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NONPARAMETRIC ESTIMATION IN RENEWAL THEORY II: SOLUTIONS OF RENEWAL-TYPE EQUATIONS

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Nonparametric estimators of solutions of renewal-type equations are proposed in terms of the empirical renewal function. Using the approach of Grübel and Pitts [*Ann. Stat.* **21** 1431–1451 (1993)], who studied asymptotic properties of the empirical renewal function, a number of properties of our estimators are established, including strong consistency, asymptotic normality and efficiency, and asymptotic validity of bootstrap confidence bounds. The results are illustrated by some particular examples.

1. Introduction. This paper concerns various properties of a nonparametric estimator of a solution of the renewal-type equation

$$Z(x) = z(x) + \int_{\mathbb{R}} Z(x-y)\,dF(y), \tag{1}$$

where F is a probability distribution on the real line and z is a bounded Borel measurable function; Z is the unknown function here. Under the assumption that the distribution F and the function z are zero for negative argument, such equations arise often and play an important role in various stochastic models, typically in stochastic processes where the process "forgets its past" at certain "renewal" points. Feller [(1971), Chapter XI] contains a detailed discussion for this case including a number of examples. Here we consider the general two-sided case where F and z can be nonzero on $(-\infty, 0)$. Although this seems rather less significant probabilistically, it turns out that the results for the one-sided case carry over without much difficulty to this extended framework.

Let $U(x) = \sum_{k=0}^{\infty} F^{\star k}(x)$ be the *renewal function* associated with F, where $F^{\star k}$ denotes that k-fold Lebesgue–Stieltjes convolution power of F. We assume throughout, unless otherwise stated, that F has a finite second and a positive first moment. These conditions ensure that the associated renewal function $U(x)$ is finite for all x [see, e.g., Gut (1988), page 89]. When F and z are zero on the negative half-axis, it is well known [see, e.g., Feller (1971), VI.6] that the function Z defined by $Z(t) = \int_{[0,\,t]} z(t-x)\,dU(x)$ is the unique solution of the *one-sided renewal equation*

$$Z(t) = z(t) + \int_{[0,\,t]} Z(t-x)\,dF(x) \tag{2}$$

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vanishing on $(-\infty, 0)$ and bounded on finite intervals. A number of quantities in applied probability satisfy (2). We mention two well-known examples from reliability theory, although some of the equations described below are also important in other contexts, such as branching processes and sequential analysis.

EXAMPLE I (The spent, residual and total lifetime). In a renewal process on $[0, \infty)$, let A_t be the time elapsed since the last renewal at time t and B_t be the time remaining until the next renewal at time t. Finally, let C_t denote the time between the last and the next renewal at time t.

Assume that $\xi > 0$ is fixed and let $Z_1(t) = P(A_t \le \xi)$, $Z_2(t) = P(B_t \le \xi)$ and $Z_3(t) = P(C_t \le \xi)$. Then, Z_i is the solution of the equation $Z_i = z_i + Z_i \star F$ for $i = 1, 2, 3$, where, for these values of i and for $t \ge 0$, $z_i(t)$ is given by

$$z_1(t) = 1_{[0,\,\xi]}(t)(1 - F(t)), \qquad z_2(t) = F(t+\xi) - F(t),$$
$$z_3(t) = 1_{[0,\,\xi]}(t)(F(\xi) - F(t));$$

here 1_A denotes the indicator function of the set A.

EXAMPLE II (Point availability). In an alternating renewal process, one envisages a device which is installed in a system at time $t = 0$ and it works until it fails. Then it takes a random time to be repaired, when it starts working again. Many examples of renewal processes are of an alternating type [see, e.g., Feller (1971)]. We assume that working times are independent identically distributed (i.i.d.) random variables with a distribution F_W, while repair times are independent with a common distribution F_R, and that working times are independent of repair times. The lifetime distribution of the process is then $F = F_W \star F_R$. Let the function Z be defined by $Z(t) = P$(the system is working at time t). The value of Z at time t is known as the *availability* of the system at time t [see, e.g., Barlow and Proschan (1975), 7.2]. Another interpretation of Z is that if repairs are not allowed, then $Z(t)$ represents the probability that the system operates without failure throughout the interval $[0, t]$. By a straightforward renewal argument we can see that Z satisfies the (one-sided) renewal equation

$$Z(t) = 1 - F_W(t) + \int_{[0,\,t]} Z(t-x)\,dF(x). \tag{3}$$

For the statistical problem of estimating Z in a renewal-type equation that we consider here, if the functional form of the underlying probability distribution F is known, parametric estimates for Z can be constructed. In many situations, however, the functional form of the distribution F, which typically represents the distribution of times between successive events in a renewal process, is unknown, thus a nonparametric approach is needed.

Assume, for instance, that a component is installed at a system and is replaced by a new one after failure. If replacements are immediate, then with the notation of Example I above, B_t is the remaining lifetime of the component in operation at time t. Assuming that lifetimes of successive components are i.i.d., and provided that a sample $X_1, X_2, \ldots, X_n$ from the lifetime distribution F is available, we can estimate the probability that the remaining lifetime of a component, B_t, does not exceed a certain value ξ.

If, on the other hand, repair times are nonzero and have a common distribution F_R, then with the setup of Example II, we can estimate the availability of the system on the basis of a sample $X_1, X_2, \ldots, X_n$ from the working time distribution F_W and a sample $Y_1, Y_2, \ldots, Y_n$ of repair times.

Our methods and results complement and extend those of Grübel and Pitts (1993), who considered the problem of estimating the renewal function U for the general two-sided case, on the basis of a sample $X_1, X_2, \ldots, X_n$ from F. They proposed as an estimator the *empirical renewal function*, $\hat{U}_n$, defined by $\hat{U}_n(x) = \sum_{k=0}^{\infty} \hat{F}_n^{\star k}(x)$, provided $\sum X_i > 0$; otherwise, $\hat{U}_n$ is defined to be identically zero. Here $\hat{F}_n(x) = n^{-1} \sum_{k=1}^{n} 1_{(-\infty, x]}(X_k)$ is the empirical distribution associated with $X_1, X_2, \ldots, X_n$. This estimator was shown to have a number of properties, including consistency, asymptotic normality and asymptotic validity of bootstrap confidence limits.

As mentioned in the beginning of the paper, here also we consider the general case where the distribution F and the function z can be nonzero for negative argument. The renewal function U corresponding to F plays a similar role in determining the solution Z of (1) as with the one-sided case. It follows from the results of Karlin (1955) and Smith (1961) that the function Z defined by $Z(x) = \int_{\mathbb{R}} z(x-y)\, dU(y)$ is the unique bounded solution of (1) such that $\lim_{x\to-\infty} Z(x) = 0$, provided that the function z satisfies the following conditions:

$$
\text{(4)} \qquad \begin{aligned} &\text{(i)} \quad z \text{ is continuous almost everywhere,} \\ &\text{(ii)} \quad \sum_{k\in\mathbb{Z}} \sup_{k<x\le k+1} |z(x)| < \infty. \end{aligned}
$$

A treatment of renewal-type equations on the whole real line, including the result above, can be found in Alsmeyer (1991); see also Rudin [(1991), Chapter 9], Bingham (1989) and Feller [(1971), VI.10]. We mention in particular that any two bounded solutions of (1) differ necessarily by a constant [Karlin (1955), Lemma 4]. The conditions we impose for the function z in the sequel imply that z satisfies (4). Thus we may from now on speak unambiguously about *the* solution of (1) and it will be implicitly understood that we refer to the unique bounded solution such that $\lim_{x\to-\infty} Z(x) = 0$. A seemingly different condition from (4) for the function z in (2), which is often used, is that z is *directly Riemann integrable* [see Feller (1971), pages 361–362; Alsmeyer (1991)]. Hinderer (1987) has shown that when z vanishes on $(-\infty, 0)$, the two notions coincide, and this is in fact true even without this assumption.

Our main effort in what follows is to extend the work of Grübel and Pitts (1993) to solutions of renewal-type equations and to translate the results for the renewal function to the solution Z of (1). We consider the case where the function z in (1) depends on F so that, based on a sample $X_1, X_2, \ldots, X_n$ from F, we estimate both U and z by $\hat{U}_n$ and $\hat{z}_n$, the plug-in estimate of z. Combining these two estimators together, we arrive at a nonparametric estimator $\hat{Z}_n = \hat{z}_n \star \hat{U}_n$ of Z. As it turns out, $\hat{Z}_n$ retains all the properties of $\hat{U}_n$, apparently in a rather stronger form. For example, since the renewal function U is unbounded, one cannot hope for uniform convergence of $\hat{U}_n$ to U on the whole real line. Under the above conditions for z and F, however, Z is in fact bounded. In Section 3, we show that $\hat{Z}_n$ converges to Z uniformly on $\mathbb{R}$ almost surely (a.s.), so that it is a strongly consistent estimator in a sense which is made precise after our definitions in Section 2.

As in Grübel and Pitts (1993), we take a functional approach and we consider the nonlinear functional Ψ that maps the (tail of the) distribution F in (1) to the unknown solution Z there. Paralleling similar results which have been developed in a parametric framework, the two ingredients for the success of the method in a nonparametric setup are the statistical properties of the input estimators, such as $\hat{F}_n$ and $\hat{z}_n$, combined with analytic properties of the map Ψ; see Gill (1989) for a general nonparametric context, Grübel and Pitts (1993) for its use in estimating the renewal function, and Pitts (1994a, b). This approach complements a similar functional view which is used in order to obtain analytic approximations for the unknown output quantity in question, as discussed in Grübel (1989); more recently, Politis and Pitts (1998) gave approximations and Taylor series expansions for the solution Z of (2).

Although we essentially extend the functional discussed in Grübel and Pitts (1993) one step further, we note that the tractability of the approach applied to a specific model requires a detailed study of the local properties of the relevant functional. In many cases, including the three equations of Example I above, our general results in Sections 3–5 apply directly to establish statistical properties of the proposed estimators. For (3), however, an ad hoc analysis is needed and this is carried out in Sections 3.2 and 4.2. The use of special arguments is not untypical of the functional approach, since this method often requires subtle analytic arguments and careful choice of topology particular to each application considered. Modulo this proviso, in this paper we demonstrate that the functional approach can indeed be applied to solutions of renewal-type equations in the function-space setting of Grübel and Pitts (1993).

We set up the necessary background in Section 2. Each of the following two sections establishes an analytic property for the map Ψ. In Section 3, we show continuity of Ψ, which yields consistency of the estimator $\hat{Z}_n$ in our first main result, Theorem 3.4. In Section 4, we consider differentiability (in an appropriate sense), which implies our second main result, Theorem 4.2, which says that the "empirical" process $\sqrt{n}(\hat{Z}_n - Z)$ is asymptotically Gaussian; this is an application of the delta method as described in Gill (1989). Theorem 4.5 gives asymptotic validity of nonparametric confidence bounds for Z based on the bootstrap. Finally, in the last section, we consider efficiency of our estimators, and in Theorem 5.4 we establish asymptotic efficiency of $\hat{Z}_n$ in the sense of van der Vaart (1988, 1991); see also Pitts (1994a).

2. Definitions and preliminaries. In our functional approach, we consider distribution functions, as well as the other functions arising in this context, as single elements in function spaces with a suitable topological structure. A well-known space to accommodate probability distributions and their empirical counterparts is the space of càdlàg functions [see, e.g., Pollard (1984), Chapter V], that is, the set of functions f: $[-\infty, \infty] \to \mathbb{R}$ which are right-continuous on $[-\infty, \infty)$ and have left-hand limits on $(-\infty, \infty]$. This space endowed with the supremum norm, $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$, is a nonseparable Banach space, D_∞. We note that any right-continuous function f: $\mathbb{R} \to \mathbb{R}$ with left-hand limits and with finite limits at $\pm\infty$ can be extended to a function in D_∞ by continuity. We identify any such function f and its extended version in D_∞. Any probability distribution function F is then in D_∞. An alternative topology which has often been considered for functions in D_∞ is the Skorohod topology [Harel, O'Cinneide and Schneider (1995) in this context; see also Dorado, Hollander and Sethuraman (1997), who studied a repair model of interest in reliability]. This, however, apart from being weaker, is also less convenient.

The renewal function U associated with a (proper) distribution F is unbounded (by the elementary renewal theorem), so that U is not an element of D_∞. Following Grübel and Pitts (1993), we consider instead weighted càdlàg spaces; for $\alpha, \beta \in \mathbb{R}$, we associate a function f on $[-\infty, \infty]$ with a new function $T_{\alpha\beta} f$ as follows:

$$T_{\alpha\beta} f(x) = \begin{cases} (1+x)^\beta f(x), & x \geq 0, \\ (1-x)^\alpha f(x), & x < 0. \end{cases}$$

Then the space of functions $D_{\alpha\beta} = \{f\colon [-\infty, \infty] \to \mathbb{R}\colon T_{\alpha\beta} f \in D_\infty\}$ with the norm on $D_{\alpha\beta}$ defined by $\|f\|_{\alpha\beta} = \|T_{\alpha\beta} f\|_\infty$ is again a nonseparable Banach space.

For purposes of measurability, in $D_{\alpha\beta}$ we take the σ-fields generated by the closed balls in the respective norm [see Pollard (1984), Chapter IV]. Note that for a probability distribution F, existence of moments of order β for some $\beta > 0$ implies that $1_{[0,\infty)} - F$ belongs to the space $D_{\beta\beta}$. Further, the renewal function belongs to the space $D_{0,-1}$. We also need the notion of weak convergence on arbitrary metric spaces. A sequence of random elements $\{X_n\}$ in a metric space converges in distribution to X if $Ef(X_n) \to Ef(X)$ as $n \to \infty$ for all real-valued, bounded, continuous measurable functions f; see again Pollard [(1984), Chapter IV]. We write $X_n \to_d X$ to denote that X_n converge in distribution to X. In the following sections, checking measurability of our estimators will be left for the reader. For measurability considerations in this context, we refer to the relevant discussions in Pollard (1984), Gill (1989) and Pitts (1994a).

As mentioned in the Introduction, a key component of the functional approach is to establish convergence properties of the input estimators. The following result strengthens the well-known Glivenko–Cantelli lemma, by imposing moment conditions on the underlying distribution F. This is Grübel and Pitts [(1993), Proposition 3.7], and it is proved by rescaling by F the corresponding result for the uniform distribution [Shorack and Wellner (1986), Section 10.2].

LEMMA 2.1. *Let $\alpha \geq 0$. Then, for a probability distribution F, the condition $\int_{\mathbb{R}} |x|^\alpha dF(x) < \infty$ implies that a.s.*

$$\lim_{n\to\infty} \|\hat{F}_n - F\|_{\alpha\alpha} = 0.$$

We shall also need the following refinement of the empirical central limit theorem [see, e.g., Pollard (1984), V2.11], for weak convergence of the *empirical process*, $\sqrt{n}(\hat{F}_n - F)$, to a Brownian bridge. This is Grübel and Pitts [(1993), Proposition 3.8].

LEMMA 2.2. *Let F be a probability distribution on the real line such that, for some $\alpha > 0$, $\int |x|^\alpha dF(x) < \infty$, and let B denote a standard Brownian bridge. Then, for any $\beta < \alpha/2$,*

$$\sqrt{n}(\hat{F}_n - F) \to_d B \circ F \quad \text{in } D_{\beta\beta},$$

where $(B \circ F)(t, \omega) = B(F(t), \omega)$.

The main technical difficulty that arises when we estimate Z rather than U in this context is that the former is not a nondecreasing function in general. This prompts us to extend the notion of a convolution, as defined in Grübel and Pitts (1993), so that it can also be defined with respect to a function of bounded variation on the line (BV, for short). The definition, along with some simple properties of the convolution operator, are given in the Appendix.

The next result gives conditions under which the solution $Z = z \star U$ of (1) is a BV function.

LEMMA 2.3. (i) *Let F be a distribution on the real line with a finite positive first moment μ such that, for some positive integer k, $F^{\star k}$ has a nonvanishing absolutely continuous component. Then, if U is the renewal function corresponding to F and $h = 1_{[a,b)}$, for $-\infty < a < b < \infty$, we have that $h \star U$ is of bounded variation on $\mathbb{R}$.*

(ii) *Assume in addition that F has finite second moment and z is a function which is absolutely integrable and of bounded variation on the line. Then $z \star U$ is also of bounded variation.*

PROOF. (i) The result follows from Rogozin [(1976), Corollary 1], where it is established that the lemma is true when h is of the form $h = 1_{[0,b)}$ for some $b > 0$.

(ii) Stone (1966) has shown that, under the conditions of the lemma, the function $V(x) = U(x) - (x/\mu)1_{[0,\infty)}(x)$ is BV. We now write

$$(z \star U)(x) = \int_{\mathbb{R}} z(x-y)\, dV(y) + \frac{1}{\mu}\int_{-\infty}^{x} z(y)\, dy.$$

It is well known that the convolution of two BV functions is also BV, which deals with the first term. Writing $z = z_1 - z_2$ with z_1 and z_2 nonnegative functions, we see that the second term is also BV. □

Let the set $\mathscr{H}$ be as in Definition A.1. For convenience of notation, we define the following.

DEFINITION 2.4. *For any function H in $\mathscr{H}$ and a fixed $c \in \mathbb{R}_+$, we define a new function $\Delta^c H\colon \mathbb{R} \to \mathbb{R}$ by*

$$\Delta^c H = 1_{[0,\,c)} \star H.$$

For $c = 1$, we write $\Delta^c H = \Delta H$.

Note that $\Delta^c H$ is well defined, that is, $1_{[0,\,c)} \star H$ always exists (for $H \in \mathscr{H}$), because the effective range of the integration is finite. Further, it is obvious from the definition that $\Delta^c H(x) = H(x) - H(x-c)$.

In what follows, we will be interested in the function ΔU, where U is the renewal function corresponding to a distribution function F. The equivalence between Blackwell's renewal theorem and the key renewal theorem [see, e.g., Feller (1971), XI] exemplifies the intimate relationship between the function ΔU and the more general class of functions of the form $z \star U$, which correspond to solutions of (2). Note that, under the conditions of Lemma 2.3 (i), ΔU qualifies as a right factor for the convolution operator according to Definition A.1. Further, we note that ΔU satisfies the equation

$$\Delta U = 1_{[0,\,1)} + \Delta U \star F. \tag{5}$$

In the next section, the results for solutions of renewal-type equations will first be proved for the function ΔU (that is, for $z = 1_{[0,\,1)}$) and then, using these, the general case will be derived.

3. Consistency.

3.1. *Continuity and strong consistency.* After these preparatory definitions and results, we now investigate how the convergence of a sequence of distributions $\{F_n\}$ to a distribution F is transmitted to the convergence of the corresponding solutions of renewal-type equations $Z_n = z_n \star U_n$ to $Z = z \star U$, provided, of course, that the functions z_n converge to z in an appropriate sense. In the simplest case, $z_n = z$ for all $n \in \mathbb{N}$, and this is the case we begin with. We follow the outline by Grübel and Pitts (1993); our results will first be established for a deterministic sequence F_n with the desired properties, and then for the empirical distribution $\hat{F}_n$.

Apart from the moment conditions to ensure that $U(x) < \infty$ for all x, we assume that the distribution F in (1) is *spread out*, that is, some convolution power of F has a nonvanishing absolutely continuous part, as in Asmussen [(1987), VI]. For the function z in (1), we assume throughout that, for some $\alpha > 1$, z is in $D_{\alpha\alpha}$. Note that this condition implies (4), and yields in particular that $z \star U$ belongs to the space D_∞. A more elaborate statement, which we use repeatedly in the sequel, is contained in Pitts (1991). Let F be a nonlattice probability distribution function on $\mathbb{R}$ with $\int x^2\, dF(x) < \infty$, $\int x\, dF(x) > 0$, and U be the associated renewal function. Assume that z is a measurable real-valued function such that, for some $\alpha > 1$, $z \in D_{\alpha\alpha}$. Then there exists a constant $c_1 = c_1(\alpha)$ such that

$$\|z \star U\|_\infty \le c_1 \|z\|_{\alpha\alpha} \|\Delta U\|_\infty. \tag{6}$$

Note that here we use the term lattice to denote a distribution which is concentrated on the integer multiples of some $\lambda > 0$. In the literature, this is sometimes called a lattice distribution with span λ and displacement 0, while a lattice distribution is defined more generally as one which puts all its mass at the points $a + r\lambda$, for $r \in \mathbb{N}$, where a and λ are not commensurate.

We are now in a position to state the following.

PROPOSITION 3.1. *For $n = 1, 2, \ldots$, let F_n be a sequence of nonlattice probability distribution functions on the real line with a finite second moment and let F be another distribution with a positive first moment μ and a finite second moment, such that F is spread out. Assume that the following hold*:

$$\int x^2 \, dF_n(x) \to \int x^2 \, dF(x), \tag{7}$$

and, for some $\alpha > 1$,

$$\lim_{n\to\infty} \|F_n - F\|_{\alpha\alpha} = 0. \tag{8}$$

Let U,U_n be the renewal functions for F,F_n, respectively. Then

$$\lim_{n\to\infty} \|\Delta U_n - \Delta U\|_\infty = 0.$$

PROOF. Let $\mu_{1,n}, \mu_{2,n}$ be the first and second moments of F_n, $n = 1, 2, \ldots$. It is easily verified that (8) implies that $\mu_{1,n} \to \mu$ as $n \to \infty$. Since the functions ΔU_n satisfy the inequality $\|\Delta U_n\|_\infty \le 1/\mu_{1,n} + \mu_{2,n}/\mu_{1n}^2$ [see, e.g., Grübel and Pitts (1993), Lemma 3.3], it follows from (7) that

$$\sup_n \|\Delta U_n\|_\infty < \infty. \tag{9}$$

Further, for $n = 1, 2, \ldots$, the functions U_n satisfy $U_n = 1_{[0,\infty)} + F_n \star U_n$, which yields that $[(1_{[0,\infty)} - F_n) \star U_n] \star \Delta U = \Delta U$. From (5) we also have, using Lemma A.2(ii),

$$\begin{aligned} [(1_{[0,\infty)} - F) \star U_n] \star \Delta U &= [(1_{[0,\infty)} - F) \star \Delta U] \star U_n \\ &= 1_{[0,1)} \star U_n = \Delta U_n, \end{aligned}$$

from which we derive

$$[(F - F_n) \star U_n] \star \Delta U = \Delta U - \Delta U_n. \tag{10}$$

The norm inequality (6) with (8) and (9) imply that $(F - F_n) \star U_n \to 0$ in D_∞. Lemma A.2 (iii) now gives that

$$\begin{aligned} \|\Delta U - \Delta U_n\|_\infty &= \|[(F - F_n) \star U_n] \star \Delta U\|_\infty \\ &\le \|(F - F_n) \star U_n\|_\infty |\mu_{\Delta U}|(\mathbb{R}), \end{aligned}$$

and the result follows from Lemma 2.3(i). □

REMARKS. (i) As in many places in renewal theory, a special treatment is needed for lattice distributions. If, for some $n \in \mathbb{N}$, F_n in the proposition is a lattice distribution concentrated on the integer multiples of some λ, then it follows directly from the renewal theorem for the lattice case [Feller (1971), XI] that $\Delta U_n(x)$ does not converge to a limit as $x \to \infty$ (unless, of course, 1 is a multiple of λ). The consequence is that ΔU_n does not belong to D_∞ and, if this happens for infinitely many n, the proposition cannot be true. On the other hand, the nonlattice assumption of F_n is sufficient to ensure that $\Delta U_n \in D_\infty$ for all n, by the renewal theorem again.

(ii) It is obvious that the proposition remains valid if the nonlattice assumption for the F_n's is replaced by the condition that "at most finitely many F_n are lattice."

Suppose now that $X_1, X_2, \ldots,$ is a sequence of independent and identically distributed random variables on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and let the distribution function of X_1 be F. In order to apply Proposition 3.1 to the sequence of empirical distributions $\{\hat{F}_n\}$, we need, in view of Remark (ii) above, to ensure that only finitely many $\hat{F}_n$ are lattice.

LEMMA 3.2. *Let F be a probability distribution on the line such that F has a nontrivial continuous part* (*i.e.*, *F is not purely atomic*), *and for $n \in \mathbb{N}$, let $\hat{F}_n$ be the empirical distribution function corresponding to the independent random variables $X_1, X_2, \ldots, X_n$, each having distribution F. Then $\mathbb{P}$-almost surely, only finitely many of the $\hat{F}_n$ are lattice.*

PROOF. The assumptions on F in the lemma imply that F can be written as $F = F_a + F_c$, where F_c is a continuous distribution whose total mass is nonzero. Let $\alpha = \int_{\mathbb{R}} dF_c(x)$. Define A_n to be the event that $\hat{F}_n$ is lattice and let, for any $x \in \mathbb{R}$, $\mathbb{Q}_x = \{rx\colon r \in \mathbb{Q}\}$. Note that $\hat{F}_n$ is lattice if and only if the ratios $X_2/X_1, \ldots, X_n/X_1$ are rational numbers. We then get, conditioning on the value of X_1, and using the independence of the X_i's,

$$\begin{aligned} \mathbb{P}(A_n | X_1 = x) &= \mathbb{P}(X_i \in \mathbb{Q}_x \text{ for } i = 2, 3, \ldots, n) \\ &= [F_a(\mathbb{Q}_x) + F_c(\mathbb{Q}_x)]^{n-1}. \end{aligned} \tag{11}$$

Since $F_c(\mathbb{Q}_x) = 0$, we obtain that $\mathbb{P}(A_n | X_1 = x) \le (1-\alpha)^{n-1}$, so that, considering now all possible values of X_1,

$$\mathbb{P}(A_n) = \int F_a(\mathbb{Q}_x)\, d\mathbb{P}(x) \le (1-\alpha)^{n-1}.$$

The Borel–Cantelli lemma now yields that $\mathbb{P}(\limsup A_n) = 0$, whence the result. □

REMARKS. (i) A trite modification is needed in (11) if $x = 0$, because in this case the first equality there fails. This can be overcome by conditioning, instead of X_1, on the value of the first nonzero of the X_i's.

(ii) If the distribution F in the lemma is only assumed nonlattice, then the result is no longer true. To see a counterexample, consider a nonlattice distribution F concentrated on the set $\mathbb{Q}$ of rational numbers; for example, if $\{r_1, r_2, \dots\}$ is an enumeration of the rationals, then F can be taken to be a purely atomic distribution assigning mass 2^{-k} to r_k. Then, with probability 1, *all* of $\hat{F}_n$ are lattice with F being obviously nonlattice.

Note that the condition "F has a nontrivial continuous part" is weaker than the condition "F is spread out" which we impose in the sequel, so that Lemma 3.2 holds if F is assumed spread out there.

We now turn to apply the results above to solutions of renewal-type equations, as asserted. The result of Proposition 3.1 can be easily generalized, so that the function $1_{[0,1)}$ is replaced by any indicator function on a half-open finite interval $[a, b)$ of the real line. This, in turn, implies that the result remains valid for any finite linear combination of such functions. Finally, by approximating any function $z \in D_{\alpha\alpha}$ by step functions, we arrive at the following.

PROPOSITION 3.3. *Let F, F_n, α be as in Proposition* 3.1 *and assume that, for some $\alpha' > 1$, z, z_n are functions in $D_{\alpha'\alpha'}$ such that*

$$\lim_{n\to\infty} \|z_n - z\|_{\alpha'\alpha'} = 0. \tag{12}$$

Then

$$\lim_{n\to\infty} \|z_n \star U_n - z \star U\|_\infty = 0.$$

The appearance of (12) in Proposition 3.3 is intuitively plausible. Convergence of both z_n and U_n to z and U, respectively, in an appropriate way is needed for the convergence of $z_n \star U_n$ to $z \star U$. The fact that U_n is "close" to U here is guaranteed by the convergence of F_n to F, as Grübel and Pitts (1993) have shown. The practical significance of (12), however, can be perhaps better understood after the following theorem, which is the main result in this section. We now substitute the sequence of functions F_n by the empirical sequence $\hat{F}_n$. Since $\hat{F}_n$ is a random sequence of functions, one wonders whether the same will be true for $\hat{z}_n$. To see the answer, recall that in (1) the function z depends most often on F. We may therefore assume that there exists a smooth function T which associates the distribution F with the function z in (1). It is, in fact, more convenient to take $1_{[0,\infty)} - F$ as the input of the map T, so that $T(1_{[0,\infty)} - F) = z$. Thus, $\hat{z}_n = T(1_{[0,\infty)} - \hat{F}_n)$ stands for the natural nonparametric estimator of z. The next theorem gives strong consistency of an estimator $\hat{z}_n \star \hat{U}_n$ for Z in (1).

THEOREM 3.4. *Let F be a spread-out distribution with finite second and positive first moment, U be the associated renewal function and $\hat{F}_n$ the empirical distribution associated with a random sample $X_1, \dots, X_n$ with distribution*

F. Let $\alpha' > 1$ *and assume that a sequence* $\{\hat{z}_n\}_{n\in N}$ *of random elements of the space* $D_{\alpha'\alpha'}$ *satisfies*

$$\lim_{n\to\infty} \|\hat{z}_n - z\|_{\alpha'\alpha'} = 0 \quad a.s.,$$

where z *is a function in* $D_{\alpha'\alpha'}$. *Then the following holds*:

$$\lim_{n\to\infty} \|\hat{z}_n \star \hat{U}_n - z \star U\|_\infty = 0 \quad a.s.$$

PROOF. Since F is assumed spread out, employing Lemma 3.2 and Remark (ii) after Proposition 3.1, we see that there is no loss of generality if we assume that, for all ω in a set $S \in \mathscr{A}$ with $\mathbb{P}(S) = 1$, $\hat{F}_n(., \omega)$ is nonlattice for all positive integers n.

Further, Lemma 2.1 shows that the $\hat{F}_n$ satisfy (8) a.s. More precisely, there is a set S' in $\mathscr{A}$, with $\mathbb{P}(S') = 1$ and such that, for $\omega \in S'$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} (1 + |x|)^\alpha |\hat{F}_n(x, \omega) - F(x)| = 0.$$

The fact that the $\hat{F}_n$ also satisfy (7) almost surely is a direct consequence of the strong law of large numbers; since $\sum_{k=1}^n X_k^2/n = \int x^2\, d\hat{F}_n(x)$, we obtain that $\int x^2\, d\hat{F}_n(x, \omega) \to \int x^2\, dF(x)$ for all ω in set S'', with $\mathbb{P}(S'') = 1$. Finally, under the conditions of the theorem, we can also find a set S''' with $\mathbb{P}(S''') = 1$ and such that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} (1 + |x|)^\alpha |\hat{z}_n(x, \omega) - z(x)| = 0$$

for all $\omega \in S'''$. Proposition 3.3 now gives that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |(\hat{z}_n \star \hat{U}_n)(x, \omega) - (z \star U)(x)| = 0,$$

for all $\omega \in S \cap S' \cap S'' \cap S'''$, and the result follows. □

REMARKS. Note that, if $F(0-) = 0$, the second moment condition can be relaxed. For indeed in this case, $\Delta\hat{U}_n(x) \le \hat{U}_n(1)$ for any $x \in \mathbb{R}$ a.s. [see, e.g., Asmussen (1987), IV, Theorem 2.4]. Next, Baxter and Li (1994) proved that $\hat{U}_n(x) \to U(x)$ a.s. for any fixed $x \ge 0$. [In fact, Baxter and Li (1994) require F to be continuous. However, for the result quoted here, this assumption is easily seen to be unnecessary.] For $x = 1$, this shows that $\sup_n \|\Delta U_n\|_\infty < \infty$, and consequently, for F one-sided, (7) in Proposition 3.1 is superfluous. Thus, the finiteness of a moment higher than the first would be sufficient for F in the theorem.

Let E be the subset of D_∞ consisting of all nonlattice probability distributions F with a finite second and a positive first moment. Further, we define $E_1 = \{1_{[0,\infty)} - F\colon F \in E\}$. Then E_1 is a subset of $D_{\alpha\alpha}$ for any $\alpha > 1$. Consider the map $\Psi\colon E_1 \to D_\infty$ defined by $\Psi(1_{[0,\infty)} - F) = T(1_{[0,\infty)} - F) \star U$. Assume that a distribution $F \in E$ is such that F is spread out. Then, Proposition 3.3 shows that the map Ψ is continuous at $1_{[0,\infty)} - F$ if T is so. Intuitively, the reason that we need continuity of T only for the map Ψ to be continuous, relies on the fact that the map Φ: $E_1 \to D_{0,-1}$ with $\Phi(1_{[0,\infty)} - F) = U$ is continuous [Grübel and Pitts (1993), Proposition 3.11; see also the first example in the next subsection].

3.2. *Applications.* We now look at some consequences of Theorem 3.4 by considering some special cases of renewal-type equations. To begin with, it is very simple to verify that the consistency result we proved in the previous subsection holds for the resulting estimator $\hat{Z}_n$ in each of the three equations in Example I in the Introduction. On the other hand, our results apply for equations over the whole real line. In the first example below, we consider such an equation and we demonstrate that results for a solution of a renewal-type equation can be utilized to deduce analogous results for the renewal function. A different example of encountering (1) in a probabilistic context can be seen by considering the renewal function in an renewal process on $\mathbb{R}$, when the distribution of the first renewal point is different from that of the subsequent ones.

EXAMPLE (a). Consider the equation

$$Z(x) = z(x) + \int_{\mathbb{R}} Z(x-t)\,dF(t), \tag{13}$$

where the function z is defined as follows:

$$z(x) = \begin{cases} \dfrac{1}{\mu}\displaystyle\int_x^{\infty} (1 - F(t))\,dt, & x \ge 0, \\ \dfrac{1}{\mu}\displaystyle\int_{-\infty}^{x} F(t)\,dt, & x < 0. \end{cases} \tag{14}$$

Then, provided that F has finite second moment, the function V defined by $V(x) = U(x) - (x/\mu)1_{[0,\infty)}(x)$ is the unique bounded solution of (13); see Alsmeyer [(1991), p. 95] and the relevant discussion in the Introduction. This equation has been studied extensively in the one-sided case; see, for example, Feller [(1971, XI.3]. Some discussion in the general case is in Smith (1960).

In order to be able to apply Theorem 3.4, we need to show that the sequence of the "empirical" elements $\hat{z}_n$ converges to z in some D-space. This follows from the proof of the following lemma.

LEMMA 3.5. *Let F be as in Proposition* 3.1, *assuming in addition that F possesses a finite moment of order β for some $\beta > 2$. Let F_n be a sequence of nonlattice probability distributions such that each F_n has finite moments of order β, and assume further that*

$$\lim_{n\to\infty} \|F_n - F\|_{\beta\beta} = 0.$$

Assume that U, U_n are the renewal functions associated with F, F_n, respectively, and define $V(x) = U(x) - (x/\mu)1_{[0,\infty)}(x)$ and $V_n(x) = U_n(x) - (x/\mu_n)$

$1_{[0,\infty)}(x)$, *where* $\mu_n > 0$ *is the first moment of* F_n. *Then we have that*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |V_n(x) - V(x)| = 0.$$

PROOF. Consider the equation (13) with z being defined as in (14). Let, for any positive integer n, the function z_n be defined by

$$z_n(x) = \begin{cases} \dfrac{1}{\mu_n} \displaystyle\int_x^\infty (1 - F_n(t))\,dt, & x \ge 0, \\ \dfrac{1}{\mu_n} \displaystyle\int_{-\infty}^x F_n(t)\,dt, & x < 0. \end{cases}$$

Then for z as in (14), we have that $V = z \star U$, and similarly, $V_n = z_n \star U_n$. By considering positive and negative values of x separately, we obtain that $\lim_{n\to\infty} \|z_n - z\|_{\alpha\alpha} = 0$. The lemma now follows from Proposition 3.3. □

THEOREM 3.6. *Assume that* F, U, V *are as in Lemma* 3.5, $\hat{F}_n$ *is the empirical distribution based on a random sample* $X_1, X_2, \ldots, X_n$ *taken from* F *and* $\hat{U}_n$ *is the empirical renewal function. For* $n = 1, 2, \ldots$, *let* $\hat{\mu}_n$ *be the average of* X_i *for* $1 \le i \le n$ *and define the function* $\hat{V}_n$ *by* $\hat{V}_n(x) = \hat{U}_n(x) - (x/\hat{\mu}_n)1_{[0,\infty)}(x)$, *provided that* $\hat{\mu}_n > 0$; *otherwise, let* $\hat{V}_n$ *be identically zero*.

Then with probability 1 *it holds that*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |\hat{V}_n(x) - V(x)| = 0.$$

PROOF. This follows immediately from Lemma 3.5 on noting that the arguments in the proof of Theorem 3.4 apply again. □

The corresponding consistency result for the empirical renewal function $\hat{U}_n$, similar to Grübel and Pitts [(1993), Theorem 2.1], is an easy deduction from above. With the assumptions of Theorem 3.6, it holds that a.s.

$$\lim_{n\to\infty} \|\hat{U}_n - U\|_{0,-1} = 0.$$

In the next example, we consider a renewal-type equation on the nonnegative half-line. We therefore assume until the end of this section that $F(0-) = 0$. To simplify the notation, we write $1-F$ rather than $1_{[0,\infty)} - F$ for a distribution F on $[0,\infty)$, and it will be tacitly understood that the function denoted by $1-F$ is identically zero on $(-\infty, 0)$.

EXAMPLE (b). Consider Example II in the Introduction about an alternating renewal process. With the notation of that example, suppose that we sample repeatedly from F_W and F_R and let $\hat{F}_{W,n}$ be the empirical distribution based on a sample $X_1, X_2, \ldots, X_n$ from F_W and $\hat{F}_{R,n}$ be the empirical distribution based on a sample $Y_1, Y_2, \ldots, Y_n$ from F_R. Assume that, for $i = 1, 2, \ldots, n$, X_i and Y_i are all defined on the same probability space

$(\Omega, \mathscr{A}, \mathbb{P})$. Then $(X_1+Y_1, X_2+Y_2, \ldots, X_n+Y_n)$ constitutes a random sample taken from F.

However, in this case, it seems more natural to estimate $F = F_W \star F_R$ by the convolution of the empirical distributions $\hat{F}_{W,n} \star \hat{F}_{R,n}$, rather than the empirical distribution $\hat{F}_n$ of F based on $(X_i + Y_i)$. A similar choice (in a different context) is made in Pitts, Grübel and Embrechts (1996); see also Theorem 5.6 here. We thus consider a function $\tilde{F}_n$ defined for $t \geq 0$ and $\omega \in \Omega$ by $\tilde{F}_n(t, \omega) = (\hat{F}_{W,n} \star \hat{F}_{R,n})(t, \omega)$, or, for brevity, $\tilde{F}_n = \hat{F}_{W,n} \star \hat{F}_{R,n}$.

In order to prove strong consistency for a nonparametric estimator for the function Z in (3) based on $\tilde{F}_n$, we need the following lemma.

LEMMA 3.7. *Let $\beta > 0$ and $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $D_{0\beta}$ such that $g_n(x) = 0$ for $x < 0$, and for some function $g \in D_{0\beta}$,*

$$\lim_{n\to\infty} \|g_n - g\|_{0\beta} = 0.$$

Assume that $\{F_n\}_{n\in N}$ is a sequence of probability distribution functions on $[0, \infty)$ with the property $\sup_n \int x^\beta \, dF_n(x) < \infty$, and F is another distribution function on $[0, \infty)$ such that

$$\lim_{n\to\infty} \|F_n - F\|_{0\beta} = 0.$$

Then for all $0 < \alpha < \beta$, we have

$$\lim_{n\to\infty} \|g_n \star F_n - g \star F\|_{0\alpha} = 0.$$

PROOF. Pitts [(1994b), Lemma 2.3] gives that, for $g \in D_{0\beta}$ and a probability distribution function F, the following norm inequality holds:

$$\|g \star F\|_{0\beta} \leq 2^\beta \|g\|_{0\beta}(\|1 - F\|_{0\beta} + 1). \tag{15}$$

Observe that this implies in particular that, under the conditions on g_n, g, F_n, F in the lemma, both $g_n \star F_n$ and $g \star F$ are in $D_{0\beta}$; the fact that they are right-continuous with left-hand limits is easily checked from dominated convergence.

We then decompose the term in the final statement of the lemma as follows:

$$g_n \star F_n - g \star F = (g_n - g) \star F_n + (g \star F_n - g \star F). \tag{16}$$

We see immediately from (15) that $[(g_n - g) \star F_n] \to 0$ in $D_{0\beta}$; thus, the convergence is also true in $D_{0\alpha}$ for any $0 < \alpha < \beta$. Considering the remaining term on the right-hand side of (16), let $\mathscr{I}$ be the space of linear combinations of indicator functions on intervals $[a, b)$, for $-\infty < a < b < \infty$. Using arguments like those in Grübel and Pitts [(1993), proof of Lemma 3.12], we see that there exists a sequence $\{h_k\}_{k\in\mathbb{N}}$ of functions in $\mathscr{I}$ vanishing on $(-\infty, 0)$ and such that, for $0 < \alpha < \beta$, $\lim_{k\to\infty} \|h_k - g\|_{0\alpha} = 0$.

Next, it is straightforward to verify that, for any function $h \in \mathscr{I}$ which is zero for negative argument,

$$\lim_{n\to\infty} \|h \star F_n - h \star F\|_{0\alpha} = 0. \tag{17}$$

We then bound the $\| \ \|_{0\alpha}$-norm of the second term in (16) as follows:

$$\|g \star F_n - g \star F\|_{0\alpha} \leq \|g \star F_n - h_k \star F_n\|_{0\alpha} + \|h_k \star F_n - h_k \star F\|_{0\alpha} + \|h_k \star F - g \star F\|_{0\alpha}.$$

Using (15), (17) and the convergence of h_k to g in $D_{0\alpha}$, we see that each of the three terms above converges to zero, taking the limits first with respect to n and then as $k \to \infty$, and this yields the result. □

We can now state the following, which gives strong consistency of $\tilde{Z}_n$.

THEOREM 3.8. *Consider an alternating renewal process with everything defined as above. Assume that for some* $\alpha > 1$, *the working and repair time distributions*, F_W *and* F_R, *respectively, satisfy*

$$\int_{[0,\infty)} x^\alpha \, dF_W(x) < \infty, \qquad \int_{[0,\infty)} x^\alpha \, dF_R(x) < \infty, \tag{18}$$

and assume further that $F = F_W \star F_R$ *is spread out.*

Let, for $n \in \mathbb{N}$, $\tilde{U}_n$ *be defined by* $\tilde{U}_n = \sum_{k=0}^{\infty} \tilde{F}_n^{\star k}$ *if* $\tilde{F}_n$ *is not concentrated at zero; in such a case, let* $\tilde{U}_n \equiv 0$. *Define also the function* $\tilde{Z}_n$ *by* $\tilde{Z}_n = (1 - \hat{F}_{W,n}) \star \tilde{U}_n$. *Then a.s.,*

$$\lim_{n\to\infty} \|\tilde{Z}_n - Z\|_\infty = 0.$$

PROOF. With probability 1, $\tilde{Z}_n$ in the statement of the lemma is the solution of (3) corresponding to $\tilde{F}_n$ (and $1-\hat{F}_{W,n}$). Further, in the remark following Theorem 3.4, we proved that when F vanishes on the negative half-line, only the finiteness of a moment of order higher than the first is needed for Propositions 3.1 and 3.3 to apply. The finiteness of such a moment for F here can be easily seen from (18). Next, Lemma 2.1 with (18) show that

$$\lim_{n\to\infty} \|\hat{F}_{W,n} - F_W\|_{0\alpha} = 0, \qquad \lim_{n\to\infty} \|\hat{F}_{R,n} - F_R\|_{0\alpha} = 0 \tag{19}$$

almost surely. We now write

$$\tilde{F}_n - F = (\hat{F}_{R,n} - F_R) - [(1 - \tilde{F}_{W,n}) \star \hat{F}_{R,n} - (1 - F_W) \star F_R].$$

Let α' be such that $1 < \alpha' < \alpha$. Since $\sup_n \int x^{\alpha'} \, d\hat{F}_n(x) < \infty$ a.s., employing (19) and Lemma 3.7, we obtain that the term in the square brackets above tends to zero in $D_{0\alpha'}$ a.s. Using (19) again, we finally deduce from the last equation that $\lim_{n\to\infty} \|\tilde{F}_n - F\|_{0\alpha'} = 0$ a.s. The theorem now follows from Proposition 3.3, employing arguments similar to those in Theorem 3.4. □

4. Asymptotic normality and the bootstrap.

4.1. *Differentiability and asymptotic normality.* In this section, we extend the results of Grübel and Pitts (1993) for the *empirical renewal process* $\sqrt{n}(\hat{U}_n - U)$ and consider the asymptotic behavior of the process $\sqrt{n}(\hat{Z}_n - Z)$. Here we make the additional assumption that the function z appearing in the renewal-type equation is of bounded variation, so that the function $Z = z \star U$ is also BV from Lemma 2.3(ii). In view of this, we can use a more direct approach, so that in particular, there is no need to establish the results for indicator functions first, as in Section 3.

We begin with the following observation. If the function z is BV and belongs to the set $D_{\alpha\alpha}$ for some $\alpha > 1$, and F_n is a sequence of probability distributions, then, using arguments similar to those in the proof of Proposition 3.1, it is easy to see that the following generalization of (10) holds:

$$[(F - F_n) \star U_n] \star (z \star U) = z \star U - z \star U_n, \tag{20}$$

provided that F, F_n have finite second and positive first moment and that F is spread out; these assumptions yield in particular, in view of Lemma 2.3, that the function $z \star U$ is an appropriate right factor for the convolution operator $\star$.

PROPOSITION 4.1. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of nonlattice probability distribution functions on the real line with a finite second and a positive first moment, and suppose that F is a spread-out distribution on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} x^2 \, dF(x) < \infty, \qquad \int_{\mathbb{R}} x \, dF(x) > 0,$$

and that (7) *holds. Assume that, for some $\alpha > 1$, there exists a function g in $D_{\alpha\alpha}$ such that*

$$\sqrt{n}(F_n - F) \to g \quad in\ D_{\alpha\alpha}. \tag{21}$$

Further, let, for $\alpha' > 1$ and for $n \in \mathbb{N}$, $\{z_n\}$ be a sequence of functions in $D_{\alpha'\alpha'}$ and z be another function in $D_{\alpha'\alpha'}$, so that z is BV. Assume additionally that there exits a function $q \in D_{\alpha'\alpha'}$ such that

$$\sqrt{n}(z_n - z) \to q \quad in\ D_{\alpha'\alpha'}. \tag{22}$$

Then

$$\sqrt{n}(z_n \star U_n - z \star U) \to (g \star U) \star (z \star U) + q \star U \quad in\ D_{\infty}.$$

PROOF. We mention first that conditions (21) and (22) in the statement of the proposition imply that $F_n - F \to 0$ and $z_n - z \to 0$ in $D_{\alpha\alpha}$ and $D_{\alpha'\alpha'}$, respectively. This shows the force of Proposition 3.3 in the course of the proof below. We now write

$$\sqrt{n}(z_n \star U_n - z \star U) = \sqrt{n}(z_n \star U_n - z \star U_n) + \sqrt{n}(z \star U_n - z \star U). \tag{23}$$

For the first term, we have

$$\begin{aligned}&\sqrt{n}(z_n \star U_n - z \star U_n) - q \star U \\ &\quad = (\sqrt{n}(z_n - z) - q) \star U_n + (q \star U_n - q \star U).\end{aligned} \tag{24}$$

We now use (6) to obtain that

$$\|(\sqrt{n}(z_n - z) - q) \star U_n\|_\infty \le c\|\sqrt{n}(z_n - z) - q)\|_{\alpha'\alpha'}\|\Delta U_n\|_\infty,$$

for some constant c, and the right-hand side of this tends to 0 because of (22). As a result, using Proposition 3.3 and (24), we deduce that

$$\sqrt{n}(z_n \star U_n - z \star U) \to q \star U \quad \text{in } D_\infty.$$

Considering the second term in (23), we employ (20), which gives

$$\sqrt{n}(z \star U_n - z \star U) = [\sqrt{n}(F_n - F) \star U_n] \star (z \star U).$$

We consequently write

$$\begin{aligned}&\sqrt{n}(z \star U_n - z \star U) - (g \star U) \star (z \star U) \\ &\quad = [(\sqrt{n}(F_n - F) - g) \star U_n] \star (z \star U) + (g \star U_n - g \star U) \star (z \star U).\end{aligned}$$

Using (6) once more, we see that $[(\sqrt{n}(F_n - F) - g) \star U_n] \to 0$ in D_∞ as $n \to \infty$, while Proposition 3.3 shows again that $g \star U_n - g \star U \to 0$ in D_∞. An application of Lemma A.2 then yields, using again that $z \star U$ is BV, that

$$\sqrt{n}(z \star U - z \star U_n) \to (g \star U) \star (z \star U)$$

in D_∞, whence the result. $\square$

We use Proposition 4.1 to obtain an appropriate differentiability property for Ψ. Recall the discussion at the end of Section 3.1 and the definition of the sets E_1 and E there. Further recall that we assume $\Psi(1_{[0,\infty)} - F) = T(1_{[0,\infty)} - F) \star U$, where $T\colon D_{\alpha\alpha} \to D_{\alpha'\alpha'}$ is defined by $T(1_{[0,\infty)} - F) = z$. Hence, we require a particular differentiability property for T, that of *Hadamard* differentiability. Gill (1989) asserts that a necessary and sufficient condition for a map ψ taking x in a Banach space $(B_1, \|\cdot\|_1)$ to an element $\psi(x)$ in another Banach space $(B_2, \|\cdot\|_2)$ to be Hadamard differentiable at $x \in B_1$ is that there exists a map ψ'_x which is linear and bounded so that

$$\frac{\|\psi(x + t_n h_n) - \psi(x) - \psi'_x(t_n h)\|_2}{t_n} \to 0 \quad \text{as } n \to \infty, \tag{25}$$

for any real sequence t_n with $\lim_{n\to\infty} t_n = 0$ and any sequence h_n in B_1 such that $h_n \to h \in B_1$. Assuming that T is Hadamard differentiable at $1_{[0,\infty)} - F$, we obtain under the conditions on $\{F_n\}$ and F in Proposition 4.1,

$$\sqrt{n}\big(\Psi(1_{[0,\infty)} - F_n) - \Psi(1_{[0,\infty)} - F)\big) \to \Psi'_{1_{[0,\infty)}-F}(g),$$

where

$$\Psi'_{1_{[0,\infty)}-F}(g) = -(g \star U) \star (z \star U) + T'_{1_{[0,\infty)}-F}(g) \star U.$$

This says that Ψ is differentiable in a certain sense at $1_{[0,\infty)} - F$, where F is a spread-out probability distribution function with corresponding z BV. This is a version of Hadamard differentiability for Ψ defined on E_1, along certain curves, that is, along those curves where $\int x^2\,dF_n(x) \to \int x^2\,dF(x)$.

The condition (21) is clearly imposed in Proposition 4.1 with a view to applying Lemma 2.2. When we substitute F_n by the sequence of empirical distributions $\hat{F}_n$, the function g in the limit of (21) becomes a Brownian bridge B scaled by the distribution F.

THEOREM 4.2. *Let F be a spread-out probability distribution with*

$$\int x\,dF(x) > 0$$

and such that, for some $\alpha > 2$, $\int |x|^\alpha\,dF(x) < \infty$. Assume further that for some $\beta, \beta' > 1$, there exists a measurable map T: $D_{\beta\beta} \to D_{\beta'\beta'}$ with $T(1_{[0,\infty)} - F) = z$, so that z is BV and such that T is Hadamard differentiable at $1_{[0,\infty)} - F$. Let $\hat{z}_n$ be defined for $\omega \in \Omega$ by $\hat{z}_n(.,\omega) = T(1_{[0,\infty)} - \hat{F}_n(.,\omega))$. Let B be a Brownian bridge and let the process $B \circ F$ be defined by $(B \circ F)(t,\omega) = B(F(t),\omega)$. We then define, suppressing the dependence on ω of the functions involved,

$$\hat{B}_n^z = \sqrt{n}(\hat{z}_n \star \hat{U}_n - z \star U) \tag{26}$$

and

$$B^z = ((B \circ F) \star U) \star (z \star U) + (T'_{1_{[0,\infty)}-F}(-B \circ F)) \star U.$$

Then we have that, in D_∞,

$$\hat{B}_n^z \to_d B^z.$$

PROOF. Since for any function g, $g \in D_{\beta\beta}$ implies that $g \in D_{\gamma\gamma}$ for any $\gamma < \beta$, there is no loss of generality if we take $\beta < \alpha/2$. Lemma 2.2 yields that, for such β,

$$\sqrt{n}(\hat{F}_n - F) \to_d B \circ F \quad \text{in } D_{\beta\beta}.$$

Let $C_{\beta\beta}(F)$ be the space of all functions in $D_{\beta\beta}$ that are continuous except possibly at the points where F jumps. Since the Brownian bridge B has continuous sample paths, $B \circ F$ is concentrated on $C_{\beta\beta}(F)$ [Pollard (1984), page 97]. On noting that this is a separable subspace of $D_{\beta\beta}$, the result is now readily verified by applying a Skorohod–Dudley–Wichura construction [Pollard (1984), page 71], as in Grübel and Pitts [(1993), Theorem 2.2], and using Proposition 4.1. □

It is easily seen that the map T in the statement of the theorem is essentially only required to be Hadamard differentiable along the appropriate curves. Also, one might only assume T to be defined in an open subset of the space $D_{\beta\beta}$ rather than throughout that space.

4.2. *Examples.* We now illustrate the use of the results so far in this section with reference to the two examples considered in the Introduction. In the first example Theorem 4.2 is directly applicable, while for the second a modification is needed and we prove differentiability of the functional in question. Since both examples involve functions which are identically zero for negative argument, we write $1 - F$ rather than $1_{[0,\infty)} - F$ for the tail of a distribution function, as in Section 3.2.

(a) Consider the three equations in Example I of Section 1. Assume that the set E_1 is as in Section 3.1 and let, for $i = 1, 2, 3$, the map T_i be defined on the set of all functions in E_1 that are zero on $(-\infty, 0)$, by $T_i(1 - F) = z_i$ with z_i defined for a fixed $\xi > 0$ as in the Introduction. Suppose that F is a distribution on $[0, \infty)$ which satisfies the conditions of Proposition 4.1. For any sequence $\{F_n\}$ of probability distribution functions such that (21) holds for some $g \in D_{\alpha\alpha}$ (with $\alpha > 1$), it is immediate to verify that T_i is Hadamard differentiable along the appropriate sequences. Thus, when we substitute F_n by the empirical distribution function $\hat{F}_n$, Theorem 4.2 shows that in each of the three cases for the spent, residual and total lifetime, respectively, the empirical process $\hat{B}_n^z$ in (26) converges in distribution, as random elements of D_∞, to a Gaussian process B^{z_i} which is given by

$$B^{z_i} = ((B \circ F) \star U) \star (z \star U) - B_\xi^i \star U,$$

where B_ξ^i is a process defined, for $i = 1, 2, 3$ and for $t \geq 0$, as follows:

$$B_\xi^1(t, .) = 1_{[0,\xi]}(t) B(F(t), .),$$

$$B_\xi^2(t, .) = B(F(t), .) - B(F(t + \xi), .)$$

and

$$B_\xi^3(t, .) = 1_{[0,\xi]}(t)(B(F(\xi), .) - B(F(t), .)).$$

(b) We continue the discussion of Example (b) in Section 3 about the estimator $\tilde{Z}_n$ in an alternating renewal process. Here we establish asymptotic normality of $\sqrt{n}(\tilde{Z}_n - Z)$. For a fixed time t, asymptotic normality of $\sqrt{n}(\tilde{Z}_n(t) - Z(t))$, in the usual sense that it converges in distribution to a normal variable, has been shown by Baxter and Li (1994).

We have seen in Section 3 that a complication arises if we are to use $\tilde{Z}_n$ rather than $\hat{Z}_n$ as an estimator of Z. In analogy with that section, here we cannot apply Theorem 4.2 to study the behavior of the empirical process $\hat{B}_n^z$, since $\hat{B}_n^z$ in (26) is expressed in terms of the empirical renewal function $\hat{U}_n$, while $\tilde{Z}_n$ is defined using $\tilde{U}_n$ (see Theorem 3.8). More explicitly, we note that since we sample from both the working and the repair time distributions, F_W and F_R, here it seems more suitable to use the pair of functions $(1 - F_W, 1 - F_R)$ rather than $1 - F_W \star F_R$ as the input of this new functional, which we denote by $\tilde{\Psi}$.

More precisely, the map $\tilde{\Psi}$ can be decomposed as follows:

$$(1 - F_W, 1 - F_R) \to (1 - F_W, 1 - F_W \star F_R) \to (1 - F_W) \star U, \tag{27}$$

where U is the renewal function corresponding to $F_W \star F_R$. We thus see that an extra step is required in proving differentiability of $\tilde{\Psi}$. This step involves establishing that the functional which maps $(1 - F_W, 1 - F_R)$ to $1 - F_W \star F_R$ is differentiable. This is achieved in the next auxiliary result, whose proof is based on Lemma 3.7.

LEMMA 4.3. *Let $\{F_n\}, \{G_n\}$ be two sequences of probability distribution functions concentrated on $[0, \infty)$ for all $n \in \mathbb{N}$, and such that F_n, G_n have finite moments of order α for some $\alpha > 1$. Assume that F, G are two other probability distribution functions on $[0, \infty)$ with finite moments of order α, and suppose that, for some functions $g, h \in D_{0\alpha}$,*

$$\sqrt{n}(F_n - F) \to g, \qquad \sqrt{n}(G_n - G) \to h \quad \textit{in } D_{0\alpha}.$$

Then, for any $0 \le \alpha' < \alpha$, the following holds:

$$\sqrt{n}(F_n \star G_n - F \star G) \to g \star G + h \star F \quad \textit{in } D_{0\alpha'}.$$

PROOF. We use the following decomposition for the difference between the two sides in the final statement of the lemma:

$$\begin{aligned}
&\sqrt{n}(F_n \star G_n - F \star G) - (g \star G + h \star F)\\
&\quad = \left\{\sqrt{n}(F_n - F) - g\right\} \star G_n\\
&\qquad + \left\{\sqrt{n}(G_n - G) - h\right\} \star F + (g \star G_n - g \star G).
\end{aligned}$$

The moment conditions of G_n imply that the sequence $\{1 - G_n\}$ is $\| \ \|_{0\alpha}$-bounded. Using (15) and the assumptions in the statement of the lemma, we thus derive that

$$\left\{\sqrt{n}(F_n - F) - g\right\} \star G_n \to 0, \qquad \left\{\sqrt{n}(G_n - G) - h\right\} \star F \to 0,$$

in the space $D_{0\alpha}$; thus, the above statements are also true in the weaker norm of the space $D_{0\alpha'}$. Finally, the fact that $g \star G_n - g \star G \to 0$ in $D_{0\alpha'}$ as $n \to \infty$ under the assumptions of the lemma, can be seen to be true from Lemma 3.7. □

In the next theorem, we obtain the asymptotic behavior of the process $\sqrt{n}(\tilde{Z}_n - Z)$. Its proof relies on the Skorohod–Dudley–Wichura theorem combined with an appropriate differentiability property for each of the two component maps of $\tilde{\Psi}$ in (27), which was given respectively in Proposition 4.1 and Lemma 4.3, and the intrinsic property of Hadamard differentiation that it obeys the chain rule; see also Pitts [(1994a), proof of Theorem 4.3].

THEOREM 4.4. *In an alternating renewal process, assume that the working and repair time distributions satisfy, for an* $\alpha > 2$,

$$\int_{[0,\infty)} x^\alpha \, dF_W(x) < \infty, \qquad \int_{[0,\infty)} x^\alpha \, dF_R(x) < \infty,$$

and assume further that $F = F_W \star F_R$ *is spread out. Let* B, B' *be two independent Brownian bridges and define a process* $B_{W,R}$ *by*

$$B_{W,R} = (B \circ F_W) \star F_R + (B' \circ F_R) \star F_W.$$

Then, with $\tilde{Z}_n$ *defined as in Theorem* 3.8, *the process* $\sqrt{n}(\tilde{Z}_n - Z)$ *converges in distribution, as random elements of* D_∞, *to the process*

$$(B_{W,R} \star U) \star ((1 - F_W) \star U) + (B \circ F_W) \star U.$$

Here U *is the renewal function associated with* F.

4.3. *Bootstrap.* A problem that has attracted interest from statisticians is how we can construct nonparametric confidence intervals for the renewal function U. Baxter and Li (1994) and Frees [(1986), using a different estimator for U, however] obtained asymptotic confidence intervals for $U(t)$ in the case where t is viewed as fixed. Grübel and Pitts (1993), exploiting the $D_{\alpha\beta}$-space framework that we have also adopted here, derived asymptotic bootstrap confidence bounds for U that hold for any $t \in \mathbb{R}$. Grübel and Pitts (1993) quote Shorack (1982) and Gill [(1989), Section 2.4] for the idea underlying the construction. Gill [(1989), Theorems 4 and 5] showed that, if F_n is an estimator sequence for F in a Banach space B_1 and ϕ is a differentiable functional from B_1 into another Banach space B_2, then the distribution of $\sqrt{n}(\phi(F_n) - \phi(F))$ is asymptotically the same as that of $\sqrt{n}(\phi(F_n^*) - \phi(F_n))$, where F_n^* is the distribution based on a sample of size n from F_n. It is noticeable that, as Gill (1989) points out, ϕ is only required to be differentiable at a single point F and not even locally.

After these considerations, it seems plausible that the construction of nonparametric asymptotic confidence bounds for $z \star U$, in the spirit of Grübel and Pitts [(1993), Theorem 2.3], should present no real difficulty, since we have established differentiability of the map Ψ. Recall the definitions of $\hat{B}_n^z$, B^z in Theorem 4.2. Let now

$$R_n(x) = \mathbb{P}(||\hat{B}_n^z||_\infty \le x), \qquad R(x) = \mathbb{P}(||B^z||_\infty \le x).$$

Since the distribution function R is unknown, we use the bootstrap to obtain confidence bounds for the output of Ψ. As in Grübel and Pitts (1993), consider, for $n \in \mathbb{N}$, a functional $\mathbb{F}_n$: $\mathbb{R}^n \to D_\infty$ which associates with each vector $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ a probability distribution function $\mathbb{F}_n(x)$ with jumps of size $1/n$ at each of the points $(x_1, \dots, x_n)$. Namely, $\mathbb{F}_n(x) = n^{-1} \sum_{i=1}^n 1_{[x_i, \infty)}$. Let $(X_1, \dots, X_n)$ be a random sample from the distribution F and $(X_{i_1}, \dots, X_{i_n})$

be a bootstrap sample from F based on the observations $X_1, \dots, X_n$. Then the bootstrap estimate $\hat{R}_n(x)$ of $R_n(x)$ is given by

$$\hat{R}_n(x) = n^{-n} \sum_{i \in I_n} 1_{[0,\, x]} \times \Big(\sqrt{n} \| \Psi\big(1_{[0,\, \infty)} - \mathbb{F}_n(X_{i_1}, \dots, X_{i_n})\big) - \Psi\big(1_{[0,\, \infty)} - \hat{F}_n\big) \|_\infty \Big),$$

where the summation is taken over all $i = (i_1, \dots, i_n)$ in the set $I_n = \{1, 2, \dots, n\}^n$.

The next theorem and its proof are merely a restatement of Grübel and Pitts [(1993), Theorem 2.3] for the current situation.

THEOREM 4.5. *Assume the conditions of Theorem* 4.2 *and let* $R, R_n, \hat{R}_n$ *be defined as above. Let* $0 < \delta < 1$ *and, for* $n = 1, 2, \dots,$ *let* $\hat{q}_n(\delta)$ *be the* δ*-quantile of the distribution* $\hat{R}_n$. *Then, provided that* $R(0) = 0$, *the following holds*:

$$\lim_{n \to \infty} \mathbb{P}\big(\| \hat{B}_n^z \|_\infty \le \hat{q}_n(\delta) \big) = \delta.$$

Similar results, concerning the successful use of the bootstrap to construct nonparametric confidence intervals in stochastic models, have appeared in Pitts (1994a) in the context of queueing theory and Pitts (1994b) for compound distributions.

A small-scale simulation study to assess the performance of the bootstrap for the distribution of the residual lifetime Z_2 in a renewal process has been carried out in Politis (1997). Using 1500 samples of size $n = 300$, it was found that the coverage probabilities of the 80% and 90% confidence bounds for Z_2 were 0.8093 and 0.9113, in both cases exceeding the nominal confidence level.

5. Efficiency. The notion of asymptotic efficiency we discuss here is that of van de Vaart (1988, 1991). Generally, suppose that $\mathscr{P}$ is a set of probability measures defined on a measurable space $(S, \mathscr{A})$ and that we estimate $\kappa(P)$, where P is a member of $\mathscr{P}$ and κ is a functional taking values in a Banach space B_1. More precisely, based on a random sample $X_1, X_2, \dots, X_n$ from P, we form an estimator sequence T_n, that is a measurable map of the data vector, $T_n = s_n(X_1, X_2, \dots, X_n)$, where s_n: $(S^n, \mathscr{A}^n) \to (B_1, \mathscr{E})$ is measurable and $\mathscr{E}$ is a σ-field of sets on B_1.

Let now ϕ: $B_1 \to B_2$ be another functional, where B_2 is a Banach space again. The main result of van de Vaart (1991) is that if T_n is asymptotically efficient for $\kappa(P)$, then $\phi(T_n)$ is asymptotically efficient for $(\phi \circ \kappa)(P)$ provided that ϕ and κ are "smooth" enough, in an appropriate sense. Smoothness of ϕ can be made precise by considering a weak version of Hadamard differentiability, as discussed in Gill (1989). Building on van der Vaart's (1991) results, and utilizing the fact that the map Ψ of the previous sections is differentiable, we prove asymptotic efficiency for our estimator $\hat{Z}_n$. We use the same framework as the one we considered to prove consistency and asymptotic normality; we regard the tail of the (unknown) probability distribution function F, $1_{[0,\infty)} - F$, as an element of a space $D_{\alpha\beta}$, while the sequence $\{1_{[0,\infty)} - \hat{F}_n\}$ plays the role of the estimator sequence, T_n.

DEFINITION 5.1. *For $\beta \geq 0$, let $\mathscr{P}_\beta$ be the set of probability measures P on the Borel σ-field of the real line with $\int |x|^{2\beta}\, dP(x) < \infty$. For such a P in $\mathscr{P}_\beta$, we define*

$$\mathscr{T}_\beta(P) = \left\{ g \in L^2(P)\colon \int g\, dP = 0,\ \int |x|^{2\beta} g^2(x)\, dP(x) < \infty \right\}.$$

Let now $\mathscr{P}_\beta(P)$ be the collection of all maps $(0,1) \to \mathscr{P}_\beta$, $t \to P_t$, such that, as $t \to 0$,

$$\int \left[\frac{1}{t}\left\{ (dP_t)^{1/2} - (dP)^{1/2} \right\} - \frac{1}{2} g (dP)^{1/2} \right]^2 \to 0, \tag{28}$$

for some function $g \in \mathscr{T}_\beta(P)$, and

$$\sup_t \int |x|^{2\beta}\, dP_t(x) < \infty.$$

We also define a map $\kappa\colon \mathscr{P}_\beta \to D_{\beta\beta}$ with $\kappa(P) = 1_{[0,\infty)} - F$, where F is the probability distribution function associated with the measure P.

The definition is similar to that in Pitts (1994a), on noting that in that paper, the probability measures considered are concentrated on $(0, \infty)$. Using Definition 5.1 and van der Vaart [(1988), Lemma 5.21], we arrive at the following which gives the required smoothness condition for κ, the map defined above.

LEMMA 5.2. *Let β be in $\mathbb{R}_+$. For any $P \in \mathscr{P}_\beta$, there exists a bounded linear map $\kappa'_P\colon \mathscr{T}_\beta(P) \to D_{\beta\beta}$ such that*

$$\frac{1}{t}(\kappa(P_t) - \kappa(P)) \to \kappa'_P(g) \quad \text{in } D_{\beta\beta},$$

for every path in $\mathscr{P}_\beta(P)$ satisfying (28). *Further, $\kappa'_P(g) = \int 1_{(\cdot,\infty)} g\, dP$.*

We now define efficiency of an estimator in our context, following van der Vaart (1991). Note that, without loss of generality, we may replace the continuous paths $\{P_t\}$ in (28) by sequences $\{P_n\}$ (setting, e.g., $n = t^{-2}$) as in Groeneboom and Wellner [(1992), page 18]. We then write P_n instead of $P_{n^{-1/2}}$ to simplify the notation. Let T_n be an estimator sequence for κ at P in the space $D_{\beta\beta}$. Assume that for every sequence $\{P_n\}$ of probability measures which satisfies (28) for some function g, it holds that, under P_n,

$$n^{1/2}\left[T_n - \kappa(P_n)\right] \to_d L, \tag{29}$$

where L is a tight probability measure on $D_{\beta\beta}$ that does not depend on the function g. Then there exists a tight Borel measure N_P on this space with

$N_P[\overline{\kappa'_P\{T(P)\}}] = 1$ and $N_P \circ b^{*-1} = N(0, \|\tilde{\kappa}_{P,b^*}\|_P^2)$, for all $b^* \in D^*_{\beta\beta}$, the dual space of $D_{\beta\beta}$; here, $\tilde{\kappa}_{P,b^*}$ is the *gradient* of κ in direction b^* [see, e.g., van der Vaart (1991)], and $\| \ \|_P$ denotes the L^2-norm with respect to the reference measure P. Moreover, there exist independent random elements G and W into the space $D_{\beta\beta}$ with its Borel σ-field, such that $\mathscr{L}(G) = N_P$, $\mathscr{L}(W)$ is tight and $L = \mathscr{L}(G+W)$; here, $\mathscr{L}(X)$ denotes the law of a random element X in a metric space. An estimator sequence T_n is called (asymptotically) efficient for κ at P if it satisfies (29) with $L = N_P$.

Let now $T_n = 1_{[0,\infty)} - \hat{F}_n$. We have the following, which gives efficiency of our input estimator. This is similar to Pitts [(1994a), Lemma 5.2]; see that paper for a proof (notice that the continuity condition for F has been dropped here, however).

LEMMA 5.3. *Let $\gamma > \beta > 0$ and assume that $\int |x|^{2\gamma}\, dF(x) < \infty$. Then T_n is efficient for κ at P in $D_{\beta\beta}$ (relative to $\mathscr{T}_\beta(P)$).*

Recall the definition of Hadamard differentiability from Section 4.1. If S is a subspace of the Banach space B_1 there and (25) holds only for sequences h_n with $h_n \to h \in S$, then the map ψ is said to be Hadamard differentiable (at x) *tangentially* to S [see Gill (1989) for details of this concept]. A simple inspection of the arguments in the proof of Theorem 4.2 shows that the functional Ψ with $\Psi(1_{[0,\infty)} - F) = z \star U$ is Hadamard differentiable (along certain curves) tangentially to the space $C_{\beta\beta}(F)$. Further, it follows from Theorem 2.1 of van der Vaart and Lemma 2.2 that $\overline{\kappa'_P\{\mathscr{T}_\beta(P)\}} \subset C_{\beta\beta}(F)$. Lemma 5.3 and a slight variation of van der Vaart [(1991), Theorem 3.1] to take account of (7), together yield the following.

THEOREM 5.4. *Consider a renewal-type equation $Z = z + Z \star F$ on the real line where F is a spread-out probability distribution with a positive first moment, $\int |x|^{2\gamma}\, dF(x) < \infty$, where $\gamma > \beta > 1$, and β, T, z and $\hat{z}_n$ are as in Theorem 4.2. Then $\Psi(T_n)$ is efficient for Z in the space D_∞ [relative to the space $\mathscr{T}_\beta(P)$].*

The result is in the same spirit as Pitts [(1994a), Theorem 5.3], who proved asymptotic efficiency of an estimator for the waiting time distribution in a $GI/G/1$ queue. In general, in the function space framework of Grübel and Pitts (1993) which we have followed throughout, once differentiability is established for a "stochastic functional," asymptotic efficiency follows along with the other statistical properties we have considered so far. For our functional Ψ, the main effort in a particular renewal-type equation is to prove that the map T is differentiable, since differentiability of Ψ then follows as we have seen in the previous sections. The next theorem illustrates this, and complements the results of Grübel and Pitts (1993).

THEOREM 5.5. *Let F be a nonlattice probability distribution function with a positive first moment, P be the associated probability measure and $\alpha > 2$.*

Assume that, for a $\gamma > 2\alpha$, $\int |x|^\gamma \, dF(x) < \infty$. *Then the empirical renewal function* $\hat{U}_n$ *is efficient for* U *in the space* $D_{0,-1}$ [*relative to* $\mathcal{T}_\alpha(P)$].

PROOF. Grübel and Pitts [(1993), Proposition 3.14] have shown that the functional Φ defined in a suitable subset of $D_{\alpha\alpha}$ which associates the tail of a distribution function F to the renewal function $U \in D_{0,-1}$ corresponding to F, is differentiable at $1_{[0,\infty)} - F$ for all F satisfying the conditions of the theorem. The result now follows, in a similar fashion with Theorem 5.4, from van der Vaart [(1991), Theorem 3.1] and Lemma 5.3. □

Note in particular that the spread-out assumption for F has been relaxed here. This is so because for the results of Grübel and Pitts (1993), this assumption is unnecessary. Compared with the results of the current paper, it appears that the relaxed assumptions for F there seem to be a consequence of the monotonicity of the renewal function U.

A different approach for Theorem 5.5 might be based on Example (a) of Section 3.2, where a normalized version of the renewal function is seen to be a solution of a renewal-type equation. This would reduce the moment conditions on F, but would require F to be a spread-out distribution.

Finally, in Example (b) of Section 3.2 for an alternating renewal process, we have chosen the function $\tilde{F}_n$ rather than the empirical distribution $\hat{F}_n$ as an estimator for the distribution F there. The following result justifies this choice as it shows that the resulting nonparametric estimator for Z is efficient.

THEOREM 5.6. *Let* $\alpha > \beta > 1$ *and assume that the working and repair time distributions* F_W *and* F_R *in an alternating renewal process satisfy*

$$\int_{[0,\infty)} x^{2\alpha} \, dF_W(x) < \infty, \qquad \int_{[0,\infty)} x^{2\alpha} \, dF_R(x) < \infty,$$

and that $F = F_W \star F_R$ *is spread out. Then* $\tilde{Z}_n$ *is efficient for the solution* Z *of* (3) *in the space* D_∞ [*relative to* $\mathcal{T}_\beta(P)$].

APPENDIX

Here we collect some standard results on functions of bounded variation on the real line and define the convolution operator $\star$ which is used throughout the paper.

Recall that a function $H: \mathbb{R} \to \mathbb{R}$ is of bounded variation on the line if and only if it can be represented as the difference between two nondecreasing bounded functions. We write $H = H_1 - H_2$ for the Jordan decomposition of H, and we may assume without loss of generality that both H_1 and H_2 are nonnegative. When we speak of BV functions, we shall always assume implicitly that they are right-continuous and that $\lim_{x\to-\infty} H(x) = 0$. If g is a Borel measurable real-valued function, the integral $\int g(x)\, dH(x)$ is then the Lebesgue–Stieltjes integral defined by $\int g(x)\, dH(x) = \int g(x)\, dH_1(x) - \int g(x)\, dH_2(x)$.

It is well known [see, e.g. Gelfand, Raikov and Shilov (1964), Section 31] that to any complex-valued function H which is BV there corresponds uniquely a complex measure μ_H. If H is a real-valued function, then μ_H is a *signed measure* and this is the case we consider here. To this measure μ_H we associate a positive measure $|\mu_H|$, the total variation measure of μ_H, defined by

$$|\mu_H|(E) = \sup \sum_i |\mu_H(E_i)|,$$

where the supremum is taken over all partitions $\{E_i\}$ of the set E; see Rudin [(1987), Chapter 6]. The following result is well known; see, for example, Rudin [(1987), Theorems 6.2 and 6.4]. Let $H: \mathbb{R} \to \mathbb{R}$ be a function which is BV and $|\mu_H|$ be the total variation measure associated with μ_H. Then $|\mu_H|$ is a positive measure such that $|\mu_H|(\mathbb{R}) < \infty$.

REMARK. Since $|\mu_H|(E) \geq |\mu_H(E)|$ for any set E, with our notation, every signed measure on the Borel σ-field associated with a BV function is, in fact, finite. In the literature, a signed measure is sometimes defined as any σ-additive set function, and our notion of a signed measure coincides with what is referred to as a *finite signed measure*.

For various proofs in the main body of the text, we need to define convolution with respect to nonmonotonic BV functions, as well as with respect to nondecreasing functions.

DEFINITION A.1. Let $\mathscr{H}$ be the set of all right-continuous functions $H: \mathbb{R} \to \mathbb{R}$ with $\lim_{x\to-\infty} H(x) = 0$ and such that H satisfies one of the following conditions:

(a) H is nondecreasing.
(b) H is of bounded variation on $\mathbb{R}$.

Let $g: \mathbb{R} \to \mathbb{R}$ be a measurable function such that the functions $y \to g(x-y)$ are H-integrable for all $x \in \mathbb{R}$. Then we define the convolution $g \star H$ as the real-valued function

$$(g \star H)(x) = \int g(x-y)\, dH(y), \qquad x \in \mathbb{R}.$$

This extends Grübel and Pitts [(1993) Definition 3.1], who considered convolutions with respect to a (possibly, unbounded) nondecreasing function, such as the renewal function.

The following readily proved lemma lists some elementary properties of the operator $\star$ defined above.

LEMMA A.2. (i) *Let H, J belong to $\mathscr{H}$. Then we have*

$$H \star J = J \star H,$$

in the sense that if one exists, then the other exists as well and they are equal.

(ii) *Let g be such that, for $H, J \in \mathscr{H}$, $g \star H$ and $g \star J$ exist. Then*

$$(g \star H) \star J = (g \star J) \star H,$$

in the sense that if one exists, so does the other and they are equal.

(iii) *Let J be a function of bounded variation and g be any function in D_∞. Then $g \star J$ belongs to D_∞ and the following holds:*

$$\|g \star J\|_\infty \le \|g\|_\infty |\mu_J|(\mathbb{R}) < \infty,$$

where $|\mu_J|$ is the total variation measure associated with the function J.

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