}\right]\mathbf{y}}. \tag{2.6}$$

The pivot that we shall use in order to construct our confidence region will be $U(\theta_1)$ in (2.4). However, the distribution of $U(\theta_1)$ depends on both B_1 and θ_1 and hence an exact confidence region for θ_1 cannot be obtained using $U(\theta_1)$. We shall proceed as follows. Let $a_\alpha(B_1, \theta_1)$ be such that

$$P\left[U(\theta_1) \le a_\alpha(B_1, \theta_1)\right] = 1 - \alpha. \tag{2.7}$$

In other words $\{\theta_1 : U(\theta_1) \le a_\alpha(B_1, \theta_1)\}$ is a $100(1-\alpha)\%$ confidence region for θ_1. Of course, this confidence region cannot be computed since $a_\alpha(B_1, \theta_1)$ depends on the nuisance parameter B_1. We shall obtain two quantities $b_\alpha(\theta_1)$ and $c_\alpha(\theta_1)$, which depend on θ_1 (but not on B_1), satisfying

$$b_\alpha(\theta_1) \le a_\alpha(B_1, \theta_1) \le c_\alpha(\theta_1) \tag{2.8}$$

for all B_1. We then have

$$\begin{aligned} P\left[U(\theta_1) \le b_\alpha(\theta_1)\right] &\le P\left[U(\theta_1) \le a_\alpha(B_1, \theta_1)\right] \\ &\le P\left[U(\theta_1) \le c_\alpha(\theta_1)\right]. \end{aligned} \tag{2.9}$$

From (2.7) and (2.9), we get

$$P\left[U(\theta_1) \le c_\alpha(\theta_1)\right] \ge 1 - \alpha. \tag{2.10}$$

In other words, the region

$$\left\{\theta_1 : U(\theta_1) \le c_\alpha(\theta_1)\right\} \tag{2.11}$$

is a conservative confidence region for θ_1 with coverage probability at least $1-\alpha$. Note that, even though $c_\alpha(\theta_1)$ depends on θ_1, the region in (2.11) can be obtained once the functional form of $c_\alpha(\theta_1)$ is known. This is the idea used in Oman (1988) to obtain a conservative confidence region. Note that the value $\hat{\theta}_1 = (Y_1'S^{-1}Y_1)^{-1}Y_1'S^{-1}\mathbf{y}$ always belongs to the confidence region (2.11).

Thus the region is nonempty. It is clearly invariant under the transformation $Y \to AY$ and $\mathbf{y} \to A\mathbf{y}$ (or equivalently $Y_1 \to AY_1$, $S \to ASA'$ and $\mathbf{y} \to A\mathbf{y}$), where A is a $p \times p$ nonsingular matrix, since $U(\theta_1)$ is an invariant statistic. The construction of $c_\alpha(\theta_1)$ in (2.11) is based on the following theorem, which is our main result.

THEOREM 1. *Consider the model* (2.2) *and let* $U(\theta_1)$ *be as defined in* (2.4). *Let* $f_q(\cdot)$ *and* $g_{r,s}(\cdot)$, *respectively, denote the probability density functions of a central chi-square random variable with q degrees of freedom and a central F random variable with* (r, s) *degrees of freedom. Also let* $F_{r,s}$ *and* $F_{r,s}(\gamma)$, *respectively, denote a central F random variable and a noncentral F random variable with noncentrality parameter* γ, *both having degrees of freedom* (r, s). *Then, for any* $a > 0$,

$$
\begin{aligned}
&P\left[F_{m,N-p-m+1} \le a\right] \\
&\quad \ge P\left[U(\theta_1) \le a\right] \\
&\quad \ge \int_0^\infty \int_0^\infty P\left[F_{m,N-p-m+1}\left(\frac{\theta_1'\theta_1 v}{1+(p-m)w/(N-p+1)}\right) \le a\right] \\
&\qquad\qquad \times f_p(v) g_{p-m,N-p+1}(w)\,dv\,dw \\
&\quad \ge \int_0^\infty P\left[F_{m,N-p-m+1}(\theta_1'\theta_1 v) \le a\right] f_p(v)\,dv.
\end{aligned} \tag{2.12}
$$

PROOF. From Fujikoshi and Nishii (1984) and Davis and Hayakawa (1987), it follows that conditionally given Y_1,

$$
\frac{N-p+1}{p-m}\mathbf{y}'\left[S^{-1} - S^{-1}Y_1\left(Y_1'S^{-1}Y_1\right)^{-1}Y_1'S^{-1}\right]\mathbf{y} \sim F_{p-m,N-p+1}(\lambda), \tag{2.13}
$$

where

$$
\lambda = \theta_1'B_1'\left[\Sigma^{-1} - \Sigma^{-1}Y_1\left(Y_1'\Sigma^{-1}Y_1\right)^{-1}Y_1'\Sigma^{-1}\right]B_1\theta_1.
$$

We note that the quantity $\mathbf{y}'[S^{-1} - S^{-1}Y_1(Y_1'S^{-1}Y_1)^{-1}Y_1'S^{-1}]\mathbf{y}$, which appears on the left-hand side of (2.13) is the quantity R considered in Fujikoshi and Nishii (1984) and Davis and Hayakawa (1987). Since $\Sigma^{-1} \ge \Sigma^{-1}Y_1(Y_1'\Sigma^{-1}Y_1)^{-1}Y_1'\Sigma^{-1}$, we have the following inequality concerning the numerator of $\Lambda(B_1, \theta_1)$ in (2.6):

$$
\begin{aligned}
&\theta_1'(B_1 - Y_1)'\Sigma^{-1}Y_1\left(Y_1'\Sigma^{-1}Y_1\right)^{-1}Y_1'\Sigma^{-1}(B_1 - Y_1)\theta_1 \\
&\qquad \le \theta_1'(B_1 - Y_1)'\Sigma^{-1}(B_1 - Y_1)\theta_1.
\end{aligned} \tag{2.14}
$$

Also, given Y_1, the noncentral F random variable

$$
(N-p+1)(p-m)^{-1}\mathbf{y}'\left[S^{-1} - S^{-1}Y_1\left(Y_1'S^{-1}Y_1\right)^{-1}S^{-1}Y_1\right]\mathbf{y}
$$

is stochastically larger than the central F random variable $F_{p-m,N-p+1}$. This fact, along with (2.14), shows that $\Lambda(B_1, \theta_1)$ in (2.6) satisfies (conditionally given Y_1)

$$\Lambda(B_1, \theta_1) \leq_{\text{st}} \frac{\theta_1'(B_1 - Y_1)'\Sigma^{-1}(B_1 - Y_1)\theta_1}{1 + (p-m)w/(N-p+1)}, \tag{2.15}$$

where w denotes a central F random variable with $(p-m, N-p+1)$ degrees of freedom and $\leq_{\text{st}}$ denotes stochastically smaller. The first inequality in (2.12) follows from (2.5) using the fact that $P[F_{r,s}(\gamma) \leq a]$ is a decreasing function of γ. Furthermore, this property, along with (2.5) and (2.15), gives the following inequality, conditionally given Y_1:

$$\begin{aligned} &P[U(\theta_1) \leq a] \\ &\geq \int_0^\infty P\left[F_{m,N-p-m+1}\left(\frac{\theta_1'(B_1 - Y_1)'\Sigma^{-1}(B_1 - Y_1)\theta_1}{1 + (p-m)w/(N-p+1)}\right) \leq a\right] \\ &\quad \times g_{p-m,N-p+1}(w)\,dw. \end{aligned} \tag{2.16}$$

Using the distribution of Y_1 given in (2.2), it follows that $(B_1 - Y_1)\theta_1 \sim N(0, (\theta_1'\theta_1)\Sigma)$. Hence

$$v = \frac{\theta_1'(B_1 - Y_1)'\Sigma^{-1}(B_1 - Y_1)\theta_1}{\theta_1'\theta_1} \sim \chi_p^2; \tag{2.17}$$

(2.16) and (2.17) together give the second inequality in (2.12). The last inequality in (2.12) follows by noting that

$$\frac{\theta_1'\theta_1 v}{1 + (p-m)w/(N-p+1)} \leq \theta_1'\theta_1 v$$

and $P[F_{r,s}(\gamma) \leq a]$ is a decreasing function of γ. This completes the proof of the theorem. □

Suppose $c_\alpha^{(1)}(\theta_1'\theta_1)$ and $c_\alpha^{(2)}(\theta_1'\theta_1)$ satisfy

$$\begin{aligned} &\int_0^\infty \int_0^\infty P\left[F_{m,N-p-m+1}\left(\frac{\theta_1'\theta_1 v}{1 + (p-m)w/(N-p+1)}\right) \leq c_\alpha^{(1)}(\theta_1'\theta_1)\right] \\ &\quad \times f_p(v) g_{p-m,N-p+1}(w)\,dv\,dw = 1 - \alpha, \end{aligned} \tag{2.18}$$

$$\int_0^\infty P\left[F_{m,N-p-m+1}(\theta_1'\theta_1 v) \leq c_\alpha^{(2)}(\theta_1'\theta_1)\right] f_p(v)\,dv = 1 - \alpha. \tag{2.19}$$

If $F_\alpha(m, N-p-m+1)$ denotes the $100(1-\alpha)$th percentile of $F_{m,N-p-m+1}$ and if $a_\alpha(B_1, \theta_1)$ is as in (2.7), we get the following inequality, using the theorem:

$$F_\alpha(m, N-p-m+1) \leq a_\alpha(B_1, \theta_1) \leq c_\alpha^{(1)}(\theta_1'\theta_1) \leq c_\alpha^{(2)}(\theta_1'\theta_1). \tag{2.20}$$

Thus we can take $b_\alpha(\theta_1) = F_\alpha(m, N-p-m+1)$ and $c_\alpha(\theta_1) = c_\alpha^{(i)}(\theta_1'\theta_1)$ ($i = 1, 2$) in (2.8). Clearly,

$$P\big[U(\theta_1) \le c_\alpha^{(i)}(\theta_1'\theta_1)\big] \ge 1-\alpha, \qquad i = 1, 2, \tag{2.21}$$

and we have the following conservative confidence regions for θ_1, for $i = 1, 2$:

$$\big\{\theta_1 \colon U(\theta_1) \le c_\alpha^{(i)}(\theta_1'\theta_1)\big\}. \tag{2.22}$$

The computation of $c_\alpha^{(i)}(\theta_1'\theta_1)$ is discussed in the next section. Note that Theorem 1 also gives an upper bound for $P[U(\theta_1) \le a]$, namely, $P[F_{m, N-p-m+1} \le a]$. In practice, this upper bound will not be useful to obtain a confidence region for θ_1.

We shall now briefly describe the construction of a confidence region for the parameter ξ in the model (1.3). Recall that $\mathbf{h}(\xi)$ is an $m \times 1$ vector and ξ is an $r \times 1$ vector, $r \le m$. Our problem now is the construction of a confidence region for ξ using Y in (1.1) and $\mathbf{y}$ in (1.3). We shall use the following canonical form, similar to (2.2):

$$\begin{gathered} Y_1 \sim N(B_1, I_m \otimes \Sigma), \qquad \mathbf{y} \sim N(B_1\mathbf{h}_1(\xi), \Sigma), \\ S \sim W_p(\Sigma, N-m), \end{gathered} \tag{2.23}$$

where

$$\mathbf{h}_1(\xi) = (XX')^{-1/2}\mathbf{h}(\xi) \tag{2.24}$$

and the other quantities in (2.23) are defined in (2.1). We shall assume that the components of $\mathbf{h}(\xi)$ [and hence those of $\mathbf{h}_1(\xi)$] are differentiable functions of ξ. Let $H(\xi)$ be the $p \times r$ matrix defined as

$$H(\xi) = Y_1 \frac{\partial \mathbf{h}_1(\xi)}{\partial \xi}. \tag{2.25}$$

In our derivation that follows, we require the assumption that $H(\xi)$ have rank r (with probability 1); see the assumption in Oman [(1988), page 179]. Similar to $U(\theta_1)$ in (2.4), define

$$\begin{aligned} U(\xi) = {} & \frac{N-m-p+1}{r}[\mathbf{y} - Y_1\mathbf{h}_1(\xi)]'S^{-1}H(\xi)\big[H(\xi)'S^{-1}H(\xi)\big]^{-1} \\ & \times H(\xi)'S^{-1}[\mathbf{y} - Y_1\mathbf{h}_1(\xi)] \\ & \times \Big[1 + \mathbf{y}'\Big\{S^{-1} - S^{-1}H(\xi)\big[H(\xi)'S^{-1}H(\xi)\big]^{-1}H(\xi)'S^{-1}\Big\}\mathbf{y}\Big]^{-1} \end{aligned} \tag{2.26}$$

Following the arguments in the proof of Theorem 1 that lead to (2.12), we get the following inequality, for any $a > 0$:

$$
\begin{aligned}
&P(F_{r,N-m-p+1} \le a)\\
&\quad \ge P(U(\xi) \le a)\\
&\quad \ge \int_0^\infty \int_0^\infty P\left[F_{r,N-m-p+1}\left(\frac{\mathbf{h}_1(\xi)'\mathbf{h}_1(\xi)v}{1+(p-r)w/(N-m-p+r+1)}\right) \le a\right] \qquad (2.27)\\
&\qquad\qquad \times f_p(v) g_{p-r,N-m-p+r+1}(w)\, dv\, dw\\
&\quad \ge \int_0^\infty P\left[F_{r,N-m-p+1}(\mathbf{h}_1(\xi)'\mathbf{h}_1(\xi)v) \le a\right] f_p(v)\, dv.
\end{aligned}
$$

The notations in (2.27) are similar to those in (2.12). Defining $c_\alpha^{(i)}(\mathbf{h}_1(\xi)'\mathbf{h}_1(\xi))$ similar to $c_\alpha^{(i)}(\theta_1'\theta_1)$ in (2.18) and (2.19), we get the following conservative confidence regions for ξ, analogous to (2.22):

$$
\{\xi : U(\xi) \le c_\alpha^{(i)}(\mathbf{h}_1(\xi)'\mathbf{h}_1(\xi))\}, \qquad i = 1, 2. \tag{2.28}
$$

Note that the regions in (2.28) are nonempty, since they contain values of ξ minimizing $(\mathbf{y} - Y_1\mathbf{h}_1(\xi))'S^{-1}(\mathbf{y} - Y_1\mathbf{h}_1(\xi))$. Such values of ξ satisfy $H(\xi)'S^{-1}(\mathbf{y} - Y_1\mathbf{h}_1(\xi)) = 0$.

REMARK 2.1. As already pointed out, our methodology in this article is similar to that in Oman (1988). As in Oman (1988), we have constructed a pivot statistic that is a projection, essentially discarding the ancillary information provided by the distance being projected. Our model is an example of a curved exponential family and the invariant unconditional analysis that we have carried out could be thought unsatisfactory. In this context, see the discussion at the end of Section 5.3 in Brown (1993).

3. Computation of the confidence region. In order to compute $c_\alpha^{(1)}(\theta_1'\theta_1)$ and $c_\alpha^{(2)}(\theta_1'\theta_1)$ satisfying (2.18) and (2.19), respectively, or to compute $c_\alpha^{(i)}(\mathbf{h}_1(\xi)'\mathbf{h}_1(\xi))$ in (2.28), we proceed as in Oman (1988). We shall first represent the integrals on the left-hand sides (LHS) of (2.18) and (2.19) as two infinite series and then approximate it using a finite number of terms. This finite sum will then be equated to $1 - \alpha$ in order to compute $c_\alpha^{(i)}(\theta_1'\theta_1)$.

We shall first give an infinite series representation for the integral on the LHS of (2.19). This is given by

$$
\begin{aligned}
\text{LHS of (2.19)} &= \frac{1}{(\theta_1'\theta_1 + 1)^{p/2}\Gamma(p/2)}\\
&\quad \times \sum_{j=0}^{\infty} \frac{1}{j!} P\left[F_{m+2j,s} \le \frac{mc_\alpha^{(2)}(\theta_1'\theta_1)}{(m+2j)}\right] \qquad (3.1)\\
&\quad \times \Gamma\left(\frac{p}{2} + j\right)\left(\frac{\theta_1'\theta_1}{\theta_1'\theta_1 + 1}\right)^j,
\end{aligned}
$$

where $s = N - p - m + 1$. The algebraic computations leading to (3.1) are very similar to those that yielded equation (3.12) in Oman (1988), and hence are omitted. A similar calculation gives

$$\text{LHS of (2.18)} = \frac{1}{\Gamma(p/2)} \sum_{j=0}^{\infty} \frac{1}{j!} P\left[F_{m+2j,s} \le \frac{mc_{\alpha}^{(1)}(\theta_1'\theta_1)}{(m+2j)}\right] \Gamma\left(\frac{p}{2}+j\right)(\theta_1'\theta_1)^j$$

$$\times \int_0^{\infty} \frac{(1+(p-m)w/(N-p+1))^{p/2}}{\left[(\theta_1'\theta_1+1)+((p-m)w/(N-p+1))\right]^{p/2+j}} \times g_{p-m,N-p+1}(w)\,dw. \tag{3.2}$$

For each specified value of $\theta_1'\theta_1$, the expressions in (3.1) and (3.2) can be evaluated numerically by truncating the summation. The integral in (3.2) has to be numerically evaluated as well. However, in order to implement the confidence regions in Section 2, it is necessary to know the forms of the functions $c_{\alpha}^{(1)}(\theta_1'\theta_1)$ or $c_{\alpha}^{(2)}(\theta_1'\theta_1)$ satisfying (2.18) and (2.19), respectively. Clearly, the functional forms of $c_{\alpha}^{(i)}(\theta_1'\theta_1)$ $(i = 1, 2)$ cannot be obtained analytically. To overcome this difficulty, we proceed as follows. Note that in practical applications, very often an upper bound (and perhaps a lower bound as well) is available on $\theta_1'\theta_1$, say,

$$\theta_1'\theta_1 \le \delta. \tag{3.3}$$

We then have the following three approaches in order to practically implement the confidence regions in Section 2.

1. Compute $c_{\alpha}^{(i)}(\theta_1'\theta_1)$ using (3.1) or (3.2), with the sums suitably truncated, for a grid of values of $\theta_1'\theta_1$ in the interval $[0, \delta]$. Once the data are available, the confidence region can be determined by computing $U(\theta_1)$ [using (2.4)] and verifying whether $U(\theta_1) \le c_{\alpha}^{(i)}(\theta_1'\theta_1)$ for a grid of values of θ_1 satisfying (3.3).
2. Replace $\theta_1'\theta_1$ in (2.18) or (2.19) by the bound δ, compute $c_{\alpha}^{(i)}(\delta)$ and obtain the confidence region

$$\left\{\theta_1 \colon U(\theta_1) \le c_{\alpha}^{(i)}(\delta)\right\}. \tag{3.4}$$

   As already mentioned, $c_{\alpha}^{(i)}(\delta)$ $(i = 1, 2)$ can be numerically evaluated, using the representations (3.1) and (3.2), by truncating the summation in the infinite series. Clearly, (3.4) is quite different from the conservative confidence region (2.22). It should be noted that the coverage probability of (3.4) is at least $1 - \alpha$ and one should feel comfortable using (3.4) if the coverage probability of (3.4) is close to $1 - \alpha$. We would also like to add that Oman (1988) implemented his conservative confidence region for the example discussed in his paper using a bound of $\theta_1'\theta_1$.
3. Compute $c_{\alpha}^{(i)}(\theta_1'\theta_1)$ for a range of values of $\theta_1'\theta_1$ in the interval $[0, \delta]$. The values of $c_{\alpha}^{(i)}(\theta_1'\theta_1)$ so obtained can be plotted and the possibility of fitting a suitable function to the plotted values can be explored. This will give, at least approximately, the functional form of $c_{\alpha}^{(i)}(\theta_1'\theta_1)$. It is this approach that we have followed for the examples in the next section.

Clearly, it is computationally advantageous to obtain $c_\alpha^{(2)}(\theta_1'\theta_1)$ satisfying (3.1) compared to $c_\alpha^{(1)}(\theta_1'\theta_1)$ satisfying (3.2). However, the region $\{\theta_1\colon U(\theta_1) \le c_\alpha^{(1)}(\theta_1'\theta_1)\}$ will have a smaller volume compared to the region $\{\theta_1\colon U(\theta_1) \le c_\alpha^{(2)}(\theta_1'\theta_1)\}$. This follows from the fact that $c_\alpha^{(1)}(\theta_1'\theta_1) \le c_\alpha^{(2)}(\theta_1'\theta_1)$; see (2.20). Nevertheless, in practice one can use the region $\{\theta_1\colon U(\theta_1) \le c_\alpha^{(2)}(\theta_1'\theta_1)\}$ if its coverage probability is close to $1 - \alpha$ in the parameter region of interest.

So far in this section, our discussion has been on the practical implementation of the confidence region (2.22). Similar observations are also applicable for the implementation of the confidence region (2.28).

The following lemma shows how the distribution of the pivotal statistics in Section 2 depend on B_1 and Σ.

LEMMA 1. *The distributions of* $U(\theta_1)$ *in* (2.4) *and* $U(\xi)$ *in* (2.26) *depend on* B_1 *and* Σ *only through* $B_1'\Sigma^{-1}B_1$.

PROOF. Since the distribution of $U(\theta_1)$ is invariant under the transformation $Y_1 \to AY_1$, $\mathbf{y} \to A\mathbf{y}$ and $S \to ASA'$, and since $(B_1'\Sigma^{-1}B_1, \theta_1)$ is a maximal invariant parameter, the lemma follows for the distribution of $U(\theta_1)$. The proof for $U(\xi)$ is similar. □

The observation in Lemma 1 can be useful for simulating the coverage probabilities of the confidence regions in Section 2. We have indeed used the above lemma in our first example in the next section.

4. Two examples. Our first example is based on the paint finish data analyzed in Brown (1982) and Brown and Sundberg (1987). In this example, x is a scalar representing viscosity of the paint samples and $\mathbf{y}$ is a bivariate observation vector consisting of two measurements on certain optical properties of the samples; see Brown [(1982), page 301] for details. Even though data on some other variables are available in Brown (1982), we shall use the data on $\mathbf{y}$ and the viscosity in our analysis, following Brown and Sundberg (1987). In Brown's data, given in Table 2 of his paper, the viscosity of each paint sample was one of three different values and these will be coded as -1, 0 and 1, as done in Brown and Sundberg (1987). Twenty-seven observations were available for calibration, and if $\mathbf{y}_i$ denotes the observation corresponding to a known viscosity x_i, the model used by Brown and Sundberg (1987) is

$$\mathbf{y}_i \sim N(\mathbf{a} + \mathbf{b}x_i, \Sigma), \tag{4.1}$$

where $\mathbf{a}$ and $\mathbf{b}$ are unknown 2×1 parameter vectors and Σ is an unknown 2×2 positive definite matrix. The problem is to construct a confidence region for the unknown viscosity, say θ, corresponding to a measurement $\mathbf{y}$, where

$$\mathbf{y} \sim N(\mathbf{a} + \mathbf{b}\theta, \Sigma). \tag{4.2}$$

Before applying our procedure for constructing a confidence region for the unknown viscosity θ, we shall first reduce the models (4.1) and (4.2) to models without the intercept term $\mathbf{a}$ and then reduce it further to the canonical form

(2.2). Writing $Y = (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{27})$ and $\mathbf{x}' = (x_1, x_2, \dots, x_{27})$, (4.1) can be expressed as

$$Y \sim N(\mathbf{a}\mathbf{1}'_{27} + \mathbf{b}\mathbf{x}', I_{27} \otimes \Sigma), \tag{4.3}$$

where $\mathbf{1}_{27}$ is a 27×1 vector of ones. Among the 27 observations used for calibration, there were 9 each corresponding to viscosity values -1, 0 and 1. Hence

$$\mathbf{1}'_{27}\mathbf{x} = 0 \quad \text{and} \quad \mathbf{x}'\mathbf{x} = 18. \tag{4.4}$$

Using the computations in Remark 2.2 in Mathew and Kasala (1994), along with (2.1) and (4.4), the canonical form (2.2) can be obtained by defining

$$\begin{aligned} &\mathbf{y}_{(1)} = \frac{1}{\sqrt{18}} Y\mathbf{x}, \qquad \mathbf{b}_1 = \sqrt{18}\,\mathbf{b}, \qquad \mathbf{y}_0 = \sqrt{\frac{27}{28}}\left(\mathbf{y} - \frac{1}{27}\sum_{i=1}^{27}\mathbf{y}_i\right), \\ &\theta_1 = \sqrt{\frac{27}{18 \times 28}}\,\theta, \qquad S = Y\left(I - \frac{1}{27}\mathbf{1}_{27}\mathbf{1}'_{27} - \frac{1}{18}\mathbf{x}\mathbf{x}'\right)Y'. \end{aligned} \tag{4.5}$$

Then

$$\mathbf{y}_{(1)} \sim N(\mathbf{b}_1, \Sigma), \qquad \mathbf{y}_0 \sim N(\mathbf{b}_1\theta_1, \Sigma), \qquad S \sim W_2(\Sigma, 25). \tag{4.6}$$

Using the 27 $\mathbf{y}_i$ values used by Brown and Sundberg (1987), we computed confidence regions for θ_1 in (4.6) for a few of the remaining $\mathbf{y}_i$ values in Brown [(1982), Table 2]. The confidence level was chosen to be 95% and we used (2.19) to compute the confidence region. Writing $d = \theta^2$ and using (4.5), we note that $\theta_1'\theta_1 = 27\theta^2/(28 \times 18) = 27d/(28 \times 18)$. Following (2.19), we computed $c(d)$ satisfying

$$\int_0^\infty P\left[F_{1,24}\left(\frac{27}{28 \times 18} d \times v\right) \le c(d)\right] f_2(v)\, dv = 0.95, \tag{4.7}$$

where $c(d)$ is used to denote $c_{0.05}^{(2)}(\theta_1'\theta_1)\,[= c_{0.05}^{(2)}(27/(28 \times 18)d)]$. Even though the viscosity assumed only three values in the data given in Table 2 of Brown (1982), coded as -1, 0 and 1 in Brown and Sundberg (1987), it is possible that the parameter space for θ is larger than $[-1, 1]$. For the evaluation of $c(d)$, it is necessary to specify a possible range of values for θ, and in our analysis, we have considered the interval $[-2, 2]$ for θ. Larger intervals can certainly be considered; see Remark 4.1 below. Since $d = \theta^2$, it is clearly enough to consider only nonnegative values of θ for the numerical evaluation of $c(d)$. In our analysis, we have evaluated $c(d)$ satisfying (4.7) for a few values of θ in the interval $[0, 2]$. Equation (3.1) was used in the computation and truncating the summation in (3.1) to eight terms (i.e., $j = 0$–7) turned out to be quite accurate for computing $c(d)$. Table 1 gives the values of $c(d)$ for a few values of d.

A plot of the above values of $c(d)$ showed that $c(d)$ is very close to being linear in d [see Zha (1995) for more details]. The fitted line (by least squares) has the equation

$$c_*(d) = 4.26311 + 0.44711d, \tag{4.8}$$

TABLE 1
Values of $c(d)$ *satisfying* (4.7)

$d = \theta^2$	0	0.5	1	1.5	2	2.5	3	3.5	4
$c(d)$	4.2600	4.4871	4.7127	4.9366	5.1585	5.3785	5.5967	5.8132	6.0279

where $c_*(d)$ denotes values on the line. From (4.7), we note that $c(0)$ is the 95th percentile of the central F distribution with degrees of freedom $(1, 24)$, which is 4.26. The intercept of the line in (4.8) is slightly larger than this value, due to the fact that the line did not fit exactly. The confidence region for θ is thus given by

$$\left\{\theta : U\left(\sqrt{\frac{27}{28 \times 18}}\,\theta\right) \le c_*(\theta^2)\right\}, \tag{4.9}$$

where $c_*(\theta^2) = c_*(d)$ is given by (4.8).

Table 2 gives the classical interval due to Brown (1982), the likelihood-based interval due to Brown and Sundberg (1987) and our region (4.9) for four $\mathbf{y}$ values. The first three of these values are taken from Brown [(1982), Table 2] and they correspond to true viscosity values -1, 0 and 1. [Note that Brown's (1982) Table 2 has 36 observations and only 27 were used to obtain $\mathbf{y}_{(1)}$ and S in (4.6). The first three $\mathbf{y}$ values in Table 2 are from among the remaining seven values.] The fourth $\mathbf{y}$ value in Table 2 corresponds to an unknown viscosity and this value is also considered in Brown and Sundberg [(1987), page 56]. We recall from Brown and Sundberg [(1987), Section 4] that the likelihood-based region is implemented using a chi-square approximation to the distribution of the relevant statistic.

We see from Table 2 that the likelihood-based region due to Brown and Sundberg (1987) is the shortest, followed by our region (4.9) and Brown's (1982) region (in cases where it is nonempty). Note that Brown's region is exact and hence its coverage probability is the assumed confidence level of 95%. Our confidence region (4.9) will have coverage probability 95% or more;

TABLE 2
95% *confidence regions for* θ *in the models* (4.2) *and* (4.3)

$\mathbf{y}$	True value of θ	Classical interval due to Brown (1982)	Likelihood-based interval due to Brown and Sundberg (1987)	The interval (4.9)
(1.78, 38.73)′	−1	(−1.65, 0.92)	(−1.27, 0.57)	(−1.41, 0.67)
(1.79, 39.83)′	0	(−1.93, 0.57)	(−1.60, 0.27)	(−1.76, 0.37)
(1.52, 35.65)′	1	(0.39, 3.12)	(0.71, 2.68)	(0.61, 2.97)
(1.94, 34.09)′	Unknown	Empty set	(−1.34, 1.16)	(−1.39, 1.23)

see Table 3. We also simulated the coverage probabilities of the likelihood-based region due to Brown and Sundberg (1987). It turns out that the coverage probability of the likelihood-based region can be much lower than 95%. We simulated the above coverage probability for the models (4.2) and (4.3) using the following parameter values: $\mathbf{a} = 0$, $\Sigma = I_2$ and $\mathbf{b} = \eta(1, 0)'$, for various values of η [such choices of the parameter values were also made for simulating the coverage probabilities of (4.9), which is explained below]. For $\theta = 1$, the coverage probabilities of the likelihood-based confidence region for values of $\eta = 1$, 100, 10,000, 50,000, 100,000 and 200,000 turned out to be 0.363, 0.377, 0.387, 0.529, 0.744 and 0.967, respectively. (This is based on 10,000 simulations.) The coverage probabilities are significantly below the assumed confidence level of 95%, except at a very large value of η, that is, a very large value of $\mathbf{b}$. This can be explained based on the chi-square approximation described in Brown and Sundberg [(1987), page 55]. This chi-square approximation is valid only when $\Sigma \to 0$. In our simulation we chose $\Sigma = I_2$. This amounts to making the transformation $\mathbf{b} \to \Sigma^{-1/2}\mathbf{b}$, which $\to \infty$ as $\Sigma \to 0$. In other words, with $\Sigma = I_2$, the coverage probability of the likelihood-based confidence region will be close to the assumed confidence level only for very large values of $\mathbf{b}$—a fact that showed up in our simulation.

Table 3 gives the simulated coverage probabilities of the interval (4.9) for the models (4.2) and (4.3), for $\mathbf{a} = \mathbf{0}$, $\Sigma = I_2$ and for various values of $\mathbf{b}$ and θ. We note that the confidence region (4.9) is translation invariant and also invariant under a nonsingular transformation and hence we can choose $\mathbf{a} = \mathbf{0}$ and $\Sigma = I_2$ without loss of generality (see the last part of the first paragraph in the Introduction). The values of $\mathbf{b}$ were chosen based on the fact that the coverage probability of the confidence region (4.9) depends on $\mathbf{b}$ only through $\mathbf{b}'\mathbf{b}$ (since $\Sigma = I_2$); see Lemma 1. Thus, for the simulation, we have chosen $\mathbf{b} = \eta(1, 0)'$, for several values of η. For a few values of η and θ, the coverage probability of (4.9) was computed based on 100,000 simulations. The coverage probability turns out to be the same at θ and $-\theta$, and hence negative values of θ are not included in Table 3. [Even though we chose values of the

TABLE 3
Simulated coverage probabilities of the interval (4.9), *based on* 100,000 *simulations, for the models* (4.2) *and* (4.3), *for* $\mathbf{a} = \mathbf{0}$, $\Sigma = I_2$ *and* $\mathbf{b} = \eta(1, 0)'$ *for different values of* η *and* θ

		η				
θ	$c_*(\theta^2)$	0.001	0.01	1	10	100
0	4.2631	0.950	0.950	0.950	0.950	0.950
0.5	4.3749	0.950	0.950	0.951	0.952	0.952
1	4.7102	0.950	0.950	0.952	0.955	0.955
1.5	5.2691	0.950	0.951	0.955	0.960	0.960
2	6.0516	0.952	0.952	0.957	0.965	0.965

parameters in the models (4.2) and (4.3) for the simulation, one can also use the canonical form (4.6).]

From the results in Table 3, it is clear that the simulated coverage probabilities are close to the assumed 95% confidence level. Table 3 gives the indication that the coverage probability is an increasing function of both $|\theta|$ and $\|\mathbf{b}\|$.

REMARK 4.1. In our calculations for Example 1, we restricted θ to lie in the interval $[-2, 2]$. It is possible to use wider or narrower intervals depending on the available information on θ. We did calculate and plot values of $c(d)$ (where $d = \theta^2$) for several values of θ outside $[-2, 2]$, that is, values of d greater than 4. The linear fit, which was quite satisfactory for the values in Table 1, turns out to be unsatisfactory for higher values of d. In fact, it turns out that $c(d) < c_*(d)$ for values of $d > 4$, where $c_*(d)$ is obtained using (4.8). Consequently, (4.9) will give a conservative confidence region even if the true value of θ is greater than 2. It may be possible to approximate $c(d)$ using a suitable nonlinear function of d when it is known that the parameter space for θ is larger than $[-2, 2]$. This will of course give a more satisfactory (i.e., narrower) confidence region compared to (4.9). We did not pursue this mainly because none of the observations used to set up the calibration curve corresponds to values of θ outside $[-1, 1]$. Hence it is reasonable to assume that the calibration curve will be used to predict values of θ within or somewhat closer to $[-1, 1]$. Consequently, the region $[-2, 2]$ was deemed satisfactory.

REMARK 4.2. The numerical computations (including the simulations) for the above example, as well as for the next example, were carried out using SAS. As already pointed out, $c(d)$ satisfying (4.7) was computed using the infinite series representation (3.1) retaining seven terms in the summation. Since $c(d) > c(0) = 4.26$ (for the above example), for any given d, $c(d)$ can be computed by considering increments of $c(0)$ and evaluating the expression in (3.1) each time, until the numerical value of (3.1) is close to 0.95, up to a desired level of accuracy. Once $c(d)$ is thus calculated for a given value of d, the computation of $c(d_1)$ for any other value $d_1 > d$ can be accomplished by considering increments of $c(d)$ instead of $c(0)$. This is so because $c(d_1) > c(d)$ whenever $d_1 > d$. We followed the above approach in order to arrive at Table 1. These observations apply to our next example as well.

Our second example deals with a situation where the model (2.23) is applicable; it is taken from Oman and Wax (1984). The same example is also discussed in Oman (1988) and in what follows, we shall use Oman's (1988) notation. The problem deals with estimating gestational age (i.e., week of pregnancy) using ultrasound measurements on two fetal bone lengths: the femur length (F) and the biparietal diameter (BPD). If ξ represents the gestational age (in weeks) and $\mathbf{y} = (F, BPD)'$, the model is [see Oman and Wax (1984)]

$$\mathbf{y} \sim N(\mathbf{a} + B\mathbf{h}(\xi), \Sigma), \quad \text{where } \mathbf{h}(\xi) = \begin{pmatrix} \xi \\ \xi^2 \end{pmatrix}, \tag{4.10}$$

where $\mathbf{a}$ is a 2×1 intercept vector, B is an unknown 2×2 matrix and Σ is an unknown 2×2 positive definite matrix. The analysis in Oman and Wax (1984) and Oman (1988) is based on the (F, BPD) measurements for 1114 women whose gestational age ξ was precisely known for values of ξ satisfying $14 \le \xi \le 41$. The problem is that of constructing a confidence region for the unknown gestational age corresponding to various (F, BPD) measurements.

Let Y denote the 2×1114 matrix of $(F, BPD)'$ values for the 1114 women and let $\mathbf{y}$ denote the $(F, BPD)'$ value for a woman whose gestational age ξ is unknown. Then $\mathbf{y}$ and the columns of Y are independent multivariate normal random vectors with

$$\begin{aligned} \mathrm{E}(Y) &= \mathbf{a}\mathbf{1}'_{1114} + BX, \qquad &\mathrm{Cov}(\mathrm{Vec}(Y)) &= I_{1114} \otimes \Sigma, \\ \mathrm{E}(\mathbf{y}) &= \mathbf{a} + B\mathbf{h}(\xi), &\mathrm{Cov}(\mathbf{y}) &= \Sigma, \end{aligned} \tag{4.11}$$

where the ith column of X is $(\xi_i, \xi_i^2)'$, ξ_i being the known gestational age for the ith woman. We shall first obtain the canonical form (2.24). For this, we shall use the computations in Remark 2.2 in Mathew and Kasala (1994) and proceed as in our previous example. Let $n = 1114$ and let Z be an $n \times n - 1$ matrix such that $((1/\sqrt{n})\mathbf{1}_n : Z)$ is an orthogonal matrix. Define

$$\begin{aligned} Y_0 &= YZ, \qquad X_0 = XZ, \qquad \mathbf{y}_0 = \left(1 + \frac{1}{n}\right)^{-1/2}\left(\mathbf{y} - \frac{1}{n}Y\mathbf{1}_n\right), \\ \mathbf{h}_0(\xi) &= \left(1 + \frac{1}{n}\right)^{-1/2}\left(\mathbf{h}(\xi) - \frac{1}{n}X\mathbf{1}_n\right). \end{aligned} \tag{4.12}$$

We note that Y_0 is a $2 \times n - 1$ matrix. Writing $N = n - 1$, we note that the quantities N, m, p and r occurring in (2.27) have values $N = 1113$, $m = 2$, $p = 2$ and $r = 1$. Similar to (2.1), now define

$$\begin{aligned} Y_1 &= Y_0 X_0'(X_0 X_0')^{-1/2}, \qquad &S &= Y_0\big(I - X_0'(X_0X_0')^{-1}X_0\big)Y_0', \\ B_1 &= B(X_0X_0')^{1/2}, &\mathbf{h}_1(\xi) &= (X_0X_0')^{-1/2}\mathbf{h}_0(\xi). \end{aligned} \tag{4.13}$$

Then

$$Y_1 \sim N(B_1, I_2 \otimes \Sigma), \qquad \mathbf{y}_0 \sim N(B_1\mathbf{h}_1(\xi), \Sigma), \qquad S \sim W_2(\Sigma, 1111). \tag{4.14}$$

Y_1 and S are essentially given in Oman [(1988), page 182]. $H(\xi)$ in (2.25) is given by

$$H(\xi) = Y_1(X_0X_0')^{-1/2}\begin{pmatrix} 1 \\ 2\xi \end{pmatrix}. \tag{4.15}$$

Let

$$d = \mathbf{h}_1(\xi)'\mathbf{h}_1(\xi). \tag{4.16}$$

Note that in order to compute d, we need the value of the matrix $X_0 X_0'$ and the vector $(1/n)X\mathbf{1}_n$ ($n = 1114$); see the definition of $\mathbf{h}_1(\xi)$ in (4.13) and $\mathbf{h}_0(\xi)$ in (4.12). These can be computed using the data in Oman and Wax (1984) and are given by

$$X_0 X_0' = \begin{pmatrix} 52{,}877.52 & 2{,}878{,}329 \\ 2{,}878{,}329 & 15{,}914{,}978 \end{pmatrix} \quad \text{and} \quad \frac{1}{n}X\mathbf{1}_n = \begin{pmatrix} 28.410233 \\ 854.607720 \end{pmatrix}. \tag{4.17}$$

In view of (2.27), in order to implement the confidence region (2.28) with $\alpha = 0.05$, we need to compute $c(d)$ satisfying

$$\int_0^\infty P[F_{1,1110}(d \times v) \le c(d)] f_2(v)\, dv = 0.95. \tag{4.18}$$

In order to evaluate $c(d)$ for various values of d, an infinite series representation similar to (3.1) was used with the summation truncated at eight terms. For a few values of ξ in the interval [14, 41], Table 4 gives the values of d and $c(d)$, where d is obtained using (4.16) and $c(d)$ satisfying (4.18) was numerically obtained.

A plot of the values in Table 4 showed that $c(d)$ is linear in d; see Zha (1995) for details. The line is given by

$$c_*(d) = 3.849868 + 7.688439d, \tag{4.19}$$

where $c_*(d)$ denotes values on the line. Using the definition of d in (4.16), we see that the confidence region for ξ is given by

$$\{\xi : U(\xi) \le 3.849868 + 7.688439\mathbf{h}_1(\xi)'\mathbf{h}_1(\xi)\}. \tag{4.20}$$

For a few arbitrarily chosen values of $\mathbf{y} = (F, BPD)'$, Table 5 gives the confidence regions for ξ. Comparing the results in Table 5 with Figure 2 in Oman (1988), we see that our confidence intervals have essentially the same length as those obtained by Oman. We noted the same for several (F, BPD)

TABLE 4

Values of d satisfying (4.16) and c(d) satisfying (4.18) for different values of ξ

ξ	d	$c(d)$
14	0.010332	3.92943
18	0.002574	3.86975
22	0.000967	3.85733
26	0.001020	3.85771
30	0.000733	3.85547
34	0.000594	3.85439
38	0.003576	3.87734
41	0.011055	3.93481

TABLE 5
The confidence region (4.20) *for* ξ *in the model* (4.11) *using Oman and Wax* (1984) *data, for a few values of* $\mathbf{y}$

$\mathbf{y}' = (\boldsymbol{F}, \boldsymbol{BPD})$	**(14, 27)**	**(32, 47)**	**(45, 62)**	**(56, 75)**	**(65, 85)**
Confidence region	(13.03, 15.95)	(18.52, 21.89)	(23.11, 27.03)	(27.64, 32.30)	(31.88, 37.64)

values from the range of values in Figure 2 in Oman (1988) and hence have reported only a few values in Table 5. We would also like to point out that the (F, BPD) values given in Table 5 are all very close to the average of several (F, BPD) values given in Table 1 in Oman and Wax (1984). The true gestational ages corresponding to these averages are 14, 20, 25, 30 and 35 weeks. It is interesting to note that these gestational age values belong to the respective confidence intervals in Table 5.

Table 6 gives the simulated coverage probability of the region (4.20). We have used the canonical form (4.14) for the simulation with $\Sigma = I_2$, $B_1 = \eta I_2$ ($\eta = 0.001$, 0.01, 1, 10 and 100) and for a few values of ξ in the interval $[14, 41]$.

The simulated coverage probabilities in Table 6 turn out to be very close to the 95% confidence level. Thus, for the two examples presented in this section, the approach described in this article turns out to be quite satisfactory for obtaining confidence regions.

5. Concluding remarks. In calibration problems where the models (1.1) and (1.2) or the models (1.1) and (1.3) are applicable, we have derived some conservative confidence regions for the parameter θ in (1.2) or the parameter ξ in (1.3). Our regions have many desirable features. They are applicable to finite samples and they possess a natural invariance property. Furthermore,

TABLE 6
Simulated coverage probabilities of the region (4.20), *based on* 100,000 *simulations*, *for the model* (4.14), *for* $\Sigma = I_2$, $B_1 = \eta I_2$ *for different values of* η *and* ξ

	$\boldsymbol{\eta}$				
$\boldsymbol{\xi}$	**0.001**	**0.01**	**1**	**10**	**100**
14	0.950	0.950	0.952	0.953	0.954
19	0.950	0.950	0.950	0.950	0.051
23	0.950	0.950	0.951	0.952	0.952
28	0.950	0.950	0.950	0.950	0.951
31	0.950	0.950	0.950	0.950	0.950
33	0.950	0.950	0.951	0.951	0.951
38	0.951	0.951	0.950	0.950	0.950
41	0.950	0.950	0.951	0.951	0.952

the pivotal quantity that we have used to construct the confidence regions is a natural choice [this should be clear from (2.3)]. In general, it is not possible to get an exact confidence region based on the pivot that we have used since its distribution depends on the unknown parameter matrix B in (1.1). It is our attempt to take care of this difficulty that resulted in the conservatism of our region. It must be pointed out that the practical implementation of our confidence region requires a certain amount of computational effort, since the functional form of $c_\alpha^{(i)}(\theta_1'\theta_1)$ in (2.22) has to be numerically obtained by computing and plotting its values for various choices of $\theta_1'\theta_1$. [The same observation also applies to the computation of $c_\alpha^{(i)}(\mathbf{h}_1(\xi)'\mathbf{h}_1(\xi))$ in (2.28).] The numerical computations can be drastically reduced by using an upper bound δ on $\theta_1'\theta_1$ and computing the confidence region using $c_\alpha^{(i)}(\delta)$, at the cost of increasing the volume of the region. The classical region due to Brown (1982) and the region derived in Mathew and Kasala (1994) are computationally simple and are applicable to finite samples. The drawbacks of these regions have already been pointed out in this article. The likelihood-based region due to Brown and Sundberg (1987), though it requires numerical evaluation of the maximum likelihood estimator, is simpler to compute compared to our regions in Section 2. However, the relevant statistic in the likelihood-based confidence region has a distribution that depends on the nuisance parameters, and the asymptotic chi-square approximation derived in Brown and Sundberg (1987) is valid only under a condition that is parameter dependent. Before recommending the likelihood-based confidence region in practice, it may be necessary to simulate its coverage probability and be convinced that the coverage probability is close to the assumed confidence level. We already saw that this need not be the case; see the discussion following Table 2. On the other hand, our confidence regions are applicable without any further conditions, as long as the assumed models are valid.

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DEPARTMENT OF MATHEMATICS
AND STATISTICS
UNIVERSITY OF MARYLAND
BALTIMORE COUNTY CAMPUS
BALTIMORE, MARYLAND 21228