OPTIMAL DESIGN WITH MANY BLOCKING FACTORS

By J. P. Morgan[1] and R. A. Bailey

Old Dominion University and Queen Mary and Westfield College

Designs for sets of experimental units with many blocking factors are studied. It is shown that if the set of blocking factors satisfies a certain simple condition then the information matrix for the design has a simple form. In consequence, a design is optimal if it is optimal with respect to one particular blocking factor and regular with respect to all the rest, in a sense which is made precise in the paper. This encompasses several previous results for optimal designs with more than one blocking factor, and applications to many other situations are given.

1. Introduction. This paper deals with the optimality problem in assigning a set of treatments to experimental units that are subject to a multiplicity of blocking factors. Over the past twenty years the area of optimal experimental design has grown rapidly, and during this time considerable effort has been devoted to developing the theory for a single blocking factor. More complicated blocking structures, while commonly encountered in practice, have received comparatively little attention in the literature and consequently have seen less progress.

In agricultural and related sciences, experimental units are frequently classified by three or more blocking factors, which are interrelated by both crossing and nesting. Variety trials with small amounts of seed and small plots are laid out in blocks with nested rows and columns [Patterson and Robinson (1989)]. In sugar-beet trials the route taken by the tractor at seeding can make distant plots similar: this may result in either a row–column structure with nested plots [Bailey (1992)] or a structure where columns are crossed with directions, in which short rows are nested [Seeger (1986)]. The former is also often used for glasshouse experiments [Darby and Gilbert (1958)], while the latter, after renaming directions as long rows, is appropriate for experiments on irrigated cotton [Williams (1986)]. Kachlicka and Mejza (1995) report a trial on irrigated potatoes in which the plots of a nested row–column structure are themselves split into row–column substructures. German agronomists deal with spatial heterogeneity by using three blocking factors which are mutually orthogonal but not fully crossed [Behrens (1956)]. All of these structures, and more, occur in forestry experiments [Williams and Matheson (1994)].

As Bailey (1993) remarked, there appears to be no coherent theory of optimal design for these complicated blocking structures. The structures are by no means new or esoteric; see Yates's (1935, 1937) discussion of factorial design

and Nelder's (1965a) list of block structures. Given the world-wide use of these structures in experiments, guidance on constructing optimal designs for them is badly needed.

There are a few instances where it has been possible to show optimality of a design for many block factors by reducing the problem to that of the optimality of the block design formed by just one of those factors. Cheng (1978) did this for multiway crosses, as we describe in Section 3.3. Bagchi, Mukhopadhyay and Sinha (1990) did it for nested row–column designs; we describe their result in this section and in Section 3.2. Our purpose is not only to bring these previous results under a common umbrella but also to expand the list of blocking structures for which this is possible.

To get an idea of where we are headed, consider the nested row–column setting consisting of b separate $p \times q$ cross classifications, that is, a nesting of rows and columns in a third nuisance factor called blocks. The standard model for the yield on the plot in row l, column m of block j is

$$Y_{djlm} = \mu + \rho_l + \gamma_m + \beta_j + \tau_{d[jlm]} + \varepsilon_{jlm},$$

ρ_l, γ_m and β_j being row, column and block effects, $\tau_{d[jlm]}$ the effect of the treatment applied by design d to plot (j, l, m), and the ε_{jlm}'s being uncorrelated random variables with mean zero and common variance. In matrix form this is

$$Y = \mu 1 + A_d \tau + Z_1 \gamma + Z_2 \rho + Z_3 \beta + \varepsilon.$$

The information matrix for estimation of τ, also called the C-matrix, is

$$C_d = A_d'\left(I - \frac{1}{p}Z_1 Z_1' - \frac{1}{q}Z_2 Z_2' + \frac{1}{pq}Z_3 Z_3'\right)A_d$$

and using $\geq$ in the sense of nonnegative definite, certainly $C_d \geq A_d'(I - (1/p)Z_1 Z_1')A_d$. Now $A_d'(I - (1/p)Z_1 Z_1')A_d$ is the information matrix for the column component design, so if there is some design d for which $A_d'(Z_2 Z_2' - (1/p)Z_3 Z_3')A_d = 0$ and the column component design is optimal, then d is optimal [Bagchi, Mukhopadhyay and Sinha (1990)]. For instance, an optimal design for $b = 6$, $v = 4$, $p = 2$, $q = 4$ is

EXAMPLE 1.   A BNRC(6, 4, 2, 4).

$$\begin{array}{cccc cccc cccc}
1 & 1 & 2 & 2 & \quad 3 & 3 & 4 & 4 & \quad 1 & 2 & 3 & 4 \\
2 & 2 & 1 & 1 & \quad 4 & 4 & 3 & 3 & \quad 3 & 4 & 2 & 1 \\
\\
1 & 2 & 3 & 4 & \quad 1 & 2 & 3 & 4 & \quad 1 & 2 & 3 & 4 \\
3 & 4 & 2 & 1 & \quad 3 & 4 & 2 & 1 & \quad 3 & 4 & 2 & 1 \,.
\end{array}$$

The condition $A_d'(Z_2 Z_2' - (1/p)Z_3 Z_3')A_d = 0$ is termed *row regularity*. The idea of regularity with respect to a blocking factor will be fully defined in Section 2; in this example regularity with respect to rows says that rows within blocks are permutations of one another. The other key property employed

in this result is orthogonality of the three blocking factors. The reader may wish to return to this example after reading the definition of orthogonality in Section 2.

Section 2, then, contains the main results: an expression for the C-matrix, and conditions for optimality, for any situation in which the set of blocking factors is orthogonal, and for which one factor has an additional nestedness property. The optimality conditions say that the component block design defined by that one factor should be optimal, and that treatments should be arranged in the pattern called *regularity* with respect to each of the other blocking factors.

The literature contains at least three other special cases of the general result to be proven here. Cheng (1978) establishes optimality of the regular generalized Youden hyperrectangles, which include the regular Youden designs as treated by Kiefer (1975). Those designs are generalized yet further in Mukhopadhyay and Mukhopadhyay (1984) to allow for empty cells in the multicross of blocking factors. In Bagchi (1988), the family of optimum nested row and column designs introduced in Bagchi, Mukhopadhyay and Sinha (1990) is generalized to a nesting of multiway cross classifications. These will be more fully explained, along with other examples and constructions, in Section 3.

2. Main results. Statement and derivation of the main optimality result require a notation for describing the structure we impose on the blocking factors that also lends itself to computation of the corresponding C-matrix. An excellent exposition of the notation to be used and accompanying concepts may be found in Tjur (1984) and its discussions, which the reader can consult for a fuller treatment. The notational conventions are those of Bailey (1984, 1985, 1996). As this approach has not been established in the optimality literature, we begin with a brief introduction.

Denote by Ω the set of N experimental units at our disposal. A factor $\phi_F : \Omega \to F$ is a mapping ϕ_F from Ω to another set F, the elements of F being the levels of the factor. Typically one discusses F without reference to the underlying mapping, it being understood. Any factor should be thought of as a partitioning of the units into classes, or "blocks." The number n_F of levels of F is the number of nonempty classes in this partition. Factor F is uniform if each member of the partition contains N/n_F units. [Some authors, including Tjur (1984) and Searle, Casella and McCulloch (1992) called such a factor "balanced." In view of the usual meaning of "balance" for block designs, we prefer to avoid confusion by calling them "uniform." See also Preece (1982).]

Labelling the units $1, 2, \ldots, N$ and the levels of F $1, 2, \ldots, n_F$, then F may also be represented by the $N \times n_F$ incidence matrix X_F where

$$(X_F)_{ij} = \begin{cases} 1, & \text{if } \phi_F(i) = j, \\ 0, & \text{otherwise.} \end{cases}$$

As will be seen, it is through the matrix X_F that F's role in the linear model is stated. The matrix P_F which projects onto the range of X_F, that is, which

transforms the data vector to be constant on levels of F, is $X_F(X_F'X_F)^-X_F'$, which for any uniform factor is $(n_F/N)X_FX_F'$.

It is useful to define two special factors U and E. The universal factor U, which corresponds to our notion of fitting an overall mean, has $\phi_U(\omega) = 1$ for all ω in Ω. The existential factor E, which identifies each and every unit, has $\phi_E(\omega) = \omega$ for all ω in Ω.

Using the above, we can now discuss relationships among and operations on factors. Factor F is said to nest factor G if every F-class is a union of G-classes. Also said "F contains G" or "G is nested in F," it is written $F \geq G$, and is the usual notation of nesting in ANOVA models. Likewise we need a notion of crossing of factors. The cross of two factors F and G, written $F \wedge G$, is the factor given by the Cartesian product of ϕ_F and ϕ_G; its classes are subsets of Ω which are constant on both F and G, commonly called the "cells" induced by the F, G cross. It can be shown that $F \wedge G$, which some authors call the infimum of F and G [Bailey (1996)] is the coarsest partition nested in both F and G. There is also a finest partition that nests both F and G, written $F \vee G$ and called the join (or supremum) of F and G. A class of $F \vee G$ contains units which are either constant on F and G, or are connected by a chain of units with the property that consecutive units in the chain differ in at most one of F and G.

As indicated in Section 1, orthogonality of the blocking factors will play an important role. Define factors F and G to be orthogonal if their projection matrices commute: $P_F P_G = P_G P_F$. In an additive linear model, commutativity of the factor projection matrices is what allows for orthogonal lines in the ANOVA table; this is the natural statistical definition of orthogonality. Tjur (1984) shows that F and G are orthogonal if and only if $P_F P_G = P_{F \vee G}$ and provides the following description of orthogonality in terms of cell counts: F and G are orthogonal if and only if the condition of proportional cell counts holds for each F, G table formed by each class of $F \vee G$.

Denote by $\mathscr{F}$ the set of blocking factors on the experimental units Ω, with incidence matrices X_F for $F \in \mathscr{F}$. Let A_d be the $N \times v$ incidence matrix for the treatment allocation defined by design $d \in \mathscr{D}$, the class of all allowable designs. Our model is

$$(1) \qquad Y = A_d\tau + \sum_{F \in \mathscr{F}} X_F \beta_F + \varepsilon,$$

where τ is the $v \times 1$ vector of treatment effects, β_F the $n_F \times 1$ vector of fixed effects of blocking factor F, and ε an $N \times 1$ random vector with mean zero and $\mathrm{var}(\varepsilon) = \sigma^2 I$. For the model (1), the information matrix C_d for design d is

$$(2) \qquad C_d = A_d'(I - P_{\mathscr{F}})A_d,$$

where $P_{\mathscr{F}}$ is the matrix that projects onto the range of $X_{\mathscr{F}}$, and $X_{\mathscr{F}}$ is the $N \times \sum n_F$ matrix of concatenation of the X_F, $F \in \mathscr{F}$. Our goal is to find a d which minimizes an optimality criterion $\Phi(C_d)$. Allowed for Φ is any nonincreasing criterion in the sense of preserving the nonnegative definite ordering, that is, if $C_1 - C_2$ is nonnegative definite then $\Phi(C_1) \leq \Phi(C_2)$. When

not explicity stated, the term "optimal" in this paper implicity refers to one or more such criteria Φ.

MAIN RESULT A. *For the model* (1), *assume that every pair of factors in $\mathscr{F}$ is orthogonal. Let F_0 be a distinguished member of $\mathscr{F}$. If $\mathscr{F}$ can be partitioned as $\mathscr{F}_0 \cup \mathscr{F}_1$ such that*

(3) $$F_0 \in \mathscr{F}_0,$$

(4) *for every $F \neq G$ in $\mathscr{F}_0$, $F_0 \leq F \vee G$ and*

(5) *for every $G \in \mathscr{F}_1$ there exists at least one $F_G \in \mathscr{F}_0$ such that $F_G \leq G$,*

then

(6) $$P_{\mathscr{F}} = P_{F_0} + \sum_{F \in \mathscr{F}_0,\, F \neq F_0} (P_F - P_{F \vee F_0}).$$

MAIN RESULT B. *Suppose the factors inducing model (1) satisfy the conditions of Main Result A. Any design d for which*

(7) *the F_0-component block design is optimal and*

(8) $$(P_F - P_{F \vee F_0})A_d = 0 \quad \text{for every } F \in \mathscr{F}_0,$$

is optimal.

The condition (8) is termed "d is F-regular in $F \vee F_0$ for each $F \in \mathscr{F}_0$," or, shortly, "d is $(\mathscr{F}, F_0)$-regular." For any F with $F \vee F_0 = U$, it is simply "d is F-regular". Precisely what does this regularity condition say? Think of rows of the matrix $P_F A_d$ as corresponding to plots, and columns to treatments. Rows of $P_F A_d$ corresponding to plots at the same level of F are identical; the ith entry of such a row is the proportion of plots with that level of F receiving treatment i. Likewise the ith entry of a row of $P_{F \vee F_0} A_d$ is the proportion of times i occurs on plots with the corresponding level of $F \vee F_0$. So $(P_F - P_{F \vee F_0})A_d = 0$ if and only if, given any treatment, its replication proportions at levels of F are constant for each fixed level of $F \vee F_0$. If the factor F is uniform, then regularity says that any given treatment i occurs the same number of times at each level of F within a level of $F \vee F_0$. That number can vary with the treatment i, and can be zero.

This usage of the term "regular" differs from that of previous authors whose work we generalize [e.g., Cheng (1978, 1979); Mukhopadhyay and Mukhopadhyay (1984)]. Here it describes a property of the treatment assignment, while in the cited papers it is used to describe a property of the blocking factors, namely that for prescribed factors F, the number of levels n_F is divisible by v. The prior usage amounts to making it possible for every treatment to be assigned with equal frequency to each level of said factors F, resulting in special cases of the general optimality result above. The broader perspective provided by the main results supports the alternative view that this property should be thought of in terms of treatment assignment patterns.

While it will always be possible to translate regularity in terms of necessary conditions on functions of the n_F's, doing so will typically grow more tedious with increasing complexity of the block structure, and will not offer any particular insight into the optimality problem.

The next section will list numerous applications of the main results for obtaining optimal designs. However, not to be overlooked is that their utility extends well beyond providing a set of conditions for optimality. The simple, compact expression for the C-matrix given by (2) and (6) is valuable regardless of whether conditions (7) and (8) can be met, a fact demonstrated by the papers referenced in section 3 as special cases of this result, wherein considerable effort is expended on calculation of C-matrices. Moreover, they also provide a method for finding efficient (not necessarily optimal) designs. If the Φ-efficiency of the F_0-component design is at least some amount e, and the regularity conditions are met, then the multifactor design also has efficiency at least e.

This section closes with the proof of the main results. In establishing optimality the nonnegative definite ordering is employed in a manner that seems to have first been formalized by Magda (1980), though the approach appears either implicitly or explicitly in many author's work before and since; see especially Kunert (1983) and many of the papers referenced in Section 3.

PROOF. Pairwise orthogonality of the factors implies that $P_F P_G = P_G P_F = P_{F \vee G}$ for every $F, G \in \mathscr{F}$. Let $V_{\mathscr{F}}$ be the column space of $X_{\mathscr{F}}$, and write $P = P_{F_0} + \sum_{F \in \mathscr{F}_0, \, F \neq F_0} (P_F - P_{F \vee F_0})$. First it will be shown that $P = P_{\mathscr{F}}$. If $\omega \in V_{\mathscr{F}}^{\perp}$, then $P_F \omega = 0$ for every $F \in \mathscr{F}$ and thus

$$P\omega = P_{F_0}\omega + \sum_{F \in \mathscr{F}_0, \, F \neq F_0} P_F \omega - \sum_{F \in \mathscr{F}_0, \, F \neq F_0} P_{F \vee F_0}\omega = - \sum_{F \in \mathscr{F}_0, \, F \neq F_0} P_F P_{F_0}\omega = 0.$$

So $P = P_{\mathscr{F}}$ provided $PP_G = P_G$ for every $G \in \mathscr{F}$. Now for any $G \in \mathscr{F}_0$,

$$PP_G = P_{F_0 \vee G} + \sum_{F \in \mathscr{F}_0, \, F \neq F_0, \, F \neq G} (P_{F \vee G} P_{F \vee F_0 \vee G}) + (P_G - P_{F_0 \vee G})$$

$$= P_G + \sum_{F \in \mathscr{F}_0, \, F \neq F_0, \, F \neq G} (P_{F \vee G} - P_{F \vee G}) = P_G,$$

using (4) to equate $F \vee F_0 \vee G$ to $F \vee G$. For $G \in \mathscr{F}_1$, find $F_G \in \mathscr{F}_0$ such that $F_G \leq G$, so that $G \vee F_G = G$. Then

$$PP_G = PP_{G \vee F_G} = PP_{F_G} P_G = P_{F_G} P_G = P_{G \vee F_G} = P_G.$$

Thus from (2), the C-matrix for estimation of τ is

$$C_d = A_d'(I - P_{\mathscr{F}})A_d = A_d'(I - P_{F_0})A_d - \sum_{F \in \mathscr{F}_0, \, F \neq F_0} A_d'(P_F - P_{F \vee F_0})A_d.$$

For each F in $\mathscr{F} \backslash F_0$, we have $P_F - P_{F \vee F_0} = P_F(I - P_{F_0})$ and so the matrix $P_F - P_{F \vee F_0}$ is nonnegative definite. Thus $\Phi(C_d) \geq \Phi(A_d'(I - P_{F_0})A_d)$. But $A_d'(I - P_{F_0})A_d$ is the information matrix for the F_0-component block design, and the result follows. $\square$

3. Applications. This section will show how a variety of results in the literature fit into the common framework of Main Results A and B, and offer extensions of these along with some new design constructions. A number of specific applications will be given in the subsections below, but we first begin with a circumstance that can be treated generally and then not again considered. We also remind the reader that in view of (1) the setting and model are fully defined by the set of factors $\mathscr{F}$, which may thus be done without mention.

Suppose there is a factor $F_0 \in \mathscr{F}$ with $F_0 \leq F$ for every $F \in \mathscr{F}$; F_0 is said to be *fully nested*. Then $P_F - P_{F \vee F_0} = P_F - P_F = 0$ for every F, and so $P_{\mathscr{F}} = P_{F_0}$, and optimal designs are just those with optimal F_0-component block designs. For instance, in a row–column setup, let $F_1 =$ rows, $F_2 =$ columns, $F_0 = F_1 \wedge F_2$, $\mathscr{F}_0 = \{F_0, F_1, F_2\}$ and $\mathscr{F}_1 = \{U\}$, with k (say) units per cell (level of F_0). Inclusion of $F_1 \wedge F_2$ in the model is fitting a cell effect. Optimality is simply a matter of finding an optimal block design for n_{F_0} blocks of size k, the arrangement of treatments in rows and columns being otherwise irrelevant. An example is Cheng and Bailey's (1991) proof of the optimality of Trojan semi-Latin squares within the equireplicate class, in which the row and column regularity of the squares plays no role. [The issues are of course quite different under different models; see Bailey (1992), concerning Trojan and other semi-Latin squares under random effects models.]

Another instance of a fully nested factor can be found in the nested BIBDs of Preece (1967). Consider a balanced incomplete block design, or BIBD (v, b, k), with v treatments in b blocks of size k. If the blocks of this BIBD can be arranged into b_1 "big blocks" of b/b_1 blocks each, such that the resulting big blocks are a BIBD with block size kb/b_1, then the so-arranged design is called a nested BIBD. With $F_0 =$ blocks, $F_1 =$ big blocks and $\mathscr{F} = \{F_0, F_1, U\}$, clearly F_0 is fully nested, and *any* arrangement of the blocks of a BIBD into equisized big blocks is an optimal design. The additional requirement that big blocks form a BIBD is not needed for this analysis, important though it may be for the analysis with recovery of interblock information [Morgan (1996), pages 944–946].

Fully nested factors are not considered in the subsections that follow, and in conjuction with this, neither are settings for which a fully nested factor distinct from E would be plausible, such as cross-classifications with multiple units per cell. So excluded is any setting with two or more experimental units identical on every blocking factor. To avoid similar trivialities, factors nested by the distinguished factor F_0 are also not allowed.

3.1. *Row–column designs.* Within the framework of Section 2, the row–column setting is identified by two orthogonal blocking factors, $F_0 =$ columns and $F_1 =$ rows, neither of which is nested in the other, and model $\mathscr{F} = \{F_0, F_1, U\}$. Conditions (4) and (5) of the main result are trivially met, and one need only ask, what is $F_0 \vee F_1$? If $F_0 \vee F_1 = U$, then the setting is *structurally complete*, that is, every combination of a level of F_0 with a level of F_1 occurs on exactly one experimental unit (this follows from the orthogonality and the

restriction of at most one unit per cell imposed just before this subsection). For structurally complete settings, $P_{F_0} - P_{F_1 \vee F_0} = P_{F_0} - P_U$, giving the following optimality conditions: (7) says the column component design is optimal, and (8) says the rows are regular in U, that is, a given treatment occurs in the same number of cells in each row. Included are Latin squares, Youden designs and all other designs subsumed in the class of regular generalized Youden designs of Kiefer (1975). Indeed, take any optimal block design with b blocks of size k. If the design is equireplicate and the number of replicates is divisible by k, then the blocks may be juxtaposed, and the treatments rearranged within blocks, to give a $k \times b$ row–column design satisfying the optimality conditions. That the rearrangement is always possible is a consequence of the theory of distinct representatives; see Hartley, Shrikhande and Taylor (1953) and Corollary 3.4 below.

What if $F_0 \vee F_1 \neq U$? This implies that the row and column layout is structurally incomplete; that is, some cells of the $n_{F_0} \times n_{F_1}$ layout contain no experimental units. Since we are demanding that rows and columns be orthogonal, the row–column layout is disconnected, connected subcomponents are sets of cells identified by different levels of $F_0 \vee F_1$, and each of these sets is a structurally complete row–column layout with at least two rows and columns (for estimability) and with one unit per cell. Regularity of rows is now regularity within levels of $F_0 \vee F_1$, any one of which need not contain all treatments. Here is an example for three treatments in a 4×6 layout using only 12 cells (using x to denote an empty cell).

EXAMPLE 2.

$$
\begin{array}{cccccc}
3 & x & 1 & 3 & x & 2 \\
x & 1 & x & x & 2 & x \\
1 & x & 3 & 2 & x & 3 \\
x & 2 & x & x & 1 & x.
\end{array}
$$

The example is an instance of the following simple theorem, which characterizes the construction of these designs.

THEOREM 3.1. *Let d_0 be an optimal block design, possibly improper. Suppose that the b blocks of d_0 can be grouped into subsets B_j such that* (a) *every block in B_j has the same size k_j and* (b) *there are nonnegative integers m_{ij} such that treatment i occurs $m_{ij}k_j$ times within the blocks of B_j. Arrange the blocks of B_j as the columns of a rectangle with k_j rows in such a way that treatment i occurs m_{ij} times in each row. Put these rectangles on the diagonal of a $p \times b$ rectangle, where $p = \sum_j k_j$, leaving the remaining cells empty. Apply any permutation to the rows of this rectangle, and any permutation to the columns. The resulting design is column-optimal and row regular with respect to columns*: *hence it is an optimal row–column design.*

Uses of this simple approach are far too numerous to list. Example 2 above started with a BIBD$(3, 6, 2)$. Aside from BIBDs and other standard optimal

block designs, another fertile class for the initial d_0 is the nonproper, binary, variance-balanced designs studied by such authors as Gupta and Jones (1983), Pal and Pal (1988) and Gupta and Kageyama (1992).

Again, the row–column layouts of all Theorem 3.1 designs based on more than one subset B_j are composed of smaller, disconnected layouts. If it is intended to compare rows or columns, this will not be appropriate, but if rows and columns are truly nuisance factors one wishes to eliminate from the analysis, then restricting to connected row–column layouts is not only unneccessary, but will typically be wasteful in the sense of requiring an excessive number of experimental units.

There is no overlap between these designs and the structurally incomplete row–column designs of Stewart and Bradley (1991). Though their designs are optimal and, like those given here, have information matrix equal to that of the column component design, their row and column blocking factors are not orthogonal.

3.2. *Nested row–column designs.* The nested row–column setting is usually presented as having b structurally complete blocks, each of p rows crossed with q columns, for a total of bp row effects and bq column effects in the model, but we do not here demand that all pq cells in each block be used. For the model fitting row, column and block effects, let $F_0 =$ columns, $F_1 =$ rows, $F_2 =$ blocks, $\mathscr{F}_0 = \{F_0, F_1\}$ and $\mathscr{F}_1 = \{F_2, U\}$. If the cells used within each block form a connected two-way layout, then $F_0 \vee F_1 = F_2$ and the optimality conditions (7) and (8) are: (i) the column component design is optimal, and (ii) rows within blocks are regular, that is, identical up to permutation. This is Theorem 3.1.1 of Bagchi, Mukhopadhyay and Sinha (1990); also see Chang and Notz (1990, 1994) and Morgan and Uddin (1993). If the column component design is a BBD, they call the optimal designs *balanced nested row–column designs*, or BNRC (b, v, p, q)'s, of which Example 1 is one such. We will use their acronym but prefer the name *bottom-stratum universally optimum nested row and column designs* suggested by Morgan and Uddin (1996). Strictly speaking, these are exactly the designs which meet Kiefer's (1975) sufficient conditions for universal optimality under the bottom-stratum analysis.

Why the change in terminology? The word "balanced" already has two established and useful meanings in the context of nested row–column designs. Both apply to designs in quite general block structures, not just nested rows and columns. One refers to variance-balance for treatments under model (1): see Singh and Dey (1979). BNRCs do have such balance, but they are not the only variance-balanced designs in this context. The other meaning is that the treatments form a BBD with respect to each blocking factor in $\mathscr{F}$: see Preece (1967) and Houtman and Speed (1983). This is appropriate when the factors in $\mathscr{F}$ are random effects and the data are analysed by combining information from two or more strata: see Nelder (1965b). As not all BNRCs meet this second notion of balance, the class has failed on both counts to deserve the appellation "balanced nested row–column designs."

The other interesting possibility in the nested row–column setup is $F_0 \vee F_1 \neq F_2$, in which case (ii) becomes: (ii)' rows within $F_0 \vee F_1$ are regular. This leads to structurally incomplete nested row–column designs, but no issue essentially different from those which arose in Section 3.1. The connected components of a Theorem 3.1 design, when considered as blocks, form an optimal nested row–column design, but with possibly different numbers of rows and columns per block (i.e., improper blocks, a situation that seems not to have been previously treated in the literature); this is formally done by including an $F_0 \vee F_1$ effect in the model.

To make clear a further aspect of the relationship between the settings of this and the preceding subsection, we state the following theorem.

THEOREM 3.2. *Let $F_0 = rows$ and $F_1 = columns$ be as defined in Section* 3.1. *Suppose that $F_0 \vee F_1 \neq U$. Define a factor F_2, called "blocks," by $F_2 = F_0 \vee F_1$. Suppose further that the number of levels of F_0 is $n_{0(1)}$ within each level of F_2, so that $n_{0(1)} = n_{F_0}/n_{F_2}$. Then the $n_{0(1)} \times n_{F_1}$ row-column layout, constructed by juxtaposing the blocks so formed from a design satisfying Theorem* 3.1, *is row-regular, and hence is an optimal structurally complete row–column design.*

The class of all designs identified as convertible to optimality by Theorem 3.2 for its row–column setting includes all designs optimal by Theorem 3.1.1 of Bagchi, Mukhopadhyay and Sinha (1990). Theorem 3.2 says, for instance, that the search for a BNRC is actually the search for a suitable partition of the columns of a regular GYD (generalized Youden design): all BNRCs reduce to regular GYDs when the blocks are juxtaposed row to row, but of course not every regular GYD will be partitionable into a BNRC; compare Theorems 8 and 9 of Morgan and Uddin (1993). Similar statements apply to any design optimal by Theorem 3.1.1 of Bagchi, Mukhopadhyay and Sinha (1990); juxtaposition of the blocks yields an optimal, structurally complete, regular row–column design. Proof of Theorem 3.2 is simple, as regularity of F_0 in $F_0 \vee F_1$ of the nested design implies regularity of F_0 when those blocks are combined.

3.3. *Multiway cross classifications.* Generalizing from the row–column designs of Section 3.1 to n-way cross-classifications requires n crossed factors $F_0, F_1, \ldots, F_{n-1}$ which, if an experimental unit is placed at each combination of these factors, forms a structurally complete $n_{F_0} \times n_{F_1} \times \cdots \times n_{F_{n-1}}$ layout with one observation per cell. While the structurally complete case is certainly not the only structure of interest, it is most certainly the simplest starting point. From there, a discussion of the possibilities for relaxation of structural completeness can be eased into, all the while staying within the framework of the main results.

Put $\mathscr{F}_0 = \{F_0, \ldots, F_{n-1}\}$ and $\mathscr{F}_1 = \{U\}$. For the structurally complete case, $F_i \vee F_j = F_j \vee F_i = U$ for $i \neq j$, so the main result gives

$$P_{\mathscr{F}} = P_{F_0} + \sum_{i=1}^{n-1}(p_{F_i} - P_U)$$

[cf. Cheng (1978), Theorem 2.1]. A design with optimal F_0-component, for which the blocks of the F_i-component, $i > 1$, are complete blocks or multiples thereof, is optimal. In particular, this is Corollary 3.1.1 of Cheng (1978) on regular generalized Youden hyperrectangles. Cheng's [(1979), Theorem 2.1] technique for construction of regular Youden hyperrectangles actually solves this case in its entirety when the optimal component block design for F_0 is equireplicate, though its consequences outside of the Youden setup seem not to have been widely recognized [cf. Jacroux and Ray (1991), Theorem 3.2]. The combinatorial result, applicable not just to BBDs but to any block design, is stated as Lemma 3.3. It will be frequently used in the sequel.

LEMMA 3.3 [Cheng (1979)]. *Let d be a equireplicate block design for v treatments in b blocks of size k. Suppose that $k = k_1 \times k_2 \times \cdots \times k_{t-1} \times \omega$ and that v divides bk/k_j for each j. Then treatments may be rearranged in blocks to form $1 \times k_1 \times k_2 \times \cdots \times k_{t-1}$ hyperrectangles with ω treatments per cell, so that the resulting $b \times k_1 \times k_2 \times \cdots \times k_{t-1}$ hyperrectangle, when identified as a design for t crossed factors $F_0, F_1, \ldots, F_{t-1}$, contains each treatment the same number of times at each of the k_i levels of F_i, for $i = 1, (1), t - 1$.*

COROLLARY 3.4. *Any equireplicate optimal block design for v treatments with b blocks of size $k = k_1 \times k_2 \times \cdots \times k_{n-1}$, for which v divides bk/k_j for each j, can be arranged into an optimal design for v treatments in a $b \times k_1 \times k_2 \times \cdots \times k_{n-1}$ complete cross of n blocking factors with one unit per cell.*

Let p be a prime and t be an integer greater than 1. Bose, Shrikhande and Bhattacharya (1953) constructed semiregular group divisible designs for kp^t treatments in p^{2t} blocks of size k whenever $k < p^t + 1$; these designs have the within-group concurrence equal to zero and the between-group concurrence equal to 1. (These designs are just the duals of the square lattice designs [Yates (1936)] and are also known as transversal designs [Beth, Jungnickel and Lenz (1986).]) For any positive integers $l_1, \ldots, l_{n-1}$ with $\sum_1^{n-1} l_i = t$, Corollary 3.4 gives a $p^{2t} \times p^{l_1} \times \cdots \times p^{l_{n-1}}$ design for p^{2t} treatments that is optimal over all equireplicate designs that are binary in at least the F_0-component [cf. Cheng and Bailey (1991), page 1670].

There are many other possibilities and we give one example (others will become apparent below). Let b_0, a subset of size k of some finite group H of order v, be an initial block for an optimal block design d_0: the blocks of d_0 are $b_0 + h$ for $h \in H$. If $k = pq$ for integers $p, q > 1$, then b_0 can be arranged in a $p \times q$ cross, and thus the blocks $b_0 + h$ juxtaposed into a $p \times q \times v$ design d which is an optimal three-way crossed design. If there are t initial blocks of size pq, the result is an optimal $p \times q \times tv$ design.

For an n-way layout with some cells empty, pairwise orthogonality of the factors in $\mathscr{F}$ requires that collapsing to any two factors leaves a row–column layout with structure as described in Section 3.1, connected components having proportional cell counts. To simplify the discussion it is henceforth assumed that cell counts are constant within connected components of each two-way

layout, and hence, by an earlier assumption, are all equal to 1. Such a structure can be described as follows. Represent the n blocking factors on the N experimental units as an $n \times N$ *block structure array* $\mathscr{S}$:

$$(\mathscr{S})_{ij} = l \iff \phi_{F_i}(j) = l;$$

that is, in X'_{F_i} the row for which column j has a 1 is row l. Then any two-rowed subarray of $\mathscr{S}$ is either an orthogonal array of strength 2 and index 1 with possibly variable number of symbols [an OAVS of strength 2; see Rao (1947, 1973); Mukhopadhyay (1981); Wang and Wu (1991) and Mukerjee and Wu (1995)], or can be partitioned columnwise into a collection of strength-two OAVSs on disjoint (in rows) sets of symbols. For instance, for the 4×6 row–column design of Example 2, the two-rowed array $\mathscr{S}$ is

$$
\begin{array}{ccccccccccccc}
F_0 & 1 & 3 & 4 & 6 & 2 & 5 & 1 & 3 & 4 & 6 & 2 & 5 \\
F_1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4.
\end{array}
$$

Columns 5, 6, 11 and 12 are on OAVS, as are the remaining columns.

With the structure set for pairwise orthogonality of the blocking factors, what are the possibilities within the restrictions set by the nesting conditions of the main results? In a two-way cross, application of Main Result A requires checking only the orthogonality. But with more than two factors, pairwise orthogonality alone is not sufficient, for it does not guarantee that one of the factors must be nested in the join of every pair. This is demonstrated by Example 3.

EXAMPLE 3. This shows three pairwise orthogonal factors on eight units, with joins also displayed. The join of any two of the factors is completely crossed with the third:

$$
\begin{array}{rcccccccc}
F_1 & 1 & 2 & 1 & 2 & 3 & 4 & 3 & 4 \\
F_2 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\
F_3 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
F_1 \vee F_2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
F_1 \vee F_3 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
F_2 \vee F_3 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2.
\end{array}
$$

Again, the aim is to identify a class or classes of structures for n blocking factors that satisfy Main Result A, and which also properly fall within the realm of structurally incomplete multiway cross-classifications. Short of a characterization, which we do not currently have, it is reasonable to seek a workable sufficient condition that may be placed on the factors, that will satisfy Main Result A and still give a class of blocking structures rich enough to be fruitful. One choice is $F_i \vee F_j = U$ for all $i \neq j$, so that trivially $F_0 < F_i \vee F_j$. We say that F_i and F_j are strictly orthogonal in this case. With this restriction, the entirety of $\mathscr{S}$ is an OAVS of strength at least 2, and the optimality result is Theorem 3.1 of Mukhopadhyay and Mukhopadhyay (1984), of which Cheng's (1978) Corollary 3.1.1 is a special case. Mukhopadhyay and

Mukhopadhyay (1984) do not address the existence question for such designs, but, aside from the regular Youden hyperrectangles, there is an obvious possibility. For an OAVS with $n + 1$ rows, let n of the rows represent blocking factors, the remaining row identifying the treatment to be placed on each experimental unit. Then each component block design has blocks which are multiples of complete blocks and are thus regular, and the design is optimal.

The next theorem offers a series of designs with OAVS structure for the blocking factors, for which the F_0-component is an optimal *incomplete* block design.

THEOREM 3.5.   *The existence of a complete set of mutually orthogonal Latin squares* (MOLS) *of order s implies the existence of a design, optimal in several senses, for s^2 treatments in a cross of up to s blocking factors, one with s^2 levels and the remaining with s levels each, in which each pair of blocking factors is strictly orthogonal.*

PROOF.   Begin with an orthogonal array for s symbols in $s + 1$ rows and s^2 columns, implied by the set of MOLS. Take the symbols to be the integers $1, 2, \ldots, s$. Permute the columns so that the last row is $(1, 2, \ldots, s) \otimes 1_s$, then delete that row and denote the resulting array by B_0. Each row of B_0 is s consecutive orderings of the s symbols.

The array $B_0 = ((b_{ij}^0))$ defines a resolvable incomplete-block design for s^2 treatments in s^2 blocks of s treatments each as follows. The columns of B_0 correspond to treatments, the rows to replicates, and the symbols in a row to s blocks, so that treatment j appears in blocks $(i - 1)s + b_{ij}^0$ for $i = 1, (1), s$. The block design so formed is a square lattice design [Yates (1936)] and so is D-Optimal and A-optimal among binary equireplicate designs [Cheng and Bailey (1991)] as well as E-optimal and MV-optimal [see Shah and Sinha (1989), page 61]. The factor defining the blocks of the block design given by B_0 is called F_0.

Next let L_l, for $l = 1, (1), s - 1$, be a set of MOLS of order s, and write $B_l = L_l \otimes 1_s' = ((b_{ij}^l))$. The B_l define blocking factors F_l as follows. Superimposing the B_l on B_0, treatment j occurs on s units; these units are, for each $i = 1, (1), s$, one unit with level $(i - 1)s + b_{ij}^0$ of F_0 and level b_{ij}^l of F_l for $l = 1, (1), s - 1$.

Since each column of B_l contains each symbol $1, 2, \ldots, s$ once, each treatment occurs once at each level of F_l, establishing regularity with respect to those blocking factors.

Since each row of B_0 is s consecutive orderings of $1, 2, \ldots, s$, each level of B_l occurs s times with each level of B_0, establishing orthogonality of F_0 with F_l. Construction of the B_l shows that each level of F_l also occurs s times with each level of $F_{l'}$ for $l' \neq l$, so the F_l are also pairwise orthogonal.   □

EXAMPLE 4.   Putting $s = 3$ in Theorem 3.5 gives a $3 \times 3 \times 9$ design for nine treatments with only three replications. The collapsed 3×9 components

are optimal row–column arrangements of a square lattice design, and the collapsed 3×3 is a semi-Latin square. Using

$$
B_0
\begin{pmatrix}
1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \\
1 & 3 & 2 & 3 & 2 & 1 & 2 & 1 & 3
\end{pmatrix}
B_1
\begin{pmatrix}
1 & 1 & 1 & 3 & 3 & 3 & 2 & 2 & 2 \\
2 & 2 & 2 & 1 & 1 & 1 & 3 & 3 & 3 \\
3 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1
\end{pmatrix}
$$

$$
B_2
\begin{pmatrix}
1 & 1 & 1 & 3 & 3 & 3 & 2 & 2 & 2 \\
3 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 \\
2 & 2 & 2 & 1 & 1 & 1 & 3 & 3 & 3
\end{pmatrix}
$$

the nine 3×3 layers are

```
1  x  x      2  x  x      3  x  x
x  7  x      x  8  x      x  9  x
x  x  4      x  x  5      x  x  6

x  5  x      x  6  x      x  4  x
x  x  1      x  x  2      x  x  3
9  x  x      7  x  x      8  x  x

x  x  8      x  x  7      x  x  9
6  x  x      5  x  x      4  x  x
x  1  x      x  3  x      x  2  x.
```

Relaxing the condition $F_i \vee F_j = U$ for all $i \neq j$ opens a myriad of possibilities; the relationships seen in Sections 3.1 and 3.2 offer a bit of insight into what can happen. As the number of factors increases, the wealth and complexity of structures which lie between the extremes of structural completeness and the pure nesting of multiway crosses grows rapidly, many of which would not satisfy our sense of what should be called a multiway cross. Some of these will be explored in Section 3.5, but first, nested multiway cross-classifications will be covered in Section 3.4

3.4. *Nested multiway crosses.* Now there are n factors $F_0, F_1, \ldots, F_{n-1}$ each nested within a factor F_n, and within each level of F_n, the nested factors $F_0, \ldots, F_{n-1}$ form an n-way cross-classification. So within each level of F_n, the incidence structure of $F_0, \ldots, F_{n-1}$ can be described by an array $\mathscr{S}$ as in Section 3.3 that can be partitioned columnwise into a collection of strength-two OAVSs, and we take $\mathscr{S}$ to be the same (aside from a change of symbols) at each F_n level. Let this n-rowed subarray be denoted by $\mathscr{S}^*$. In Main Result A, $\mathscr{F}_0 = \{F_0, \ldots, F_{n-1}\}$ and $\mathscr{F}_1 = \{F_n, U\}$.

The structurally complete case is $\mathscr{S}^*$ being an OAVS of strength n. As in Section 3.3, the immediate obvious generalization is to take $\mathscr{S}^*$ as an OAVS of strength at least 2; regardless of the strength, the importance of the OAVS property is that it forces $F_i \vee F_j = F_n$ for all $i \neq j \in \{0, 1, \ldots, n-1\}$.

Thus the main result says that the F_0 component block design should be optimal, and that the F_i-component should be regular in $F_0 \vee F_i = F_n$. When the F_0-component design is a BBD, this is Theorem 3.1 of Bagchi (1988). Bagchi (1988) calls these designs *balanced nested multiway designs*, abbreviated BNMW $(b, n, v; p_1, p_2, \ldots, p_n)$ where $b = n_{F_n}$ and $p_i = n_{F_{i-1}}/b$. For reasons elucidated earlier, we shall use the same acronym to mean *bottom-stratum universally optimal nested multiway design*. A BNRC(b, v, p, q) is a BNMW$(b, 2, v; q, p)$.

Bagchi (1988) offers some constructions for three-way crosses based on $\mathscr{S}^*$ being an OAVS of strength 2. Here are given some general series for the nested n-way setting, and some series of nested three-way crosses for the structurally complete setting.

THEOREM 3.6. *Existence of a BIBD(v, b_0, v_1) and of a BNMW$(b_1, n, v_1; p_1, p_2, \ldots, p_n)$ implies the existence of a BNMW$(b_1 b_0, n, v; p_1, \ldots, p_n)$.*

Included within the class of BNMWs are the regular in $n - 1$ directions Youden hyperrectangles, Latin hypercubes, F-hyperrectangles, and the multiway crosses of Theorem 3.5, from which many new designs can be produced. Theorem 3.6 generalizes Theorem 2 of Morgan and Uddin (1993).

THEOREM 3.7. *Existence of a BNMW$(b, n, v; p_1, p_2, \ldots, p_n)$ for which b is a multiple of the integer s implies the existence of a BNMW$((b/s), n, v; sp_1, p_2, \ldots, p_n)$.*

THEOREM 3.8. *Existence of a BNMW$(b, n, v; p_1, p_2, \ldots, p_n)$ and of a BNMW$(b, n, v; p'_1, p_2, \ldots, p_n)$ implies the existence of a BNMW$(b, n, v; p_1 + p'_1, p_2, \ldots, p_n)$.*

Theorems 3.7 and 3.8 generalize Theorems 8 and 9 of Morgan and Uddin (1993).

THEOREM 3.9. *Existence of a BNRC(b, v, p, q) with $p = p_2 p_3$ implies the existence of a BNMW$(b, 3, v; q, p_2, p_3)$.*

PROOF. Each block of the BNRC is row-regular; consecutive sets of p_2 rows are the $p_2 \times p_1$ layers of a block of the BNMW.

Theorem 3.9 says that BNRCs can be "folded" into optimum three-way designs. The existence results for BNRCs in Bagchi, Mukhopadhyay and Sinha (1990) and in Morgan and Uddin (1993), thus provide many nested three-way designs. Here is one such result found by starting from Corollary 3.2.2 and Theorem 7 of those two papers, respectively.

COROLLARY 3.10. *For $v = tq + 1$ a prime power, there is a BNMW$(b, 3, v; p_1, p_2, p_3)$ with $b = tv$, $p_1 = q$, p_2 and p_3 for any $p_2 p_3 \le q$. For $v = 2tq + 1$ a*

prime power, there is a BNMW$(b, 3, v; p_1, p_2, p_3)$ with $b = tv$, $p_1 = q$, p_2 and p_3 for any $p_2 p_3 \leq q$.

The techniques of Theorem 3.9 and that of Theorem 3.7 are quite general, being applicable whenever the starting designs have the same block structure as BNMWs, have (not necessarily universally) optimal F_0-component designs and are regular in the other factors. For instance, if one starts with this optimal nest of two 4×6's for six treatments (rows are regular and the column component design is an optimal group-divisible design):

$$
\begin{array}{cccccc} 1 & 3 & 5 & 2 & 4 & 6 \\ 3 & 5 & 1 & 6 & 2 & 4 \\ 5 & 1 & 3 & 4 & 6 & 2 \\ 6 & 2 & 4 & 3 & 5 & 1 \end{array}
\qquad
\begin{array}{cccccc} 6 & 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 6 & 2 \\ 5 & 2 & 4 & 6 & 1 & 3 \\ 4 & 3 & 6 & 5 & 2 & 1 \end{array}
$$

then one can get an optimal design for six treatments in two $2 \times 2 \times 6$ blocks (compare Theorem 3.9) or in one $2 \times 2 \times 12$ block (compare Theorem 3.7).

The final construction in this section, like the designs used to produce Corollary 3.10, employs the difference technique over the finite fields. Theorem 3.10 combines BIBRCs [cf. Singh and Dey (1979)] to obtain three-way BNMWs.

THEOREM 3.11. *Let $v = smt+1$ be a prime or prime power, with $m = s/a+1$ for some integer a. Then there exists a BNMW $(tv, 3, v; s+a, s/a, s)$.*

PROOF. Let x be a primitive element of GF_v. Form the $(s/a) \times s$ array,

$$
B_{ij} = x^{(j-1)(m-1)t+(i-1)} \begin{pmatrix} x^t & x^{(m+1)t} & \cdots & x^{[(s-1)m+1]t} \\ x^{2t} & x^{(m+2)t} & \cdots & x^{[(s-1)m+2]t} \\ \vdots & \vdots & \ddots & \vdots \\ x^{(m-1)t} & x^{(2m-1)t} & \cdots & x^{(sm-1)t} \end{pmatrix},
$$

$j = 1, (1), s+a$; $i = 1, (1), t$. For fixed i, stack $B_{i1}, B_{i2}, \ldots, B_{i, s+a}$ to form a $(s+a) \times (s/a) \times s$ array $\mathscr{S}_i$. The $\mathscr{S}_i$'s are a set of initial arrays for the stated design. Verification is a routine exercise in differences. □

EXAMPLE 5. The four 3×3 layers of the initial block for a BNMW(13, 3, 13; 3, 3, 4) are

$$
\begin{array}{ccc} 2 & 6 & 5 \\ 4 & 12 & 10 \\ 8 & 11 & 7 \end{array}
\quad
\begin{array}{ccc} 3 & 9 & 1 \\ 6 & 5 & 2 \\ 12 & 10 & 4 \end{array}
\quad
\begin{array}{ccc} 11 & 7 & 8 \\ 9 & 1 & 3 \\ 5 & 2 & 6 \end{array}
\quad
\begin{array}{ccc} 10 & 4 & 12 \\ 7 & 8 & 11 \\ 1 & 3 & 9. \end{array}
$$

3.5. *Settings with a mixture of nested and crossed factors.* Section 3.3 starts with n factors $F_0, \ldots, F_{n-1}$ and goes on to study multiway crosses which are either structurally complete or structurally incomplete with $F_i \vee F_j = U$ for all $i \neq j$. Relaxing this pairwise condition while staying within the framework of Main Result A opens the doors to a wide variety of structures. One such

class, incorporating both nesting and crossing, is explored in Theorem 3.12 below.

With two crossed factors, $F_0 \vee F_1 \neq U$ was shown in Sections 3.1 and 3.2 to be equivalent to a nesting of complete crossings. The designs of Theorem 3.12, depending on the parameter t, generalize this relationship; they are stepping stones between the structurally complete case and the full nesting of multidimensional crosses discussed in Section 3.4. While a full nesting of crosses has $F_i \vee F_j \neq U$ for all $i \neq j$, this condition is not sufficient for such a nesting, as shown by Example 3 above.

The block structure of Theorem 3.12 designs is described in the proof. An equivalent description from a different perspective may be found in Section 3.7.

THEOREM 3.12. *Let d_0 be an r_0-resolvable optimal block design for v treatments in b blocks of size $k = k_1 \times k_2 \times \cdots \times k_{n-1}$. Write $r = bk/v$ and $b_0 = r/r_0$. If $v|(b/b_0)\prod_{j \neq i}^{t-1} k_j$ for $i = 1, \ldots, t-1$ and $v|(bk/k_i)$ for all i, then d_0 can be arranged to produce an optimal design in a structurally incomplete $b \times b_0 k_1 \times \cdots \times b_0 k_{t-1} \times k_t \times \cdots \times k_{n-1}$ cross-classification.*

PROOF. Divide the blocks of d_0 into b_0 groups so that each treatment occurs r_0 times in each group. Applying Lemma 3.3, the blocks of each group can be arranged into a $(b/b_0) \times k_1 \times \cdots \times k_{t-1}$ hyperrectangle with $w = k_t \times \cdots \times k_{n-1}$ treatments per cell. This is a nesting of t-dimensional crosses with regularity in $t-1$ of the dimensions of each cross, defining t crossed factors: F_0 with b levels and F_i with $b_0 k_i$ levels for $i = 1, \ldots, t-1$. Now thinking of the cells of the crosses as $b \times k_1 \times \cdots \times k_{t-1}$ "blocks" of size w, Lemma 3.3 can be applied again to form $n - t$ additional crossed factors which cross all of $F_0, \ldots, F_{t-1}$. Then F_i will have k_i levels for $i \geq t$, and the design will be F_i-regular for $i \geq t$. In the Main Results, $\mathscr{T}_0 = \{F_0, F_1, \ldots, F_{n-1}\}$, $\mathscr{F}_1 = \{U\}$, $F_i \vee F_j = U$ if $i \geq t$ or $j \geq t$, and $F_0 \leq F_i \vee F_j \neq U$ otherwise. □

For an application of Theorem 3.12, let $v = 4t + 1$ be a prime power. Then there is a BIBD with $b = 2v$, $k = (v-1)/2$ that is k-resolvable [Bose (1939)], and hence a universally optimal structurally incomplete $b \times 2k_1 \times 2k_2 \times \cdots \times 2k_{t-1} \times k_t \times \cdots \times k_{n-1}$ design for any factorization $k = k_1 \times k_2 \times \cdots \times k_{n-1}$.

The relationship between Theorem 3.12 designs and a nested set of multidimensional crosses is seen by fitting another blocking parameter $G = F_0 \vee F_1 \vee \cdots \vee F_{t-1}$. This gives a nested t-way design and an additional $n - t$ factors completely crossed with the t nested factors. Upon adding G to $\mathscr{F}_1$, the main results still hold. Viewed relative to Corollary 3.4, the resolvability with the stricter divisibility condition allows this partial nesting to be achieved.

Designs for this setting can also be found by separating blocks into subsets, so that the first divisibility condition of Theorem 3.12 is satisfied for the number of treatments within each subset. A method for doing this for three crossed factors is given in Theorem 3.13, but first an explicit expression for the structure array $\mathscr{S}$ when $n = 3$ and $t = 2$ will be given. Also included is

the factor $G = F_0 \vee F_1$ described above. Let R_0, R_1, R_2 be the three rows of an OAVS of strength 3 and index 1, describing the three-way cross of F_0, F_1, F_2 within a cell of G. Let n_i be the number of symbols in R_i, which are taken to be the integers $1, 2, \ldots, n_i$. Letting

$$\mathscr{S}_j = \begin{pmatrix} 1_{n_0 n_1 n_2} + (j-1) \\ R_0 + (j-1)n_0 \\ R_1 + (j-1)n_1 \\ R_2 \end{pmatrix},$$

the array is

(9) $$\mathscr{S} = (\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_{n_G}).$$

THEOREM 3.13. *Let $B_1, B_2, \ldots, B_s$ be initial blocks for an optimal block design with sv blocks, with each B_j containing k elements from some Abelian group H or order v. Suppose $k = k_1 k_2$ and the B_j's may be arranged as a $s \times k_1 \times k_2$ array such that the $k_1 \times k_2$ layers are the B_j's, and the $s \times k_2$ layers each contain the same number, say m_i, of elements from the coset H_i of a subgroup H_0 of order v_0. Then there is a set of v/v_0 three-way crosses of size $v_0 s \times k_1 \times k_2$ optimal for the block structure (9).*

The routine proof is omitted, for the method can be seen in an example.

EXAMPLE 6. The block $(0, 1, 4, 6)$ (mod 12) generates a group divisible design with $b = v = 12, \lambda_1 = 2 = \lambda_2 + 1$ [R109 of Clatworthy (1973)]. The subgroup of order 4 is generated by multiples of 3, and $(0, 1, 4, 6)$ (mod 3) $= (0, 1, 1, 0)$. With $k_1 = k_2 = 2$, the 2×2 layers of the three $4 \times 2 \times 2$ crosses are

$$G = 1 \quad \begin{matrix} 0 & 1 & 3 & 4 & 6 & 7 & 9 & 10 \\ 4 & 6 & 7 & 9 & 10 & 0 & 1 & 3, \end{matrix}$$

$$G = 2 \quad \begin{matrix} 1 & 2 & 4 & 5 & 7 & 8 & 10 & 11 \\ 5 & 7 & 8 & 10 & 11 & 1 & 2 & 4, \end{matrix}$$

$$G = 3 \quad \begin{matrix} 2 & 3 & 5 & 6 & 8 & 9 & 11 & 0 \\ 6 & 8 & 9 & 11 & 0 & 2 & 3 & 5. \end{matrix}$$

The 2×4 layers within a level of G are regular, and the 2×2 layers are the blocks of the optimal [Shah and Sinha (1989), page 61] F_0 component. Here F_2 has only two levels; its blocks are the union of the three first rows in the three G-components displayed above and the union of the three second rows.

Other settings with this mixing of nested and crossed factors that fit within the framework of Main Result A can be generated by increasing the complexity of the nesting. As an example, take two completely crossed factors G_1 and G_2 (then $G_1 \wedge G_2$ has $n_{G_1} n_{G_2}$ cells). In each cell nest a complete crossing of three factors F_0, F_1, F_2. Now further take the levels of F_2 to have the same

effects at each level of G_1, and those of F_0 and F_1 to be the same at each level of G_2. That is, F_2 is crossed with G_1 and F_0, F_1 are crossed with G_2. Here $\mathscr{F}_0 = \{F_0, F_1, F_2\}$, $\mathscr{F}_1 = \{G_1, G_2\}$ and $F_1 \vee F_2 = U \geq F_0$, so the main result holds. Explicitly, the structure array, following the (9) notation, is $\mathscr{S} = (\mathscr{S}_{11}, \mathscr{S}_{12}, \ldots, \mathscr{S}_{n_{G_1} n_{G_2}})$ where

$$
(10) \qquad \mathscr{S}_{ij} = \begin{pmatrix} 1_{n_0 n_1 n_2} + (i-1) \\ 1_{n_0 n_1 n_2} + (j-1) \\ R_0 + (i-1)n_0 \\ R_1 + (i-1)n_1 \\ R_2 + (j-1)n_2 \end{pmatrix}
$$

(also see Section 3.7). An interesting characteristic of this setting is that $F_2 \vee G_2 = G_2 \not\geq F_0$, so that G_2 cannot be included in $\mathscr{F}_0$. Heretofore, all examples have satisfied $F \in \mathscr{F}_1 \Rightarrow F \geq F_0$, so that their inclusion in $\mathscr{F}_1$ has been for the convenience of avoiding null terms $P_F - P_{F \vee F_0}$ in the expression for $P_{\mathscr{F}}$. This structure shows that $\mathscr{F}_1$ is a necessary part of Main Result B. Designs for this structure can be constructed with methods similar to those of the preceding two theorems.

3.6. *Gerechte designs.* Row and column designs in which there is an additional blocking factor corresponding to spatially compact regions are called gerechte designs. Their development in the statistical literature has been sporadic and somewhat confused; for an overview with a rigorous treatment of many of the issues involved in their use and analysis see the papers of Bailey, Kunert and Martin (1990, 1991).

In this section we confine ourselves to gerechte designs with rectangular regions. For a $p \times q$ structurally complete row–column layout with one experimental unit per cell, let p_1 and q_1 be integers satisfying $p_1 | p$ and $q_1 | q$. The additional blocking factor partitions the experimental units into smaller, $p_1 \times q_1$ row-column layouts (see Figure 1). In doing so, the model allows for local additivity of rows and columns while also providing the ability to fit a departure from additivity from one area to another. The structure array for
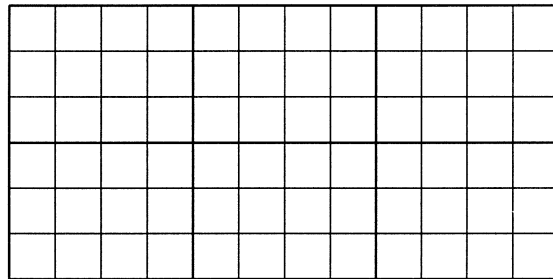

FIG. 1. *A rectangular gerechte layout with $p = 6$, $q = 12$, $p_1 = 3$ and $q_1 = 4$.*

the three blocking factors $F_0 = $ areas, $F_1 = $ rows and $F_2 = $ columns is

$$F_1 \qquad\qquad\qquad\qquad (1, 2, \ldots, p) \otimes 1_q,$$

$$F_2 \qquad\qquad\qquad\qquad 1_p \otimes (1, 2, \ldots, q),$$

$$F_0 \quad 1_{p_1} \otimes z \otimes 1_{q_1}, \ \ 1_{p_1} \otimes \left(z + \frac{q}{q_1}\right) \otimes 1_{q_1}, \ldots, 1_{p_1} \otimes \left(z + \left(\frac{p}{p_1} - 1\right)\frac{q}{q_1}\right) \otimes 1_{q_1},$$

where $z = (1, 2, \ldots, q/q_1)$. An alternative description is given in Section 3.7.

We allow the additional generalization that there may be b of these row–column blocks with the same structure, that is, a nesting of gerechte blocks with rectangular regions. Using F_3 for the nesting factor "blocks," put $\mathscr{F}_0 = \{F_0, F_1, F_2\}$ and $\mathscr{F}_1 = \{F_3, U\}$. Now $F_0 < F_1 \vee F_2 = F_3$, and Main Result B says that an optimal design may be found by arranging treatments so that the area-component block design is optimal, and so that each of rows and columns is regular within blocks.

Optimal area-component designs will typically be equireplicate, so that for an optimal design with a single ($b = 1$) block, the row and column regularity will demand that block be a doubly regular Youden design. When $p = q = v$, the block will be a Latin square. Thus do we arrive at the problem of partitioning doubly regular Youden designs into subrectangles so that those subrectangles form an optimal block design. The problem for areas being complete blocks, or multiples thereof, is solved by Theorem 3.14.

THEOREM 3.14. *For v treatments and for any p, q both of which are multiples of v, there is a doubly regular $p \times q$ Youden design that is an optimum gerechte design with rectangular areas of size $p_1 \times q_1$, for any p_1, q_1 satisfying $p_1 | p, q_1 | q$, and $v | p_1 q_1$.*

PROOF. Let $p_1^* = \operatorname{lcm}(q_1, v)/q_1$, so that p_1^* divides p_1. Form a $p_1^* \times q_1$ array S by writing the integers $(0, 1, \ldots, v - 1)$ row-wise until the array is filled:

$$S = \begin{pmatrix} 0 & 1 & \cdots & q_1 - 1 \\ q_1 & q_1 + 1 & \cdots & 2q_1 - 1 \\ \vdots & \vdots & \ddots & \vdots \\ (p_1^* - 1)q_1 & (p_1^* - 1)q_1 + 1 & \cdots & p_1^* q_1 - 1 \end{pmatrix}.$$

The design is the partitioned matrix $L = L_{ij}$ where $L_{ij} = S + i + (j - 1)q_1 - 1$ for $i = 1, (1), p/p_1^*$ and $j = 1, (1), q/q_1$, and all entries are evaluated (mod v).

That the $p_1 \times q_1$ areas are multiply complete blocks follows from S itself being so.

Row regularity will follow from the first row being a multiply complete block. But the first row is just q/v consecutive copies of $(1, 2, \ldots, v)$. Likewise, column regularity will follow from the first column being multiply complete. The p_1^* elements of the first column of S are $(0, q_1, \ldots, (p_1^* - 1)q_1)$, which is the additive subgroup of order p_1^* generated by v/p_1^*. Adding $i = 1, (1), p/p_1^*$ gives each treatment symbol $(p/p_1^*)/(v/p_1^*) = p/v$ times. $\quad\square$

Constructing a Theorem 3.14 design on each block of a BIBD will produce an optimal nested gerechte design.

EXAMPLE 7. The following is a nested gerechte design for five treatments in five 4×4 blocks with 2×2 areas. The first block is an optimal 4×4 for four treatments, and is Yates' (1951) corner design,

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

2	3	4	5
4	5	2	3
3	2	5	4
5	4	3	2

3	4	5	1
5	1	3	4
4	3	1	5
1	5	4	3

4	5	1	2
1	2	4	5
5	4	2	1
2	1	5	4

5	1	2	3
2	3	5	1
1	5	3	2
3	2	1	5

.

Many other species of optimal Gerechte designs may be found by working with the conditions outlined above, but the construction problem will be pursued no further here. A final example shows that optimal partially balanced designs can be accommodated in the areas.

EXAMPLE 8. This is an optimal gerechte Latin square for 12 treatments with areas of size 3×3. The area design is group-divisible with between-group concurrence equal to one more than the within-group concurrence, so it is optimal in several senses [see Shah and Sinha (1989), page 61, and Cheng and Bailey (1991)].

6	B	2	7	C	3	8	9	4	5	A	1
C	3	7	9	4	8	A	1	5	B	2	6
4	8	9	1	5	A	2	6	B	3	7	C
8	C	1	5	B	4	6	A	3	7	9	2
B	4	5	A	3	6	9	2	7	C	1	8
3	6	A	2	7	9	1	8	C	4	5	B
7	A	4	6	9	1	5	C	2	8	B	3
9	1	6	C	2	5	B	3	8	A	4	7
2	5	C	3	8	B	4	7	A	1	6	9
5	9	3	8	A	2	7	B	1	6	C	4
A	2	8	B	1	7	C	4	6	9	3	5
1	7	B	4	6	C	3	5	9	2	8	A

.

3.7. *Simple orthogonal block structure descriptions.* This subsection offers descriptions of the three most complicated block structures introduced in Sections 3.5 and 3.6 in terms of the simple orthogonal block structures of Nelder (1965a). Readers familiar with that work may find that this approach adds clarity.

The structure studied in Theorems 3.12 and 3.13 is part of the simple orthogonal block structure,

$$\left[b_0 \to \left(\frac{b}{b_0} * k_1 * \cdots * k_{t-1} \right) \right] * [k_t * \cdots * k_{n-1}].$$

Consider the complete cross of factors $P_1, \ldots, P_{n+1}$ with $b_0, b/b_0, k_1, \ldots, k_{n-1}$ levels. Then $G = P_1$; for $0 \le i \le t-1$, factor F_i is $P_1 \wedge P_{i+2}$; while for $t \le i \le n-1$ we have $F_i = P_{i+2}$.

For the structure array generated by (10) the related simple orthogonal block structure is

$$[n_{G_1} \to (n_0 * n_1)] * [n_{G_2} \to n_2].$$

It may be defined by the complete cross of factors $P_1, \ldots, P_5$, such that $G_1 = P_1$, $F_0 = P_1 \wedge P_2$, $F_1 = P_1 \wedge P_3$, $G_2 = P_4$ and $F_2 = P_4 \wedge P_5$.

For the nested set of gerechte blocks studied in Section 3.6, the coordinate version is

$$b \to \left[\left(\frac{p}{p_1} \to p_1 \right) * \left(\frac{q}{q_1} \to q_1 \right) \right].$$

Given a complete cross of factors $P_1, \ldots, P_5$, we have $F_3 = P_1$, $F_1 = P_1 \wedge P_2 \wedge P_3$, $F_2 = P_1 \wedge P_4 \wedge P_5$ and $F_0 = P_1 \wedge P_2 \wedge P_4$.

4. Comments. Main Result B says that optimal designs for multiple blocking factors may be found by starting with an optimal block design, the blocks of which will represent the levels of the distinguished factor F_0, and arranging treatments within those blocks so that regularity is attained with respect to each of the other blocking factors in $\mathscr{T}_0$. When an optimal block design is not known, starting with an efficient component design and arranging for regularity will produce an efficient design for the multifactor setting.

Of course, the specific optimality conditions presented here, like those of Cheng (1978), Bagchi, Mukhopadhyay and Sinha (1990), and others can be satisfied only for certain sets of parameter values. This does not make the results useless for other sizes of design. Leeming (1998) shows how to adapt the result of Bagchi, Mukhopadhyay and Sinha (1990) to obtain a nested row–column design which, while it cannot satisfy the optimality conditions exactly, is far more efficient than competing designs which satisfy intuitive criteria of optimality. And for the many parameter combinations that are covered, the reach of Main Result B is extensive and suggests that closer combinatorial study of known optimal block designs will be fruitful.

REFERENCES

BAGCHI, S. (1988). On the optimality of nested multiway designs. In *Probability, Statistics, and Design of Experiments* (R. R. Bahadur, ed.) 23–31. Wiley Eastern, New Delhi.

BAGCHI, S., MUKHOPADHYAY, A. C. and SINHA, B. K. (1990). A search for optimal nested row-column designs. *Sankhyā Ser. B* **52** 93–104.

BAILEY, R. A. (1984). Discussion of "Analysis of variance models in orthogonal designs," by T. Tjur. *Internat. Statist. Rev.* **52** 65–77.

BAILEY, R. A. (1985). Factorial design and Abelian groups. *Linear Algebra Appl.* **70** 349–368.

BAILEY, R. A. (1992). Efficient semi-Latin squares. *Statist. Sinica* **2** 413–437.

BAILEY, R. A. (1993). Recent advances in experimental design in agriculture. *Bull. Internat. Statist. Inst.* **55** 179–193.

BAILEY, R. A. (1996). Orthogonal partitions in designed experiments. *Designs, Codes and Cryptography* **8** 45–77.

BAILEY, R. A., KUNERT, J. and MARTIN, R. J. (1990). Some comments on gerchte designs I. Analysis for uncorrelated errors. *J. Agron. Crop Sci.* **165** 121–130.

BAILEY, R. A., KUNERT, J. and MARTIN, R. J. (1991). Some comments on gerchte desings II. Randomization analysis, and other methods that allow for inter-plot dependence. *J. Agron. Crop Sci.* **166** 101–111.

BEHRENS, W. U. (1956). Die Eignung verschiedener Feldversuchsanordnungen zum Ausgleich der Bodenunterschiede. *Zeitschrift für Acker- und Pflanzenbau* **101** 243–278.

BETH, T., JUNGNICKEL, D. and LENZ, H. (1986). *Design Theory*. Cambridge Univ. Press.

BOSE, R. C. (1939). On the construction of balanced incomplete block designs. *Ann. Eugen.* **9** 353–399.

BOSE, R. C., SHRIKANDE, S. S. and BHATTACHARYA, K. N. (1953). On the construction of group divisible incomplete block designs. *Ann. Math. Statist.* **24** 167–195.

CHANG, J. Y. and NOTZ, W. I. (1990). A method for constructing universally optimal block designs with nested rows and columns. *Utilitas Math.* **38** 263–276.

CHANG, J. Y. and NOTZ, W. I. (1994). Some optimal nested row-column designs. *Statist. Sinica* **4** 249–263.

CHENG, C.-S. (1978). Optimal designs for the elimination of multi-way heterogeneity. *Ann. Statist.* **6** 1262–1272.

CHENG, C.-S. (1979). Construction of Youden hyperrectangles. *J. Statist. Plann. Inference* **3** 109–118.

CHENG, C.-S. and BAILEY, R. A. (1991). Optimality of some two-associate class partially balanced incomplete-block designs. *Ann. Statist.* **19** 1667–1671.

CLATWORTHY, W. H. (1973). *Tables of Two-Associate-Class Partially Balanced Designs*. National Bureau of Standards, Washington, D.C.

DARBY, L. A. and GILBERT, N. (1958). The Trojan square. *Euphytica* **7** 183–188.

GUPTA, S. and JONES, B. (1983). Equireplicate balanced block deisgns with unequal block sizes. *Biometrika* **70** 433–440.

GUPTA, S. and KAGEYAMA, S. (1992). Variance balanced designs with unequal block sizes and unequal replications. *Utilitas Math.* **42** 15–24.

HARTLEY, H. O., SHRIKHANDE, S. S. and TAYLOR, W. B. (1953). A note on incomplete block designs with row balance. *Ann. Math. Statist.* **24** 123–126.

HOUTMAN, A. M. and SPEED, T. P. (1983). Balance in designed experiments with orthogonal block structure. *Ann. Statist.* **11** 1069–1085.

JACROUX, M. and RAY, R. S. (1991). On the determination and construction of optimal row-column designs having unequal row and column sizes. *Ann. Inst. Statist. Math.* **43** 377–390.

KACHLICKA, D. and MEJZA, S. (1995). Cox's design in repeated row-column design with split plots and a control. *Listy Biometryczne* **32** 81–92.

KIEFER, J. (1975). Construction and optimality of generalized Youden designs. In *A Survey of Statistical Design and Linear Models* (J. N. Srivastava, ed.) 333–353. North-Holland, Amsterdam.

KUNERT, J. (1983). Optimal designs and refinement of the linear model with applications to repeated measures designs. *Ann. Statist.* **11** 247–257.

LEEMING, J. A. (1998). Comparing a resolvable 7/(2×4) nested row and column design with two non-resolvable designs. Preprint.

MAGDA, C. G. (1980). Circular balanced repeated measurements designs. *Comm. Statist. Theory Methods* **9** 1901–1918.

MORGAN, J. P. (1996). Nested designs. In *Handbook of Statistics 13: Design and Analysis of Experiments* (S. Ghosh and C. R. Rao, eds.) 939–976. North-Holland, Amsterdam.

MORGAN, J. P. and UDDIN, N. (1993). Optimality and construction of nested row and column designs. *J. Statist. Plann. Inference* **37** 81–93.

MORGAN, J. P. and UDDIN, N. (1996). Optimal blocked main effects plans with nested rows and columns and related designs. *Ann. Statist.* **24** 1185–1208.

MUKERJEE, R. and WU, C. F. J. (1995). On the existence of saturated and nearly saturated asymmetrical orthogonal arrays. *Ann. Statist.* **23** 2102–2115.

MUKHOPADHYAY, A. C. (1981). Orthogonal main effects plans with variable numbers of levels for factors. In *Combinatorics and Graph Theory. Lecture Notes in Statist.* **885** 347–357. Springer, New York.

MUKHOPADHYAY, A. C. and MUKHOPADHYAY, S. (1984). Optimality in a balanced incomplete multiway heterogeneity set up. In *Statistics: Applications and New Directions* (J. K. Ghosh and J. Roy, eds.) 466–477. Indian Statistical Institute, Calcutta.

NELDER, J. A. (1965a). The analysis of randomized experiments with orthogonal block structure. I. Block structure and the null analysis of variance. *Proc. Roy. Soc. London Ser. A* **283** 147–162.

NELDER, J. A. (1965b). The analysis of randomized experiments with orthogonal block structure. II. Treatment structure and the general analysis of variance. *Proc. Roy. Soc. London Ser. A* **283** 163–178.

PAL, S. and PAL, S. (1988). Nonproper variance balanced designs and optimality. *Comm. Statist. Theory Methods* **17** 1685–1695.

PATTERSON, H. D. and ROBINSON, D. L. (1989). Row-and-column designs with two replicates. *J. Agric. Sci.* **112** 73–77.

PREECE, D. A. (1967). Nested balanced incomplete block designs. *Biometrika* **43** 479–486.

PREECE, D. A. (1982). Balance and designs: another terminological tangle. *Utilitas Mathematica* **21C** 85–186.

RAO, C. R. (1947). Factorial experiments derivable from combinatorial arrangements of arrays. *J. Roy. Statist. Soc. (Supp.)* **9** 128–139.

RAO, C. R. (1973). Some combinatorial problems of arrays and applications to design of experiments. In *A Survey of Combinatorial Theory* (J. N. Srivastava, ed.) 349–359. North-Holland, Amsterdam.

SEARLE, S. R., CASELLA, G. and MCCULLOCH, C. E. (1992). *Variance Components*. Wiley, New York.

SEEGER, P. (1986). Design and analysis of experiments with sugar beet seeds. *J. Roy. Statist. Soc. Ser. C* **35** 262–268.

SHAH, K. R. and SINHA, B. K. (1989). *Theory of Optimal Designs*. Springer, Berlin.

SINGH, M. and DEY, A. (1979). Block designs with nested rows and columns. *Biometrika* **66** 321–326.

STEWART, F. P. and BRADLEY, R. A. (1991). Some universally optimum row-column designs with empty nodes. *Biometrika* **78** 337–348.

TJUR, T. (1984). Analysis of variance models in orthogonal designs. *Internat. Statist. Rev.* **52** 33–65.

WANG, J. C. and WU, C. F. J. (1991). An approach to the construction of asymmetrical orthogonal arrays. *J. Amer. Statist. Assoc.* **86** 450–456.

WILLIAMS, E. R. (1986). Row and column designs with contiguous replicates. *Austral. J. Statist.* **28** 154–163.

WILLIAMS, E. R. and MATHESON, A. C. (1994). *Experimental Design and Analysis for Use in Tree Improvement.* CSIRO, Melbourne.

YATES, F. (1935). Complex experiments. *J. Roy. Statist. Soc. (Supp.)* **2** 181–247.

YATES, F. (1936). A new method of arranging variety trials involving a large number of varieties. *J. Agric. Sci.* **26** 424–455.

YATES, F. (1937). The design and analysis of factorial experiments. Technical communication 35. Imperial Bureau of Soil Science, Harpenden.

YATES, F. (1951). Bases logiques de la planification des expériences. *Ann. Inst. H. Poincaré* **12** 97–112.

DEPARTMENT OF MATHEMATICS AND STATISTICS
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23529
E-MAIL: jmorgan@odu.edu

SCHOOL OF MATHEMATICAL SCIENCES
QUEEN MARY AND WESTFIELD COLLEGE
LONDON E1 4NS
UNITED KINGDOM
E-MAIL: R.A.Bailey@qmw.ac.uk