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EFFICIENT ESTIMATION OF A DENSITY IN A PROBLEM OF TOMOGRAPHY

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The aim of tomography is to reconstruct a multidimensional function from observations of its integrals over hyperplanes. We consider the model that corresponds to the case of positron emission tomography. We have n i.i.d. observations from a probability density proportional to Rf, where Rf stands for the Radon transform of the density f. We assume that f is an N-dimensional density such that its Fourier transform is exponentially decreasing. We find an estimator of f which is asymptotically efficient; it achieves the optimal rate of convergence and also the best constant for the minimax risk.

1. Introduction. Tomography is a particular inverse problem where the aim is to invert the Radon operator which maps a function into its integrals over hyperplanes. This problem appears in different fields such as medical image processing, nuclear medicine and radar theory. One of the first examples where tomography appeared is the case of radiology by use of X-rays where one wants to reconstruct the internal structure of a body. More details about tomography may be found in Natterer (1986).

In this paper, we consider the problem of positron emission tomography in a statistical setup. We suppose that we have n independent and identically distributed (i.i.d.) observations from a probability density proportional to Rf, where Rf stands for the Radon transform of the density f. The problem is to estimate f. Positron emission tomography is discussed in detail in Johnstone and Silverman (1990). Other statistical results for the problems of X-ray tomography and positron emission tomography are given in Korostelev and Tsybakov (1989, 1991, 1993), Donoho and Low (1992) and Donoho (1995). These papers propose estimators which attain the optimal rates of convergence, in minimax sense, on the Sobolev and Besov classes of functions. Beyond the rate of convergence, it is interesting to consider the problem of finding exact minimax constants and efficient estimators attaining these constants.

The problem of exact constants in a nonparametric minimax framework is a difficult problem which has been studied by Pinsker (1980), Nussbaum (1985) and Korostelev (1993) for the Gaussian white noise model and nonparametric regression. They define estimators which are asymptotically efficient, that is, attain the exact constants in the asymptotics of the minimax risk. Such a problem has been studied in the inverse problem of convolution in Ermakov (1989) and Efromovich (1997).

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The mathematical ideas here are close to Golubev and Levit (1996), Golubev, Levit and Tsybakov (1996) and Guerre and Tsybakov (1998) which define efficient estimators for the problem of estimating a univariate probability density and its derivatives or nonparametric regression on the class of analytic functions even in the case of pointwise estimation.

In this paper [as in Cavalier (1998) for the model of X-ray tomography], we obtain results on exact constants in an N-dimensional inverse problem. We consider a model of positron emission tomography and we study the class of densities f admitting an exponentially decreasing Fourier transform. In this class we define an estimator which is asymptotically efficient: it attains the optimal rate and constant in asymptotics of the minimax risk.

Except for the definition of the class of functions, the N-dimensional setting is not so different from the one-dimensional case. On the other hand, the inverse problem of tomography is really different from models such as probability density, nonparametric regression or Gaussian white noise. The indirect observations and the form of the Radon transform result in some difficulties, especially in getting the lower bound on the minimax risks.

In Section 2, we define precisely the model and the class of densities. Then we construct our estimator. Theorem 1 gives the rate of convergence and the constant for the mean-squared error of the estimator and proves asymptotical normality. In Theorem 2 we slightly modify the class of densities in order to obtain a uniform result. Theorem 3 proves that our estimator is asymptotically efficient. In Section 3 we give the proofs.

2. Problem statement and results. Let $L : \mathbb{R}^N \to \mathbb{R}$, be a probability density, $N \geq 2$. Define the Radon transform Rf of f as

$$Rf(s, u) = \int_{v:(v,\, s)=u} f(v)\, dv,$$

where $u \in \mathbb{R}$, $s \in S^{N-1}$, $S^{N-1} = \{v \in \mathbb{R}^N, |v| = 1\}$ is the unit sphere in $\mathbb{R}^N$, $|v|$ is the Euclidean norm of v, and $(\cdot, \cdot)$ is the scalar product in $\mathbb{R}^N$. The function $Rf(s, u)$ is defined on the cylinder $\mathscr{Z} = S^{N-1} \times \mathbb{R}$. The set of points $v \in \mathbb{R}^N$ such that $(v, s) = u$ is a hyperplane in $\mathbb{R}^N$, characterized by (s, u). The Radon transform is a suitable tool for the problem of tomography, because $Rf(s, u)$ represents the integral of f over the hyperplane $\{v \in \mathbb{R}^N\colon (v, s) = u\}$. For general properties of the Radon transform, we refer to Natterer (1986).

Now define the class of functions $\mathscr{A}_\gamma^N(L)$, where γ and L are positive constants. We say that a function f on $\mathbb{R}^N$ belongs to $\mathscr{A}_\gamma^N(L)$ if and only if, f is a density, $f(v) > 0, \forall v \in \mathbb{R}^N$, f is continuous, and

$$\left(\frac{1}{2\pi}\right)^N \int_{\mathbb{R}^N} |\hat{f}(\omega)|^2 \exp(2\gamma|\omega|)\, d\omega \leq L, \tag{1}$$

where $\hat{f}$ is the Fourier transform of the function f,

$$\hat{f}(\omega) = \int_{\mathbb{R}^N} f(v) e^{i(v,\, \omega)}\, dv.$$

The class of functions $\mathscr{A}_\gamma^N(L)$ is a multivariate generalization of the class of real functions admitting a bounded analytical continuation into a certain strip of the complex plane (the class of "analytic" functions). For more details on this type of function in the case $N = 1$, see Achieser [(1967), page 251] and Timan [(1963), page 137] Such a class of functions in the case $N = 1$, was first considered in the context of statistical problems by Ibragimov and Hasminskii (1983, 1984). Then Golubev and Levit (1996), Golubev, Levit and Tsybakov (1996) and Guerre and Tsybakov (1998) proved that it is possible to obtain exact constants on this class of functions for different problems of estimation.

Consider now the model, which corresponds to the problem of positron emission tomography. We have that Rf (up to a certain constant) is a probability density on the cylinder $\mathscr{Z} = S^{N-1} \times \mathbb{R}$. We have an unobservable probability density f on $\mathbb{R}^N$, and we want to reconstruct it from i.i.d. observations from a density proportional to Rf. More details on this type of models can be obtained in Johnstone and Silverman (1990).

Let us give the precise definitions. Let f be a probability density on $\mathbb{R}^N$. Suppose that (S_i, U_i) are n i.i.d. observations from a probability density $(1/\rho_N)Rf(s,u)$ on $\mathscr{Z} = S^{N-1} \times \mathbb{R}$, where ρ_N is the surface area of the sphere S^{N-1}. Remark that $(1/\rho_N)Rf(s,u)$ is a density since $\int_{\mathbb{R}} Rf(s,u)\,du = \int_{\mathbb{R}^N} f(v)\,dv = 1$.

We want to construct an estimator $f_n^*(x)$ of $f(x)$ at a fixed point $x \in \mathbb{R}^N$, using the observations (S_i, U_i), $i = 1, \dots, n$.

The following lemma, which is an essential tool in tomography, can be found in Natterer [(1986), page 11] and Korostelev and Tsybakov [(1993), page 238].

LEMMA 1. *Projection theorem. Let $f \in L_1(\mathbb{R}^N) \cap L_2(\mathbb{R}^N)$, then*

$$\widehat{Rf}(s,t) = \hat{f}(ts),\ t \in \mathbb{R}, \qquad s \in S^{N-1},$$

where $\widehat{Rf}(s,t)$ is the Fourier transform of $Rf(s,u)$ over the argument u only.

Define the function

$$K_{\delta_n}(u) = (2\pi)^{-N} \rho_N \int_0^{1/\delta_n} r^{N-1} \cos(ur)\,dr, \qquad u \in \mathbb{R}, \tag{2}$$

where $1/\delta_n = 1/2\gamma \log n$. Note that its Fourier transform is

$$\hat{K}_{\delta_n}(t) = (2\pi)^{1-N} \frac{\rho_N}{2} |t|^{N-1} I_{\delta_n}(t), \qquad t \in \mathbb{R}, \tag{3}$$

where $I_{\delta_n}(t)$ is the indicator function of the set $\{t : |t| \le 1/\delta_n\}$.

The function K_{δ_n} is used as a kernel of the estimator. One may remark that K_{δ_n} does not satisfy all the standard properties of kernels. For example, conditions of the Bochner lemma [see Parzen (1962)] are not satisfied. In fact, this function K_{δ_n} is called a band-limited filter. This filter has already been used in the context of tomography by Natterer (1986) and Korostelev and Tsybakov (1991, 1993). It has a compactly supported Fourier transform. Using Lemma 1 one may understand that the factor $|t|^{N-1}$ has been chosen in order to give K_{δ_n} the property to approximately invert the Radon transform.

Another interesting remark concerns the bandwidth δ_n. It is very large compared to usual bandwidths which are polynomially decreasing as $n \to \infty$. The reason is that the class $\mathscr{A}_\gamma^N(L)$ contains very smooth functions and hence the bandwidth should be chosen larger than usual.

Define the estimator

$$f_n^*(x) = \frac{1}{n}\sum_{i=1}^{n} K_{\delta_n}((S_i, x) - U_i), \qquad x \in \mathbb{R}^N.$$

Let $R^\#$ be the dual operator of R as defined in Natterer [(1986), page 14] for any $h \in L_1(\mathscr{Z})$,

$$R^\# h(v) = \int_{S^{N-1}} h(s, (s, v))\, ds, \qquad v \in \mathbb{R}^N,$$

then,

$$R^\# Rf(x) = \int_{S^{N-1}} Rf(s, (s, x))\, ds.$$

In words, $R^\# Rf(x)$ represents the integrals of f over all the hyperplanes passing through point x. Observe that as $f \in \mathscr{A}_\gamma^N(L)$, $R^\# Rf(x) < \infty$ (see the proof of Theorem 1). Note that $R^\# Rf(x) > 0; \forall x, \forall f \in \mathscr{A}_\gamma^N(L)$ since $f > 0$ everywhere on $\mathbb{R}^N$. The following theorem gives the rate of convergence and the asymptotic constant for the estimator f_n^* of the function f.

THEOREM 1. *For any fixed $x \in \mathbb{R}^N$ and any $f \in \mathscr{A}_\gamma^N(L)$ we have, as $n \to \infty$,*

$$E_f[(f_n^*(x) - f(x))^2] = \frac{(2\pi)^{-2N}\pi\rho_N}{2N-1}\left(\frac{1}{2\gamma}\right)^{2N-1} R^\# Rf(x) \times \frac{(\log n)^{2N-1}}{n}(1 + o(1)). \tag{4}$$

Furthermore, $(f_n^*(x) - f(x))$ is asymptotically normal,

$$\left(C^* R^\# Rf(x)\frac{(\log n)^{2N-1}}{n}\right)^{-1/2}(f_n^*(x) - f(x)) \xrightarrow{\mathscr{D}} \mathscr{N}(0, 1), \qquad n \to \infty. \tag{5}$$

where $C^* = ((2\pi)^{-2N}\pi\rho_N)/(2N-1)(1/2\gamma)^{2N-1}$, and $\to^{\mathscr{D}}$ denotes the convergence in distribution.

REMARK 1. This result proves that $f_n^*(x)$ converges to $f(x)$ with the "almost parametric" rate $\left((\log n)^{2N-1}/n\right)^{1/2}$ and with the given asymptotic constant. In the case of direct observations, Golubev and Levit (1996) obtained a rate of convergence of the form $\log n/n$. Thus, the smooth nature of the functions confines the loss due to indirect observations to powers of the logarithm.

REMARK 2. Due to the definition of $\mathscr{A}_\gamma^N(L)$ we obtain the exact constant of the estimator and not only the rate of convergence. Indeed, the main property here is that the bias term will be of smaller order than the stochastic term and then will vanish in the asymptotic of the risk (see proof of Theorem 1). As a consequence we may remark that the estimator $f_n^*(x)$ does not depend on the constant L of the class $\mathscr{A}_\gamma^N(L)$ since this constant appears only in the bias term. Thus, we may choose L arbitrary large to enlarge the class of functions.

Now consider uniform result. In Theorem 1, the function f and then $R^\# Rf(x)$ are all fixed. In the case of uniform result over the class $\mathscr{A}_\gamma^N(L)$, one may construct in $\mathscr{A}_\gamma^N(L)$ a sequence of functions $\{f_i\}$ such that $R^\# Rf_i(x)$ tends to zero too fast as $i \to \infty$. Thus, it is possible to obtain a uniform result only if we slightly modify the definition of the class to avoid the case of densities f for which $R^\# Rf(x)$ is small. Define

$$\mathscr{A}_\gamma^N(L, \alpha_n) = \{f \in \mathscr{A}_\gamma^N(L)\colon R^\# Rf(x) \geq \alpha_n\}, \tag{6}$$

where $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty}(\alpha_n(\log n)^{1/3}) = \infty$. Define also

$$\psi_n(f, x) = \left(\frac{(2\pi)^{-N}\pi\rho_N}{2N-1}\left(\frac{1}{2\gamma}\right)^{2N-1} R^\# Rf(x)\frac{(\log n)^{2N-1}}{n}\right)^{1/2}. \tag{7}$$

Thus, with this new definition of the class of functions, we have the following result.

THEOREM 2. *For any fixed $x \in \mathbb{R}^N$, we have*

$$\lim_{n\to\infty} \sup_{f\in\mathscr{A}_\gamma^N(L,\alpha_n)} \mathrm{E}_f\left[\left(\frac{f_n^*(x) - f(x)}{\psi_n(f, x)}\right)^2\right] = 1.$$

REMARK 3. This result proves that $f_n^*(x)$ converges uniformly to $f(x)$ with the normalization $\psi_n(f, x)$. The term conditions on α_n in (6) ensure that when the sup is taken over $\mathscr{A}_\gamma^N(L, \alpha_n)$ the bias of $f_n^*(x)$ will remain asymptotically negligible with respect to the stochastic term and that the normalized stochastic term will tend to 1 (see the proof of Theorem 2).

The remaining part of the study concerns what we call the lower bound. We prove that the normalization (7) is asymptotically exact, and thus f_n^* is an asymptotically efficient estimator. This is shown by the following theorem.

THEOREM 3. *For a fixed $x \in \mathbb{R}^N$, we have*

$$\liminf_{n\to\infty} \inf_{\bar f_n} \sup_{f\in\mathscr{A}_\gamma^N(L,\alpha_n)} \mathrm{E}_f\left[\left(\frac{\bar f_n(x) - f(x)}{\psi_n(f, x)}\right)^2\right] \geq 1,$$

where $\inf_{\bar f_n}$ denotes the infimum over all the estimators of $f(x)$.

Thus, we have constructed an estimator $f_n^*(x)$ of the density f for the problem of positron emission tomography, which is not only rate optimal but also asymptotically efficient.

One of the main interests of this study is that the method used could be generalized to other inverse problems, say the class of homogeneous operator K defined in Donoho (1995) and the deconvolution. In fact, the main point is to have some relation between the Fourier transform of Kf and of f as we have here in Lemma 1.

Another comment is about the use of the class of "analytical" functions $\mathscr{A}_\gamma^N(L)$. Since we have no restrictions on γ, functions in the class $\mathscr{A}_\gamma^N(L)$ may be in some sense as nonsmooth as we want. The main drawback is not the form of the class $\mathscr{A}_\gamma^N(L)$ but the fact that γ is supposed fixed. Thus a natural improvement of the study would be to construct adaptive estimators in the spirit of Lepski and Levit (1998) and possibly to obtain adaptive efficient estimators.

3. Proofs.

PROOF OF THEOREM 1. In nonparametric estimation, the usual way is to decompose the risk of a kernel estimator into a bias term and a stochastic term. Then, the bandwidth is chosen in order to balance these two terms and thus one obtains the optimal rate of convergence. Here the problem is different since the bias term and the variance cannot be balanced. This difference comes from the exponential term in the definition of $\mathscr{A}_\gamma^N(L)$ instead of usual polynomial term. Thus, we choose a large bandwidth and the bias will disappear compared to the stochastic term.

The risk of the estimator is decomposed in two parts, a bias and a stochastic term,

$$\begin{aligned}\mathrm{E}_f\big[(f_n^*(x)-f(x))^2\big] &= (\mathrm{E}_f(f_n^*(x))-f(x))^2\\ &\quad+\mathrm{E}_f\big[(f_n^*(x)-\mathrm{E}_f(f_n^*(x)))^2\big]=b_n^2(x)+\sigma_n^2(x).\end{aligned}\tag{8}$$

The bias $b_n(x)$ is equal to

$$b_n(x)=\mathrm{E}_f(f_n^*(x))-f(x)=\frac{1}{\rho_N}\int_{S^{N-1}}\int_{\mathbb{R}}K_{\delta_n}((s,x)-u)Rf(s,u)\,dv\,ds-f(x).$$

Due to the properties of the inverse Fourier transform and of the convolution,

$$\int_{\mathbb{R}}K_{\delta_n}((s,x)-u)Rf(s,u)\,du=(2\pi)^{-1}\int_{\mathbb{R}}\hat{K}_{\delta_n}(t)\widehat{Rf}(s,t)e^{-it(s,x)}\,dt,\tag{9}$$

where $\widehat{Rf}(s,t)$ is a Fourier transform of $Rf(s,u)$ over the argument u only. Using Lemma 1 (Projection theorem), (3) and (9) we get

$$\begin{aligned}(10)\qquad &\frac{1}{\rho_N}\int_{S^{N-1}}\int_{\mathbb{R}} K_{\delta_n}((s,x)-u)Rf(s,u)\,dv\,ds\\ &\quad=\int_{S^{N-1}}\frac{(2\pi)^{-N}}{2}\int_{\mathbb{R}}|t|^{N-1}I_{\delta_n}(t)\hat{f}(ts)e^{-it(s,x)}\,dt\,ds\\ &\quad=(2\pi)^{-N}\int_{\mathbb{R}^N} I_{\delta_n}(|\omega|)\hat{f}(\omega)e^{-i(\omega,x)}\,d\omega.\end{aligned}$$

Now we need the following technical lemma (given without proof) which concerns functions in the class $\mathscr{A}_\gamma^N(L)$.

LEMMA 2. *Let* $f\in\mathscr{A}_\gamma^N(L)$. *Then*

$$(11)\qquad |\hat{f}(\omega)|\le 1\qquad \forall\omega\in\mathbb{R}^N,$$

$$(12)\qquad \int_{\mathbb{R}^N}|\hat{f}(\omega)|\,d\omega\le Q,$$

for every $v\in\mathbb{R}^N$ *we have the inverse Fourier transform*

$$(13)\qquad f(v)=\left(\frac{1}{2\pi}\right)^N\int_{\mathbb{R}^N}\hat{f}(\omega)e^{-i(v,\omega)}\,d\omega$$

and

$$(14)\qquad \sup_{f\in\mathscr{A}_\gamma^N(L)} R^{\#}Rf(x)\le\frac{\rho_N+Q}{\pi},$$

where Q *is a positive constant which depends only on* γ, L, N.

Therefore, using (10) and (13) of Lemma 2 we obtain

$$\begin{aligned}b_n(x)&=(2\pi)^{-N}\int_{\mathbb{R}^N} I_{\delta_n}(|\omega|)\hat{f}(\omega)e^{-i(\omega,x)}\,d\omega-f(x)\\ &=(2\pi)^{-N}\int_{\mathbb{R}^N} I\left(|\omega|>\frac{1}{\delta_n}\right)\hat{f}(\omega)e^{-i(\omega,x)}\,d\omega.\end{aligned}$$

Thus, by the Cauchy–Schwarz inequality,

$$|b_n(x)|^2\le(2\pi)^{-2N}\int_{\mathbb{R}^N}|\hat{f}(\omega)|^2e^{2\gamma|\omega|}\,d\omega\cdot\int_{\mathbb{R}^N} I\left(|\omega|>\frac{1}{\delta_n}\right)e^{-2\gamma|\omega|}\,d\omega.$$

Since $f\in\mathscr{A}_\gamma^N(L)$, we have (1), and

$$\begin{aligned}\int_{\mathbb{R}^N} I\left(|\omega|>\frac{1}{\delta_n}\right)e^{-2\gamma|\omega|}\,d\omega&=\int_{S^{N-1}}\int_0^\infty t^{N-1}I\left(t>\frac{1}{\delta_n}\right)e^{-2\gamma t}\,dt\,ds\\ &\le c_1\left(\frac{1}{\delta_n}\right)^{N-1}e^{-2\gamma(1/\delta_n)}(1+o(1)),\end{aligned}$$

as $\varepsilon \to 0$, where c_1 is a positive constant. We obtain

$$|b_n(x)|^2 \le c_2 \frac{(\log n)^{N-1}}{n}(1+o(1)) \quad \text{as } n \to \infty, \tag{15}$$

where c_2 is a positive constant and $1/\delta_n = 1/2\gamma \log n$.

The variance term is

$$\begin{aligned}\sigma_n^2(x) &= \mathrm{E}_f\Bigg[\bigg(\frac{1}{n}\sum_{i=1}^{n} K_{\delta_n}((S_i, x) - U_i)\\ &\qquad - \frac{1}{\rho_N}\int_{S^{N-1}}\int_{\mathbb{R}} K_{\delta_n}((s,x)-u)Rf(s,u)\,du\,ds\bigg)^2\Bigg]\\ &= \frac{1}{n}\mathrm{Var}\, K_{\delta_n}((S,x)-U)\\ &= \frac{1}{n}\mathrm{E}_f[K_{\delta_n}^2((S,x)-U)]\\ &\quad - \frac{1}{n}\Big(\mathrm{E}_f[K_{\delta_n}((S,x)-U)]\Big)^2,\end{aligned} \tag{16}$$

where (S, U) is a random variable on $\mathscr{Z}$ with the density $(1/\rho_N)Rf(s,u)$. However,

$$\begin{aligned}(\mathrm{E}_f[K_{\delta_n}((S,x)-U)])^2 &= \bigg(\frac{1}{\rho_N}\int_{S^{N-1}}\int_{\mathbb{R}} K_{\delta_n}((s,x)-u)Rf(s,u)\,du\,ds\bigg)^2\\ &= (2\pi)^{-2N}\bigg(\int I_{\delta_n}(|\omega|)\hat{f}(\omega)e^{-i(\omega,x)}\,d\omega\bigg)^2,\end{aligned}$$

as shown in (10). Thus, as $n \to \infty$,

$$\begin{aligned}(\mathrm{E}[K_{\delta_n}((S,x)-U)])^2 &\le (2\pi)^{-2N}\int_{\mathbb{R}^N} I_{\delta_n}(|\omega|)e^{-2\gamma|\omega|}\,d\omega\\ &\qquad\times \int_{\mathbb{R}^N} |\hat{f}(\omega)|^2 e^{2\gamma|\omega|}\,d\omega = O(1),\end{aligned} \tag{17}$$

where $O(1)$ is uniform in $f \in \mathscr{A}_\gamma^N(L)$. Now study

$$\mathrm{E}[K_{\delta_n}^2((S,x)-U)] = \frac{1}{\rho_N}\int_{S^{N-1}}\int_{\mathbb{R}} K_{\delta_n}^2((s,x)-u)Rf(s,u)\,du\,ds. \tag{18}$$

Fix $s \in S^{N-1}$ and consider $R_s f(u) = Rf(s,u)$ as a function of u. Denote

$$G(u) = \frac{2N-1}{\pi}\bigg(\int_0^1 r^{N-1}\cos(ur)\,dr\bigg)^2.$$

By use of the Parseval equality and (3), we have $\int_{\mathbb{R}} G(u)\,du = 1$. We can remark, using integration by parts in the definition of $G(\cdot)$, that, as $u \to \infty$,

$$G(u) \le \frac{c_3}{u^2}, \tag{19}$$

where c_3 is a positive constant. Denote $G_\delta(u) = (1/\delta)G(u/\delta)$ for $\delta > 0$.

LEMMA 3. *For all $f \in \mathscr{A}_\gamma^N(L)$ we have*

$$\int_{S^{N-1}} |Rf(s,u) - Rf(s,y)|\,ds \le \frac{\rho_N + 2Q}{2\pi}|u-y| \quad a.e.\ u, y \in \mathbb{R},$$

where Q is the positive constant defined in Lemma 2, which depends only on γ, L and N.

By use of the inverse Fourier transform properties and of Lemma 1 we obtain the lemma. □

The following lemma is similar to the Bochner lemma [see Parzen (1962)].

LEMMA 4. *We have, as $\delta \to 0$,*

$$\int_{S^{N-1}} (G_\delta * R_s f)((s,x))\,ds = \int_{S^{N-1}} Rf(s,(s,x))\,ds(1+o(1)) + O(\delta^{1/3}),$$

where $o(1)$ and $O(\cdot)$ are uniform in $f \in \mathscr{A}_\gamma^N(L)$ and $*$ denotes the convolution.

PROOF. Since $\int_{\mathbb{R}} G(u)\,du = 1$, we have

$$\begin{aligned}
&\left|\int_{S^{N-1}}\int_{\mathbb{R}} G_\delta(u) Rf(s,(s,x)-u)\,du\,ds - \int_{S^{N-1}} Rf(s,(s,x))\,ds\right| \\
&\quad\le \int_{S^{N-1}}\int_{\mathbb{R}} G_\delta(u)\Big|Rf(s,(s,x)-u) - Rf(s,(s,x))\Big|\,du\,ds \\
&\quad\le \int_{S^{N-1}}\int_{|u|\le\Delta} G_\delta(u)\Big|Rf(s,(s,x)-u) - Rf(s,(s,x))\Big|\,du\,ds \qquad (20)\\
&\qquad + \int_{S^{N-1}}\int_{|u|>\Delta} \frac{Rf(s,(s,x)-u)}{|u|}\frac{|u|}{\delta}G\Big(\frac{u}{\delta}\Big)\,du\,ds \\
&\qquad + \int_{S^{N-1}}\int_{|u|>\Delta} Rf(s,(s,x))G_\delta(u)\,du\,ds,
\end{aligned}$$

for any $\Delta > 0$. From Lemma 3 we have

$$\begin{aligned}
&\int_{S^{N-1}}\int_{|u|\le\Delta} G_\delta(u)\Big|Rf(s,(s,x)-u) - Rf(s,(s,x))\Big|\,du\,ds \\
&\quad = \int_{|u|\le\Delta} G_\delta(u)\int_{S^{N-1}}\Big|Rf(s,(s,x)-u) - Rf(s,(s,x))\Big|\,ds\,du \\
&\quad \le \frac{\rho_N + 2Q}{2\pi}\int_{|u|\le\Delta} G_\delta(u)|u|\,du \le \frac{\rho_N+2Q}{2\pi}\Delta \to 0 \quad \text{as } \Delta \to 0.
\end{aligned}$$

Since $G(u) \le c_3 u^{-2}$, $u \to \infty$ and Rf is integrable we obtain

$$\begin{aligned}
&\int_{S^{N-1}}\int_{|u|>\Delta} \frac{Rf(s,(s,x)-u)}{|u|}\frac{|u|}{\delta}G\Big(\frac{u}{\delta}\Big)\,du\,ds \\
&\quad \le \frac{\rho_N}{\Delta}\sup_{|u|>\Delta/\delta}|uG(u)| \le \frac{c_3\rho_N\delta}{\Delta^2} \to 0 \quad \text{as } \delta \to 0.
\end{aligned}$$

Moreover since $G \in L_1(\mathbb{R}^N)$, using (14) of Lemma 2 we have as $\delta \to 0$,

$$\int_{S^{N-1}} \int_{|u|>\Delta} Rf(s,(s,x))G_\delta(u)\,du\,ds \le \int_{S^{N-1}} Rf(s,(s,x))\,ds \int_{|u|>\Delta/\delta} G(u)\,du \le \frac{\rho_N + Q}{\pi} \int_{|u|>\Delta/\delta} G(u)\,du \to 0.$$

Using these results with $\Delta = \delta^{1/3}$ and taking the limit in (20) we obtain the lemma. □

Remark that

$$\frac{1}{\rho_N} \int_{\mathbb{R}} K^2_{\delta_n}((s,x)-u)Rf(s,u)\,du = (2\pi)^{-2N} \rho_N \frac{\pi}{2N-1} \left(\frac{1}{\delta_n}\right)^{2N-1} (G_{\delta_n} * R_s f)((s,x)). \tag{21}$$

Together (18), (21) and Lemma 4 yield

$$\mathrm{E}\left[K^2_{\delta_n}((S,x)-U)\right] = (2\pi)^{-2N} \rho_N \frac{\pi}{2N-1} R^\# Rf(x) \left(\frac{1}{\delta_n}\right)^{2N-1} \times (1+o(1)) + O\left(\left(\frac{1}{\delta_n}\right)^{2N-4/3}\right),$$

as $\delta_n \to 0$. Thus, as $n \to \infty$, by use of (16) and (17), we have

$$\sigma_n^2(x) = C^* R^\# Rf(x) \frac{(\log n)^{2N-1}}{n}(1+o(1)) + O\left(\left(\frac{(\log n)^{2N-4/3}}{n}\right)\right), \tag{22}$$

where $o(1)$ and $O(\cdot)$ are uniform in $f \in \mathscr{A}_\gamma^N(L)$.
Finally, using (8), (15) and (22) we obtain (4).

Equation (5) concerning the asymptotic normality of the estimator $f_n^*(x)$ follows from (15), (22) and the Liapounov condition [see Loève (1977), page 287].

Let $X_i = n^{-1} K_{\delta_n}((S_i, x) - U_i)$. Then we have $s_n^2 = \sum_{i=1}^n \mathrm{Var}(\xi_i) = \sigma_n^2(x)$. Using (2), observe that

$$|X_i| \le \frac{\rho_N \delta_n^{-N}}{(2\pi)^N N n} = c_3 \frac{(\log n)^N}{n}, \tag{23}$$

where c_3 is a positive constant. Thus, the Liapounov ratio tends to zero for $\alpha > 0$,

$$s_n^{-1-(\alpha/2)} \sum_{i=1}^n \mathrm{E}|X_i|^{2+\alpha} \to 0 \quad \text{as } n \to \infty.$$

Therefore, from Loève (1977) we have that as $n \to \infty$,

$$s_n^{-1} \sum_{i=1}^n (X_i - \mathrm{E}X_i) \xrightarrow{\mathscr{D}} \mathscr{N}(0,1).$$

Note that

$$s_n^{-1}\sum_{i=1}^{n}(X_i - \mathrm{E}X_i) = \frac{f_n^*(x) - \mathrm{E}(f_n^*(x))}{\sigma_n(x)} = \frac{f_n^*(x) - f(x)}{\sigma_n(x)} - \frac{b_n(x)}{\sigma_n(x)}.$$

Using (15) and (22) we obtain the asymptotic normality of $f_n^*(x)$.

PROOF OF THEOREM 2. The proof of Theorem 2 is similar to that of Theorem 1. We only mention the point where there is some difference. Using (7), (8), (15) and (22) we find

$$\begin{aligned}
\mathrm{E}_f\left[\left(\frac{f_n^*(x) - f(x)}{\psi_n(f, x)}\right)^2\right] &= \frac{\sigma_n^2(x)}{\psi_n^2(f, x)} + \frac{b_n^2(x)}{\psi_n^2(f, x)} \\
&= (1 + o(1)) + O\left(\frac{1}{(\log n)^{1/3}}\right)\frac{1}{R^\# Rf(x)} \\
&\quad + O\left(\frac{1}{(\log n)^N}\right)\frac{1}{R^\# Rf(x)} \quad \text{as } n \to \infty,
\end{aligned}$$

where $o(1)$ and $O(\cdot)$ are uniform in $f \in \mathscr{A}_\gamma^N(L, \alpha_n)$. It remains to remark that the definition of α_n in (6) implies that

$$\sup_{f\in\mathscr{A}_\gamma^N(L,\alpha_n)} \frac{1}{(\log n)^{1/3} R^\# Rf(x)} \to 0 \quad \text{as } n \to \infty.$$

PROOF OF THEOREM 3. The idea of the proof is standard. We define a hardest parametric subfamily in $\mathscr{A}_\gamma^N(L, \alpha_n)$. Then, we use the Van Trees inequality to get the lower bound for this family. The main problem in the context of tomography is to get this parametric subfamily and to be sure that it belongs to $\mathscr{A}_\gamma^N(L, \alpha_n)$.

Let

$$H_a(u) = (2\pi)^{-N}\int_0^\infty \frac{r^{N-1}}{1 + a^{-1}\sinh^2 \gamma r}\cos(ur)\,dr, \qquad u \in \mathbb{R}, \tag{24}$$

where $a = a_n = n^\alpha$, with $0 < \alpha < 1$. Then, the Fourier transform

$$\hat{H}_a(t) = \frac{(2\pi)^{1-N}}{2}\frac{|t|^{N-1}}{1 + a^{-1}\sinh^2 \gamma t}, \qquad t \in \mathbb{R}. \tag{25}$$

Let

$$g_a(v) = \frac{(2\pi)^{1-2N}}{2}\int_{\mathbb{R}^N} \frac{|w|^{N-1}}{1 + a^{-1}\sinh^2 \gamma|w|}\cos((v, w))\,dw, \quad v \in \mathbb{R}^N. \tag{26}$$

Then

$$\hat{g}_a(w) = \frac{(2\pi)^{1-N}}{2}\frac{|w|^{N-1}}{1 + a^{-1}\sinh^2 \gamma|w|}, \qquad w \in \mathbb{R}^N. \tag{27}$$

Thus, if $h_a(v) = g_a(x - v)$, we have

$$Rh_a(s, u) = H_a((s, x) - u). \tag{28}$$

Equation (28) is obtained, for example, by use of Lemma 1, (25) and (27). In fact, H_a is close from K_{δ_n} with $1/\delta_n = \log a/2\gamma$.

Introduce the probability density

$$f_0(v) = \frac{\kappa}{(\beta^2 + |v - x|^2)^\nu} \qquad \forall v \in \mathbb{R}^N, \tag{29}$$

where $\nu = (N + 1)/2$, $\beta > 0$ is arbitrary and $\kappa > 0$ is a normalizing factor such that $\int_{\mathbb{R}^N} f_0(v)\, dv = 1$. Standard properties of the Bessel kernel [see, e.g., Adams and Hedberg (1996), page 11] yield

$$\hat{f}_0(w) = e^{-\beta|w|} e^{i(x, w)}, \qquad w \in \mathbb{R}^N. \tag{30}$$

Thus, for $\beta > \gamma$ large enough, $f_0 \in \mathscr{A}_\gamma^N(L/4)$.

Consider now the following family of functions:

$$f_c(v) = f_0(v) + ch_a(v),$$

where c is a real parameter. We know that f_0 belongs to $\mathscr{A}_\gamma^N(L/4)$, thus for $f_c \in \mathscr{A}_\gamma^N(L)$ we need

$$\begin{aligned} c^2 \int_{\mathbb{R}} |\hat{h}_a(\omega)|^2 e^{2\gamma|\omega|}\, dw &= c^2 \frac{(2\pi)^{2-2N}}{4} \rho_N \int_0^\infty \frac{t^{3N-3}}{(1 + a^{-1} \sinh^2 \gamma t)^2} e^{2\gamma t}\, dt \\ &\le \frac{(2\pi)^N L}{4}. \end{aligned} \tag{31}$$

We will need the following lemma which is close to the one in Golubev and Levit (1996).

LEMMA 5. *We have, for $m \ge 0$,*

$$\int_0^\infty \frac{t^m e^{2\gamma t}}{(1 + a^{-1} \sinh^2 \gamma t)^2} dt \le c_1' a (\log a)^m,$$

$$\int_0^\infty \frac{t^m}{1 + a^{-1} \sinh^2 \gamma t} dt \le c_2' (\log a)^m,$$

$$\int_0^\infty t^m \left| \frac{1}{(1 + a^{-1} \sinh^2 \gamma t)^2} - I\left(t \le \frac{\log a}{2\gamma}\right) \right| dt \le c_3' (\log a)^m,$$

$$\int_0^\infty t^m \left| \frac{1}{1 + a^{-1} \sinh^2 \gamma t} - I\left(t \le \frac{\log a}{2\gamma}\right) \right| dt \le c_4' (\log a)^m,$$

where c_1', c_2', c_3' and c_4' are positive constants.

By use of Lemma 5 and (31), we have that f_c satisfies the property (1) for all c such that

$$|c| \le C_a = \frac{\rho}{\sqrt{a}(\log a)^{(3N-3)/2}} \tag{32}$$

for some sufficiently small $\rho > 0$. Clearly f_c is continuous. Using (26), (32) and Lemma 5, observe that for any fixed $v \in \mathbb{R}^N$ we have

$$|ch_a(v)| \le C_a |h_a(v)| \to 0 \quad \text{as } a \to \infty. \tag{33}$$

Thus, since $f_0(v) > 0$ we have that as $a \to \infty$, $f_c(v) > 0$ when v belongs to any compact interval in $\mathbb{R}^N$. Furthermore, using results similar to (4) and (5) in Golubev and Levit (1996) based on formulas 3.981.1 and 3.983.1 in Gradshteyn and Ryzhik (1980), we can remark that $h_a(v)$ is exponentially decreasing as $|v| \to \infty$. By definition of f_0 in (29), we have that f_0 decreases as a polynomial as $|v| \to \infty$. Thus for a large enough, for all c such that $|c| \le C_a$, f_c is a strictly positive function on $\mathbb{R}^N$. Moreover, f_c is a density for all c such that $|c| \le C_a$ since

$$\int_{\mathbb{R}^N} f_c(v)\, dv = \int_{\mathbb{R}^N} f_0(v)\, dv + c \int_{\mathbb{R}^N} h_a(v)\, dv = 1 + c\hat{g}_a(0) = 1,$$

by use of (27). Thus, $f_c \in \mathscr{A}_\gamma^N(L)$, for all $|c| \le C_a$.

It follows from (28) that

$$Rf_c(s, u) = Rf_0(s, u) + cH_a((s, x) - u).$$

Using Lemma 1, (30) and the inverse Fourier transform, we obtain

$$\begin{aligned} Rf_0(s,u) &= \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} \widehat{Rf}_0(s,t) e^{-itu}\, dt = \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} \hat{f}_0(ts) e^{-itu}\, dt \\ &= \int_{\mathbb{R}} \exp(-\beta|t|) \exp(-it(u-(s,\ x)))\, dt = \frac{\beta}{\pi} \frac{1}{(u-(s,x))^2+\beta^2}. \end{aligned} \tag{34}$$

Thus,

$$Rf_c(s, u) = \frac{\beta}{\pi} \frac{1}{(u-(s,x))^2+\beta^2} + cH_a((s, x) - u). \tag{35}$$

Using (35) we have for all $|c| \le C_a$,

$$R_c(x) = R^{\#} Rf_c(x) = \int_{S^{N-1}} Rf_c(s, (s, x))\, ds = R_0 + c\rho_N H_a(0),$$

where $R_0 = \rho_N/(\pi\beta)$. Note that $R_c(x)$ does not depend on x. We denote therefore $R_c(x) = R_c$. We obtain by use of (24), Lemma 5 and definition of C_a in (32),

$$\sup_{|c| \le C_a} |c\rho_N H_\alpha(0)| = o(1) \quad \text{as } a \to \infty.$$

Thus, $R_c = R^{\#} Rf_c(x) = R_0(1 + o(1)) \ge \alpha_n$, for $n \to \infty$. We conclude that for a large enough, f_c belongs to $\mathscr{A}_\gamma^N(L, \alpha_n)$, $\forall |c| \le C_a$. Moreover, we have

$$\psi_n(f_c, x) = \psi_n(f_0, x)(1 + o(1)) \qquad \forall |c| \le C_a, \quad \text{as } n \to \infty. \tag{36}$$

Furthermore, $R_0 > 0$ and $R_c > 0$, and it is easy to see that

$$\sup_{|c|\le C_a}\left|\frac{1}{R_c}-\frac{1}{R_0}\right| = o(1) \quad \text{as } a\to\infty. \tag{37}$$

The lower bound for estimating $f(x)$ will be based on the Van Trees inequality [Van Trees (1968), page 72] and Gill and Levit (1995).

Let $\lambda_0(c)$ be a probability density on the interval $[-1, 1]$ with a finite Fisher information

$$I_0 = \int_{-1}^{1} \frac{(\lambda_0'(c))^2}{\lambda_0(c)}\,dc,$$

such that $\lambda_0(-1) = \lambda_0(1) = 0$ and $\lambda_0(c)$ is continuously differentiable for $|c| < 1$. The density $\lambda(c) = \lambda_a(c) = C_a^{-1}\lambda_0(C_a^{-1}c)$ defined on the interval $[-C_a, C_a]$, is a prior density with Fisher information $I(\lambda) = I_0 C_a^{-2}$. Finally, denote $I(c)$ the Fisher information of the family of densities $(1/\rho_N)Rf_c$ on $S^{N-1}\times\mathbb{R}$.

We have by use of Van Trees' inequality [see Van Trees (1968)],

$$\begin{aligned}\inf_{\bar f_n}\sup_{f\in\mathscr{A}_\gamma^N(L,\alpha_n)} \mathrm{E}_f\big[(\bar f_n(x)-f(x))^2\big] &\ge \inf_{\bar f_n}\sup_{|c|<C_a}\mathrm{E}_{f_c}\big[(\bar f_n(x)-f_c(x))^2\big]\\ &\ge \inf_{\bar f_n}\int_{-C_a}^{C_a}\mathrm{E}_{f_c}\big[(\bar f_n(x)-f_c(x))^2\big]\lambda(c)\,dc \ge \frac{(\int_{-C_a}^{C_a}((\partial f_c(x))/\partial c)\lambda(c)\,dc)^2}{n\int_{-C_a}^{C_a} I(c)\lambda(c)\,dc + I(\lambda)}.\end{aligned}$$

Thus here we obtain, using (36) and (37),

$$\begin{aligned}&\inf_{\bar f_n}\sup_{f\in\mathscr{A}_\gamma^N(L,\alpha_n)}\mathrm{E}_f\left[\left(\frac{\bar f_n(x)-f(x)}{\psi_n(f,x)}\right)^2\right]\\ &\qquad\ge \frac{(\int_{-C_a}^{C_a}((\partial f_c(x))/\partial c)\lambda(c)\,dc)^2}{n\int_{-C_a}^{C_a} I(c)\lambda(c)\,dc + I(\lambda)}(\psi_n(f_0,x)(1+o(1)))^{-2},\end{aligned} \tag{38}$$

where $o(1)$ tends to zero uniformly on $|c|\le C_a$ as $a\to\infty$. After some calculations based on (26) and Lemma 5, we obtain as $a\to\infty$,

$$\frac{\partial f_c(x)}{\partial c} = g_a(0) = C^*(\log a)^{2N-1}(1+o(1)). \tag{39}$$

LEMMA 6. *The Fisher information of the family of densities* $\{(1/\rho_N)Rf_c$ *is as* $a\to\infty\}$,

$$I(c) = \frac{C^*}{\rho_N^2}(\log a)^{2N-1}\int_{S^{N-1}}\frac{1}{Rf_c(s,(s,x))}\,ds(1+o(1)). \tag{40}$$

PROOF. The Fisher information of the family $\big((1/\rho_n)Rf_c\big)_{|c|\le C_a}$ is

$$I(c) = \int_{S^{N-1}}\int_{\mathbb{R}} \left(\frac{\partial \log(1/\rho_n)Rf_c(s,u)}{\partial c}\right)^2 \frac{1}{\rho_N} Rf_c(s,u)\,du\,ds$$

$$= \frac{1}{\rho_N}\int_{S^{N-1}}\int_{\mathbb{R}} \frac{H_a^2((s,x)-u)}{Rf_c(s,u)}\,du\,ds \tag{41}$$

$$= \frac{1}{\rho_N}\int_{S^{N-1}}\int_{\mathbb{R}} \frac{H_a^2(u)}{Rf_c(s,(s,x)-u)}\,du\,ds.$$

Denote for $t \in \mathbb{R}$,

$$D(t) = \frac{1}{Rf_c(s,(s,x)-t)} - \frac{1}{Rf_c(s,(s,x))} - \left(\frac{\partial}{\partial u}\frac{1}{Rf_c(s,u)}\right)_{u=(s,x)} t.$$

Thus, if $\big((\partial^2/\partial u^2)(1/Rf_c(s,u))\big)_{u=(s,x)-t}$ is bounded uniformly in t then $D(t)t^{-2}$ is bounded too. We have

$$\left|\left(\frac{\partial^2}{\partial u^2}\frac{1}{Rf_c(s,u)}\right)_{u=(s,x)-t}\right| \le \left|\frac{Rf_c{}''(s,(s,x)-t)}{Rf_c(s,(s,x)-t)^2}\right| + \left|\frac{Rf_c'(s,(s,x)-t)^2}{Rf_c(s,(s,x)-t)^3}\right|, \tag{42}$$

where Rf_c' and $Rf_c{}''$ denote the derivative with respect to the second argument only. Observe that (34) implies

$$Rf_0(s,(s,x)-t) = \frac{\beta}{\pi}\frac{1}{t^2+\beta^2}. \tag{43}$$

Note that if t belongs to a compact interval, we can directly bound $((\partial^2/\partial u^2)$ $(1/Rf_c(s,u)))_{u=(s,x)-t}$ since by use of (24), (32) and Lemma 5 we have for a fixed $t \in \mathbb{R}$,

$$\sup_{|c|\le C_a} |cH_a(t)| \to 0 \quad \text{as } a \to \infty,$$

and the same result for the derivatives of H_a. Thus, we can replace Rf_c by Rf_0 in (42) to obtain the bound.

For $|t| \to \infty$ we use the fact that H_a and its derivatives are exponentially decreasing. Thus, $Rf_c(s,(s,x)-t)$ (resp. Rf_c', $Rf_c{}''$) is of order $|t|^{-2}$ (resp. $|t|^{-3}$, $|t|^{-4}$).

Applying this, we obtain as $a \to \infty$, $\forall t \in \mathbb{R}$,

$$\left|\left(\frac{\partial^2}{\partial u^2}\frac{1}{Rf_c(s,u)}\right)_{u=(s,x)-t}\right| \le c_4 \qquad \forall |c| \le C_a,$$

where c_4 is a positive constant. Hence $D(t)t^{-2}$ is bounded and therefore using the Taylor formula we obtain since H_a^2 is symmetric,

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \frac{H_a^2(u)}{Rf_c(s,(s,x)-u)}\,du - \frac{1}{Rf_c(s,(s,x))}\int_{\mathbb{R}} H_a^2(u)\,du\right| \\
&\quad = \left|\int_{\mathbb{R}} D(u)u^{-2}u^2 H_a^2(u)\,du\right| \qquad (44)\\
&\quad \le c_5 \int_{\mathbb{R}} u^2 H_a^2(u)\,du = O\left(\left(\int_{\mathbb{R}} \hat{H}_a'(t)\,dt\right)^2\right),
\end{aligned}$$

where $\hat{H}_a'$ is the weak derivative of $\hat{H}_a$ and c_5 is a positive constant. After calculation of $\hat{H}_a'$ and the use of Lemma 5 we obtain

$$\int_{\mathbb{R}} u^2 H_a^2(u)du = O\big((\log a)^{2N-2}\big) \quad \text{as } a \to \infty. \qquad (45)$$

Finally, using (45) and (44) we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \frac{H_a^2(u)}{Rf_c(s,(s,x)-u)}\,du &= \frac{1}{Rf_c(s,(s,x))}\int_{\mathbb{R}} H_a^2(u)\,du \qquad (46)\\
&\quad + O\big((\log a)^{2N-2}\big) \quad \text{as } a \to \infty.
\end{aligned}$$

From Plancherel's equality, (25) and Lemma 5 we obtain as $a \to \infty$,

$$\begin{aligned}
\int_{\mathbb{R}} H_a^2(u)\,du &= \frac{(2\pi)^{1-2N}}{4}\int_{\mathbb{R}} \frac{|t|^{2N-2}}{(1+a^{-1}\sinh^2 \gamma t)^2}\,dt \\
&= \frac{(2\pi)^{-2N}\pi}{2N-1}\left(\frac{1}{2\gamma}\right)^{2N-1}(\log a)^{2N-1}(1+o(1)),
\end{aligned}$$

and then using (46) and (41) we obtain the lemma. □

By definition of Rf_c in (35) and (37), observe that $Rf_c(s,(s,x))$ does not depend on $s \in S^{N-1}$ and

$$\int_{S^{N-1}} \frac{1}{Rf_c(s,(s,x))}\,ds = \frac{\rho_N^2}{R_0}(1+o(1)) \quad \text{as } a \to \infty.$$

Thus, using definitions of a, $I(\lambda)$, C^*, (32), (38), (39) and Lemma 6, we obtain

$$\begin{aligned}
&\inf_{\bar{f}_n} \sup_{f \in \mathscr{A}_\gamma^N(L,\alpha_n)} \mathrm{E}_f\left[\left(\frac{\bar{f}_n(x)-f(x)}{\psi_n(f,x)}\right)^2\right] \\
&\quad \ge \frac{\big(C^*\alpha^{2N-1}(\log n)^{2N-1}(1+o(1))\big)^2 \psi_n^{-2}(f_0,x)(1+o(1))}{\big(C^*\alpha^{2N-1}(1/R_0)n(\log n)^{2N-1}(1+o(1))\big) + \rho^{-2}I_0 n^{\alpha}(\log n)^{3N-3}},
\end{aligned}$$

as $n \to \infty$, which implies the result since α can be chosen arbitrarily close to 1.

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