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MARGINAL DENSITIES OF THE LEAST CONCAVE MAJORANT OF BROWNIAN MOTION

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A clean, closed form, joint density is derived for Brownian motion, its least concave majorant, and its derivative, all at the same fixed point. Some remarkable conditional and marginal distributions follow from this joint density. For example, it is shown that the height of the least concave majorant of Brownian motion at a fixed time point has the same distribution as the distance from the Brownian motion path to its least concave majorant at the same fixed time point. Also, it is shown that conditional on the height of the least concave majorant of Brownian motion at a fixed time point, the left-hand slope of the least concave majorant of Brownian motion at the same fixed time point is uniformly distributed.

1. Introduction. In order restricted inference, many estimators can be expressed as functionals of least concave majorants of functions. This is primarily due to the fact that an antitonic regression problem can be solved in terms of a least concave majorant of a cumulative sum diagram [see Barlow et al. (1972) for clarification]. Two famous examples of such estimators are the Grenander estimators of the distribution function and the density function in the monotone density problem. Grenander (1956) showed that the maximum likelihood estimator of a monotone decreasing density defined on $\Re^+$ is the left-continuous density associated with the distribution function given by the least concave majorant of the empirical distribution function. Thus, the Grenander estimator of the distribution function is the least concave majorant of the empirical distribution function. Prakasa Rao (1969) and Groeneboom (1985) have discussed some of the asymptotics for the Grenander estimator at a fixed point where the slope of the true density is negative. Groeneboom (1985) points out that when, in fact, the density function is uniform, the properly normalized Grenander estimator of the density at a fixed point converges in distribution to the left-hand slope of the least concave majorant of a Brownian bridge at a specified point. Similarly, if the true density is uniform, the convergence of the properly normalized Grenander estimator of the distribution function is to the least concave majorant of a Brownian bridge, properly time-transformed.

Other examples where the estimator is based upon a least concave majorant include nonparametric maximum likelihood estimators of distribution functions under simple stochastic ordering in the continuous case [see Brunk et al. (1966), Robertson and Wright (1981) and Dykstra (1982)], and maxi-

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mum likelihood estimators of distribution functions and density functions under likelihood ratio ordering in the continuous case [see Dykstra et al. (1995)]. A substantial amount of work has also been done on estimation problems involving monotone hazard rates (IFR and DFR) which can be phrased in terms of least concave majorants [Marshall and Proschan (1965)]. Additional estimators based upon least concave majorants can be found in Robertson et al. (1988). For boundary cases–cases where equality in fact holds, the limiting behavior of the aforementioned estimators is typically described in terms of functionals of the least concave majorant of a Brownian bridge process. Note that uniformity in the monotone density problem is a boundary case. Hence, in order to understand the limiting behavior of these estimators under a boundary case, it is useful to have some understanding of the behavior of the least concave majorant of a Brownian bridge process.

Groeneboom (1983) has presented many remarkable facts about the least concave majorant of Brownian motion in a wide ranging and comprehensive effort. In particular, Groeneboom has shown that Brownian motion is determined by its least concave majorant vertex points [finite in every finite, closed interval of $(0,\infty)$] in the sense that given the vertex points, independent (rescaled) Brownian excursions (Brownian bridges conditioned to be nonpositive) fitted between the vertex points give sample paths which are distributionally equivalent to those of Brownian motion. In his paper, Groeneboom defines a pure jump process with independent, non-stationary increments and right continuous, non-decreasing paths $\tau(a)$ that is closely related to the slope process $S(t)$ of the least concave majorant of Brownian motion. From this process, he is able to derive the form of the density of $S(t)$ at single values of t for both Brownian motion and the Brownian bridge.

Pitman (1983) has published results closely related to those of Groeneboom concerning the distribution of the vertex points of the least concave majorant of Brownian motion and has expressed his results in a very appealing manner. To be more precise, Pitman has shown that after placing an increasing tangent line to the least concave majorant of Brownian motion, the sequence of slopes of the concave majorant following the tangent point forms a Markov chain with uniform transition probabilities (with comparable results for the slopes preceding the tangent point). Moreover, the slopes preceding the tangent point, and the slopes following the tangent points are independent, and conditional on all the slopes, the segment lengths of the least concave majorant have independent gamma distributions.

A key tool underlying the results of both Groeneboom and Pitman is the Brownian path decomposition results of Williams (1974). In particular, suppose that a linear function bt, $b > 0$, is subtracted from the Brownian motion $\{W(t) : t \geq 0\}$ to give negative drift. Then if γ_b is the last time that the negative drifting Brownian motion $\{W(t) - bt : t \geq 0\}$ achieves its overall maximum, the processes

$$\{W(t) : 0 \leq t \leq \gamma_b\} \text{ and } \{W(\gamma_b + s) - W(\gamma_b) - bs : s \geq 0\}$$

are independent. Williams shows that the latter process can be identified with a three dimensional Bessel process with drift b.

This path decomposition is very closely associated with the independent increments property of the $\{\tau(a) : a > 0\}$ process discussed in Groeneboom(1983). Bass(1983) has discussed results similar to Groeneboom's concerning the Brownian excursions between the vertex points of the least concave majorant by using the decomposition of general Markov processes at splitting times.

In the following section we develop some notation and state some rescaling and transformation properties of the least concave majorant of Brownian motion. Section 3 is the major contribution of this paper. In Section 3, we give a closed-form, tri-variate density of the least concave majorant of Brownian motion at time $t = 1$, its derivative at $t = 1$, and the distance from the Brownian motion path to its least concave majorant at $t = 1$. This tri-variate density is amazingly sparse. Results of Section 2 will imply the behavior of the least concave majorant of Brownian motion, or the least concave majorant of a Brownian bridge, at any fixed timepoint t is a simple transformation of this tri-variate density. Marginal distributions of this tri-variate distribution are also closed-form and clean. In Section 4, we give some distributions of locations and hitting times associated with the least concave majorant processes. Finally, in Section 5, we compare and contrast the limiting processes of the maximum likelihood estimators of an unknown distribution function known to satisfy, respectively, a simple stochastic ordering and a likelihood ratio ordering with a standard uniform distribution. By results of Section 3, these limiting processes have identical marginal distributions at the same time point, but are very different as processes.

2. Properties of the LCM of Brownian motion. We begin with notation. Suppose that $\{W(t) : t \geq 0\}$ is a standard Brownian motion process and $\{W_0(t) : 0 \leq t \leq 1\}$ is a Brownian bridge process. Let $\{K(t) : t \geq 0\}$ denote the least concave majorant over the positive halfline of the process $\{W(t) : t \geq 0\}$ and let $\{K_0(t) : 0 \leq t \leq 1\}$ denote the least concave majorant over the unit interval of the process $\{W_0(t) : 0 \leq t \leq 1\}$. See Figure 1 for a realization of W_0 and its corresponding least concave majorant K_0, paying special attention to the slopes and heights of K_0 and the distance between K_0 and W_0. We now state some interesting theorems regarding the process $\{K(t) : t \geq 0\}$ and the process $\{K_0(t) : 0 \leq t \leq 1\}$.

THEOREM 2.1. *Let $\{W(t) : t \geq 0\}$ be a standard Brownian motion process. Just as*

$$\{W(t) : t \geq 0\} \stackrel{d}{=} \left\{\sqrt{x}\, W\left(\frac{t}{x}\right) : t \geq 0\right\}$$

holds, then

$$\{K(t) : t \geq 0\} \stackrel{d}{=} \left\{\sqrt{x}\, K\left(\frac{t}{x}\right) : t \geq 0\right\}$$

also holds, for fixed $x > 0$.

FIG. 1. *A linearly interpolated realization of a Brownian bridge path at* 1001 *equally spaced points in the interval* $[0, 1]$ *with the least concave majorant of the linearly interpolated Brownian bridge path. The circles represent locations where the two paths touch.*

Thus, a properly linearly time-transformed, rescaled least concave majorant of Brownian motion process is also a least concave majorant of a Brownian motion process. The above theorem is proved quite easily by use of the following lemma which gives an expression for the value of a least concave majorant at a specified point.

LEMMA 2.2. *Suppose g is a function defined on a set containing the interval $[a, b]$, where a can equal $-\infty$ and/or b can equal ∞. Let g^* denote the least concave majorant taken over the interval $[a, b]$ of g. Then for $t \in [a, b]$,*

$$g^*(t) = \sup_{a\le u\le t}\sup_{t\le v\le b}\left\{\frac{(v-t)\,g(u)+(t-u)\,g(v)}{v-u}\right\}$$

where $\frac{0}{0}$ is defined as $g(t)$ when $u = v = t$.

The preceding lemma is based upon the principle that the least concave majorant of a function can be determined by considering all possible secant segments, a secant segment being a line segment connecting any two points of the function. The height of the least concave majorant at a fixed point will be the supremum of the heights at that point of all appropriate (defined at the fixed point) secant segments. This lemma also implies that the least concave majorant of the sum of any function f and a linear function l is equal to the sum of the linear function l and the least concave majorant of f.

THEOREM 2.3. *Let* $\{W(t) : t \geq 0\}$ *be a standard Brownian motion process and let* $\{W_0(t) : 0 \leq t \leq 1\}$ *be a Brownian bridge process. Just as*

$$\{W(t) : t \geq 0\} \stackrel{d}{=} \left\{(1+t)\, W_0\left(\frac{t}{1+t}\right) : t \geq 0\right\}$$

and

$$\{W_0(t) : 0 \leq t \leq 1\} \stackrel{d}{=} \left\{(1-t)\, W\left(\frac{t}{1-t}\right) : 0 \leq t \leq 1\right\}$$

holds, then

$$\{K(t) : t \geq 0\} \stackrel{d}{=} \left\{(1+t)\, K_0\left(\frac{t}{1+t}\right) : t \geq 0\right\}$$

and

$$\{K_0(t) : 0 \leq t \leq 1\} \stackrel{d}{=} \left\{(1-t)\, K\left(\frac{t}{1-t}\right) : 0 \leq t \leq 1\right\}$$

holds.

These equalities (in distribution) show how we can transform the least concave majorant of a standard Brownian motion process into the least concave majorant of a Brownian bridge process and vice versa. The transformation will be referred to as *Doob's transformation of the concave majorants*.

3. Marginal densities of the LCM of Brownian motion. To aid in the understanding of the behavior of the processes $\{K(t) : t \geq 0\}$ and $\{K_0(t) : 0 \leq t \leq 1\}$, we examine a joint distribution which will describe the marginal behavior of these processes. The joint distribution we will consider is that of the triple $(K(1), K(1) - W(1), K'(1))$ where $K(1)$ is the height of the least concave majorant of the Brownian motion process at time 1, $K(1) - W(1)$ is the distance between the Brownian motion process and its least concave majorant at time 1, and $K'(1)$ is defined to be the left-hand slope at time 1 of the least concave majorant of the Brownian motion process. Equivalently, $K'(1)$ is almost surely the derivative of the process $\{K(t) : t \geq 0\}$ at time 1. This joint distribution will illuminate the behavior of the process $\{K(t) : t \geq 0\}$ about some open neighborhood of time 1. Properties of the least concave majorant of Brownian motion stated in the previous section will imply the joint distributions of $(K(x), K(x) - W(x), K'(x))$, $x > 0$, and $(K_0(x_0), K_0(x_0) - W_0(x_0), K_0'(x_0))$, $0 < x_0 < 1$, are simple transformations of the joint distribution of $(K(1), K(1) - W(1), K'(1))$. Specifically

$$(K(x), K(x) - W(x), K'(x)) \stackrel{d}{=} \sqrt{x}\left(K(1), K(1) - W(1), \frac{1}{x} K'(1)\right)$$

by Theorem 2.1. Similarly,

$$\begin{aligned}&(K_0(x_0), K_0(x_0) - W_0(x_0), K_0'(x_0))\\ &\quad\stackrel{d}{=} \sqrt{x_0(1-x_0)}\left(K(1), K(1) - W(1), \frac{1}{1-x_0}\left(\frac{1}{x_0}K'(1) - K(1)\right)\right)\end{aligned}$$

by Theorem 2.3. These transformations will be discussed in greater detail later. We now give the joint density of $(K(1), K(1) - W(1), K'(1))$, followed by its proof. The authors of this paper originally derived this result using only linear boundary crossing probabilities of Brownian motion along with the fact that Brownian motion is a Gaussian process with the Markov property. The referee suggested the more succinct proof which we provide.

THEOREM 3.1. *The joint density of* $(K(1), K(1) - W(1), K'(1))$ *is given by*

$$g_{K(1),K(1)-W(1),K'(1)}(y_1, y_2, a) = 4y_2(y_1 + y_2)\phi(y_1 + y_2),\ 0 \le a \le y_1, y_2 \ge 0$$

where ϕ *is the standard normal density.*

PROOF. By (2.22) in Groeneboom (1983), the joint density of the random vector $(N_t^-, N_t^+, Y(t), W(N_t^-), W(N_t^+))$ at (t_1, t_2, y, x_1, x_2) is given by

$$\frac{4a(x_1 - at_1)_+}{t_1(t-t_1)(t_2-t)}\frac{y_+^2}{\sigma}\phi\left(\frac{y}{\sigma}\right)\frac{\phi\left(\frac{x_1}{\sqrt{t_1}}\right)\phi\left(\frac{x_2-x_1}{\sqrt{t_2-t_1}}\right)}{\sqrt{t_1(t_2-t_1)}}$$

where ϕ denotes the standard normal density, $\sigma^2 = (t_2 - t)(t - t_1)/(t_2 - t_1)$, N_t^- and N_t^+ the jump times of the slope process preceding and following t respectively, and $Y(t) = K(t) - W(t)$.

Denote the value of $K(t)$ by x and the value of $K'(t)$ by a. Make the change of variables $(x_1, x_2) \to (a, x)$:

$$x = \frac{(t_2 - t)x_1 + (t - t_1)x_2}{t_2 - t_1}, \qquad a = \frac{x_2 - x_1}{t_2 - t_1}.$$

The absolute value of the Jacobian of this transformation is $(t_2 - t_1)$. Thus, the density of $(N_t^-, N_t^+, Y(t), K(t), K'(t))$ at (t_1, t_2, y, x, a) is given by

$$\frac{4a(x - at) + }{t_1\sqrt{t_1(t_2 - t_1)}}\frac{y_+^2}{\sigma^3}\phi\left(\frac{y}{\sigma}\right)\phi\left(\frac{x - a(t - t_1)}{\sqrt{t_1}}\right)\phi\left(a\sqrt{t_2 - t_1}\right),$$

noting that $x_1 - at_1 = x - at$.

Now let $t = 1$, $U = (1 - N_1^-)/N_1^-$, and $V = N_1^+ - 1$. Moreover, let $g(u, v, y, x, a)$ be the density of $(U, V, Y(1), K(1), K'(1))$. Then the density of

$(Y(1), K(1), K'(1))$ is given by

$$\int_0^\infty \int_0^\infty g(u,v,y,x,a)\,du\,dv$$
$$= 4a(x-a)y^2 \exp\left[-\frac{1}{2}(x^2+y^2)\right]$$
$$\times \int_0^\infty \int_0^\infty \frac{u+v+uv}{(2\pi uv)^{3/2}} \exp\left[-\frac{1}{2}\left\{y^2\left(\frac{1}{u}+\frac{1}{v}\right)+(x-a)^2u+a^2v\right\}\right]du\,dv.$$

The result follows from two well known relations from the theory on passage times of Brownian motion. If T_a is the first passage time

$$T_a = \min\{t : W(t) = a\}, \qquad a > 0$$

with density

$$f_{T_a}(u) = \frac{a}{\sqrt{2\pi u^3}} \exp\left\{-\frac{a^2}{2u}\right\}, \qquad u > 0$$

where W is a standard Brownian motion, then its Laplace transform is

$$E[\exp\{-\lambda T_a\}] = \int_0^\infty \frac{a}{\sqrt{2\pi u^3}} \exp\left\{-\lambda u - \frac{a^2}{2u}\right\} = \exp\left\{-\sqrt{2\lambda}a\right\}, \qquad \lambda \geq 0$$

see Itô and McKean [(1974), relation (2) on page 25 and relation (5) on page 26]. Taking $\lambda = a_1/2$ and $a = \sqrt{a_2}$, we obtain the first formula given by

$$\int_0^\infty (2\pi x)^{-1/2} x^{-1} \exp\left\{-\tfrac{1}{2}(a_1 x + a_2/x)\right\} dx = \exp\{-\sqrt{a_1 a_2}\}/\sqrt{a_2}$$

and from this we obtain the second formula through differentiation with respect to a_1 which is given by

$$\int_0^\infty (2\pi x)^{-1/2} \exp\left\{-\tfrac{1}{2}(a_1 x + a_2/x)\right\} dx = \exp\{-\sqrt{a_1 a_2}\}/\sqrt{a_1}$$

for $a_1, a_2 > 0$. □

The joint density of Theorem 3.1 given above is amazingly simple. In addition, some rather remarkable joint, marginal, and conditional distributions follow from the joint density of $(K(1), K(1) - W(1), K'(1))$. The remainder of this section is dedicated to these marginal, joint, and conditional distributions. We first consider the joint distribution of $(K(1), K(1) - W(1))$.

THEOREM 3.2. *The joint density function of* $(K(1), K(1) - W(1))$ *is given by*

$$f_{K(1),K(1)-W(1)}(y_1, y_2) = 4y_1 y_2 (y_1 + y_2)\,\phi(y_1 + y_2), \qquad y_1, y_2 \geq 0$$

where ϕ *is the standard normal pdf.*

From this joint distribution, we obtain the following remark.

REMARK 3.3. We have the following relation:

$$K(1) - W^{+}(1) = \min(K(1), K(1) - W(1))$$

and the density of the quantity $\min(K(1), K(1) - W(1))$ is given by

$$f(y) = 8y\overline{\Phi}(2y) + 8y^2\phi(2y), \; y \geq 0$$

where $W^{+}(t) \equiv W(t) \cdot I(W(t) > 0)$, and $\overline{\Phi}$ is the standard normal survival function.

It is apparent from the joint density given in Theorem 3.2 that the marginal distributions of $K(1)$ and $K(1) - W(1)$ are equal. Let us now look at the marginal distributions of $K(1)$ and $K(1) - W(1)$.

THEOREM 3.4. *The marginal distribution function and marginal density function of* $K(1)$ *and* $K(1) - W(1)$ *are given respectively by*

$$\begin{aligned} F_{K(1)}(y) &= P(K(1) \leq y) = F_{K(1)-W(1)}(y) = P(K(1) - W(1) \leq y) \\ &= 1 - 2\left(1 - y^2\right)\overline{\Phi}(y) - 2y\phi(y), \; y \geq 0 \end{aligned}$$

and

$$f_{K(1)}(y) = f_{K(1)-W(1)}(y) = 4y\overline{\Phi}(y), \; y \geq 0$$

where $\overline{\Phi}$ *is the standard normal survival function and* ϕ *is the standard normal pdf.*

See Figure 2 for a graph of the density of $K(1)$. We give the moments of $K(1)$ in the following remark.

REMARK 3.5. The moments of $K(1)$ are given by

$$E\left[K(1)^r\right] = \begin{cases} (r-1)(r-3)\cdots 1\left[2\dfrac{r+1}{r+2}\right], & \text{if } r \geq 0, \text{ even}, \\ (r-1)(r-3)\cdots 2\left[\dfrac{4}{\sqrt{2\pi}}\dfrac{r+1}{r+2}\right], & \text{if } r \geq 1, \text{ odd}, \end{cases}$$

with mean and variance given respectively by $\frac{4}{3}\sqrt{\frac{2}{\pi}} = 1.06385$ and $\frac{3}{2} - \frac{32}{9\pi} = 0.36823$.

REMARK 3.6. Given $x \geq 0$ and $0 \leq x_0 \leq 1$, then

$$(K(x), K(x) - W(x)) \stackrel{d}{=} \sqrt{x}\,(K(1), K(1) - W(1))$$

and

$$(K_0(x_0), K_0(x_0) - W_0(x_0)) \stackrel{d}{=} \sqrt{x_0(1 - x_0)}\,(K(1), K(1) - W(1))$$

by Theorems 2.1 and 2.3.

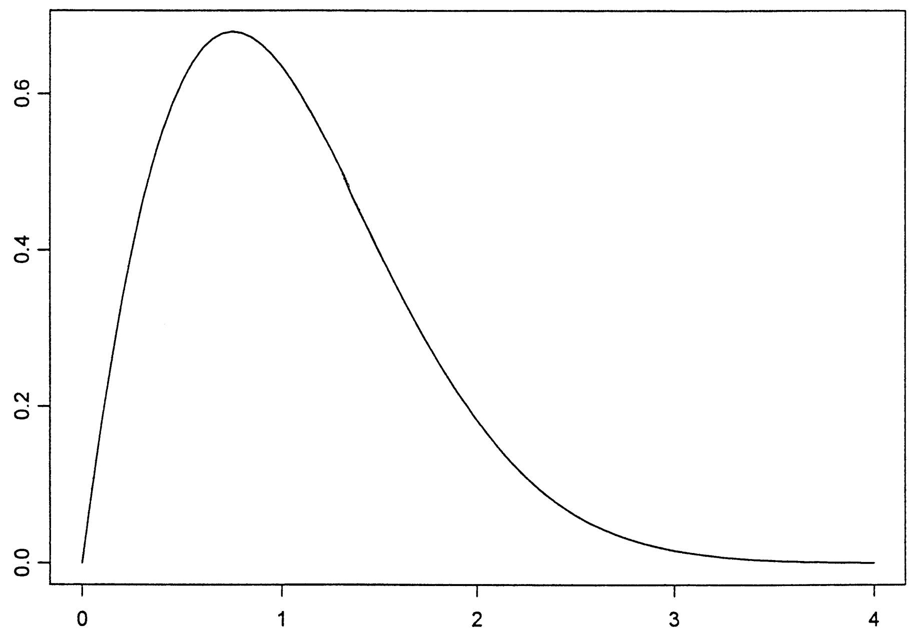

FIG. 2. *A graph of the density* $f_{K(1)}(y) = 4y\overline{\Phi}(y)$, $y > 0$, *over the interval* $[0, 4]$.

Theorem 3.4 and the preceding remark tell us that the process $\{K_0(t) : 0 \le t \le 1\}$ and the process $\{K_0(t) - W_0(t) : 0 \le t \le 1\}$ have the same marginal distributions. However, the two processes are very different as processes. A more in depth comparison is provided in a later section where it is shown that these two processes are competing limiting distributions in the estimation of cumulative distribution functions (cdfs) under different stochastic orderings in the one sample case. We will now turn our attention to the joint distribution of the pair $(K(1), K'(1))$.

THEOREM 3.7. *The joint density of* $(K(1), K'(1))$ *is given by*

$$f_{K(1),K'(1)}(y, a) = 4\overline{\Phi}(y),\ 0 \le a \le y$$

where $\overline{\Phi}$ *is the standard normal survival function.*

Rather amazingly, the argument for the slope in the joint density for $(K(1), K'(1))$ only appears in the support. This leads us to the following remark.

REMARK 3.8. Given $x > 0$,

$$(K(x), K'(x)) \stackrel{d}{=} \left(\sqrt{x}K(1), \frac{1}{\sqrt{x}}K'(1)\right)$$

by Theorem 2.1. Then, by Theorem 3.10,

$$K'(x)\,\big|[K(x)=y] \overset{d}{=} U\left(0, \frac{y}{x}\right)$$

and

$$\frac{K'(x)}{K(x)} \overset{d}{=} U\left(0, \frac{1}{x}\right).$$

Therefore, given $K(x)=y$, $K'(x)$ is uniformly distributed over all its possibilities.

Theorem 3.7 and the preceding remark state the behavior of the least concave majorant of Brownian motion over some open neighborhood of x. In a broader context, consider the following physical interpretation. Fix $x>0$ and let W be a random standard Brownian motion path. Suppose we start with a line L which majorizes W (almost surely there exists such a line). We will allow the line L to pivot about x and/or to slide vertically. If we pull the line down at x as far as possible, keeping in mind that W will act as an obstacle, then Theorem 3.7 and the preceding two remarks tell us the distribution of the resulting position of the line. $K(x)$ is the height of the final line at x and $K'(x)$ is the slope of the line.

We now turn our attention to the joint distribution of $(K_0(x_0), K_0'(x_0))$, $0<x_0<1$. The following theorem describes in detail how we can transform the joint distribution of $(K(1), K'(1))$ to obtain the joint distribution of $(K_0(x_0), K_0'(x_0))$. The joint density of $(K_0(x_0), K_0'(x_0))$ is given in the theorem immediately following.

REMARK 3.9. Given $0<x_0<1$,

$$\begin{aligned}
&(K_0(x_0), K_0'(x_0)) \\
&\quad\overset{d}{=} \left[(1-x_0)\left(\frac{x_0}{1-x_0}\right), \frac{\partial}{\partial t}\left\{(1-t)K\left(\frac{t}{1-t}\right)\right\}\Big|_{t=x_0}\right] \\
&\quad\overset{d}{=} \left[(1-x_0)\left(\frac{x_0}{1-x_0}\right), \frac{1}{1-x_0}K'\left(\frac{x_0}{1-x_0}\right) - K\left(\frac{x_0}{1-x_0}\right)\right] \\
&\quad\overset{d}{=} \left[\sqrt{x_0(1-x_0)}K(1), \frac{1}{\sqrt{x_0(1-x_0)}}K'(1) - \sqrt{\frac{x_0}{1-x_0}}K(1)\right] \\
&\quad\overset{d}{=} \sqrt{1-x_0}\left[\sqrt{x_0}K(1), \frac{1}{1-x_0}\left(\frac{1}{\sqrt{x_0}}K'(1) - \sqrt{x_0}K(1)\right)\right] \\
&\quad\overset{d}{=} \sqrt{1-x_0}\left[K(x_0), \frac{1}{1-x_0}\left(K'(x_0) - K(x_0)\right)\right]
\end{aligned}$$

by Theorems 2.3 and 2.1.

THEOREM 3.10. *Given $0 < x_0 < 1$, the joint distribution of $(K_0(x_0), K_0'(x_0))$ is given by*

$$f_{K_0(x_0),K_0'(x_0)}(y,a) = 4\overline{\Phi}\left(\frac{y}{\sqrt{x_0(1-x_0)}}\right), \ -\frac{1}{1-x_0}y \le a \le \frac{1}{x_0}y, \qquad y \ge 0$$

where $\overline{\Phi}$ is the standard normal survival function.

Again, the argument for the slope in the joint density for $(K_0(x_0), K_0'(x_0))$ only appears in the support. This leads us to the following remark.

REMARK 3.11. Given $0 < x_0 < 1$,

$$K_0'(x_0)\,\big|[K_0(x_0) = y] \overset{d}{=} U\left(-\frac{y}{1-x_0}, \frac{y}{x_0}\right)$$

and

$$\frac{K_0'(x_0)}{K_0(x_0)} \overset{d}{=} U\left(-\frac{1}{1-x_0}, \frac{1}{x_0}\right).$$

Therefore, given $K_0(x_0) = y$, $K_0'(x_0)$ is distributed uniformly over all its possibilities. Note that $P\left(K_0'(x_0) < 0\,\big|[K_0(x_0) = y]\right) = x_0$. Thus, the height of the least concave majorant process at x_0, or $K_0(x_0)$, is independent of the sign of the slope of the least concave majorant process at x_0, or *sign* $\left[K_0'(x_0)\right]$.

Theorem 3.10 and the preceding remark state the behavior of the least concave majorant of a Brownian bridge over some open neighborhood of x_0. In a broader context, consider the following physical interpretation. Fix x_0 in $(0, 1)$ and let W_0 be a random Brownian bridge path. Suppose we start with a line L which majorizes W_0 (almost surely there exists such a line). We will allow the line L to pivot about x_0 and/or to slide vertically. If we pull the line down at x_0 as far as possible, keeping in mind W_0 will act as an obstacle, then Theorem 3.10 and the preceding remark tell us the distribution of the resulting position of the line. $K_0(x_0)$ is the height of the final line at x_0 and $K_0'(x_0)$ is the slope of the line.

Recall, $K'(x)$ is almost surely the derivative of the process $\{K(t) : t \ge 0\}$, evaluated at x. Groeneboom (1983) derives the density of $K'(x)$. By Theorem 2.1, it follows that $K'(x) \overset{d}{=} \frac{1}{\sqrt{x}} K'(1)$. Therefore, it suffices to analyze the distribution of $K'(1)$. We state the density of $K'(1)$ in the following theorem and then analyze some of the properties of this distribution.

THEOREM 3.12. *The distribution function and density function of $K'(1)$ are given respectively by*

$$\begin{aligned} F_{K'(1)}(a) &= P(K'(1) \le a) \\ &= 1 - 2\left(1 + a^2\right)\overline{\Phi}(a) + 2a\phi(a), \qquad a \ge 0 \end{aligned}$$

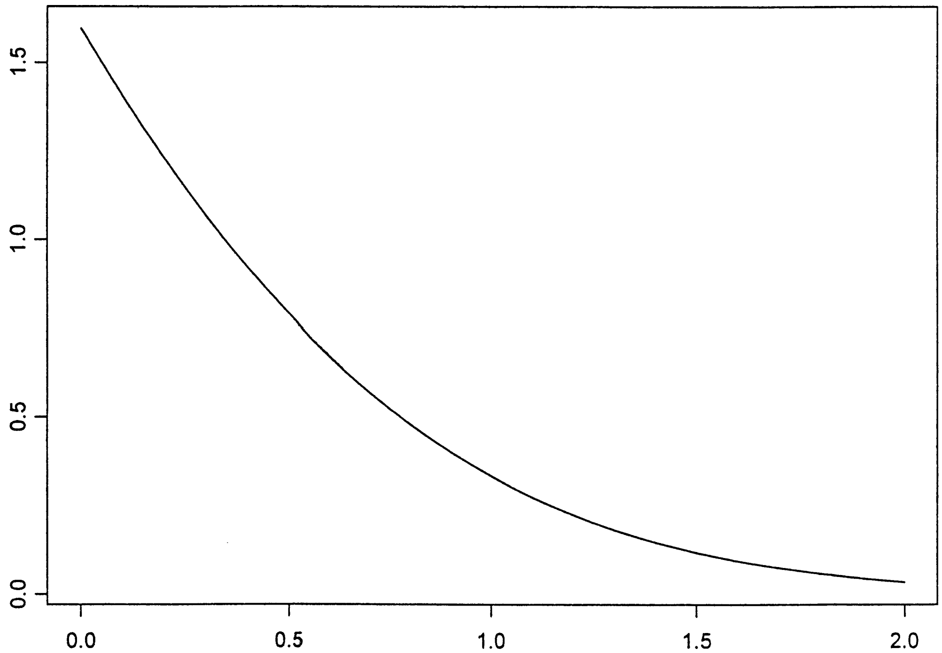

FIG. 3. *A graph of the density* $f_{K'(1)}(a) = 4\phi(a) - 4a\overline{\Phi}(a)$, $a > 0$, *over the interval* $[0, 2]$.

and

$$\begin{aligned} f_{K'(1)}(a) &= 4\int_a^{\infty} \overline{\Phi}(z)\ dz, \qquad a \geq 0 \\ &= 4\phi(a) - 4a\overline{\Phi}(a), \qquad a \geq 0 \end{aligned}$$

where $\overline{\Phi}$ *is the standard normal survival function and* ϕ *is the standard normal pdf.*

See Figure 3 for a graph of the density of $K'(1)$. Clearly, the density function of $K'(1)$ is strictly monotone decreasing and convex over the positive half line. We give the moments of $K'(1)$ in the following remark.

REMARK 3.13. The moments of $K'(1)$ are given by

$$E\left[K'(1)^r\right] = \begin{cases} (r-1)(r-3)\cdots 1\left[\dfrac{2}{r+2}\right], & \text{if } r \geq 0, \text{ even}, \\ (r-1)(r-3)\cdots 2\left[\dfrac{4}{\sqrt{2\pi}}\dfrac{1}{r+2}\right], & \text{if } r \geq 1 \text{ odd}, \end{cases}$$

with mean and variance given respectively by $\frac{4}{3\sqrt{2\pi}} = 0.53192$ and $\frac{1}{2} - \frac{8}{9\pi} = 0.21706$.

The process $\{K(t) : t \geq 0\}$ can be thought of as being centered around the function

$$g(t) = E[K(t)] = \frac{8}{3\sqrt{2\pi}}\sqrt{t},\ t \geq 0$$

both in terms of its heights and its slopes. For example, if we let $K^{(1)}, K^{(2)}, \ldots, K^{(n)}$ be a sample of independent least concave majorant of Brownian motion paths, and define $\overline{K} = \frac{1}{n}\sum_{i=1}^{n} K^{(i)}$, then for $x > 0$, $\overline{K}(x)$ converges almost surely to $g(x)$ as $n \to \infty$. In addition, $\overline{K}'(x)$ converges almost surely to $E[K'(x)]$, which amazingly happens to be $g'(x)$, as $n \to \infty$. Finally, by the multivariate central limit theorem,

$$\begin{aligned}
&\sqrt{n}\left(\begin{pmatrix} \overline{K}(x) \\ \overline{K}'(x) \end{pmatrix} - \begin{pmatrix} g(x) \\ g'(x) \end{pmatrix}\right) \\
&\quad \xrightarrow{d} N\left(\underset{\sim}{0}, \begin{bmatrix} \operatorname{Var}(K(x)) & \operatorname{Cov}(K(x), K'(x)) \\ \operatorname{Cov}(K(x), K'(x)) & \operatorname{Var}(K'(x)) \end{bmatrix}\right) \\
&\quad \overset{d}{=} N\left(\underset{\sim}{0}, \begin{bmatrix} x\left(\frac{3}{2} - \frac{32}{9\pi}\right) & \frac{1}{2}\left(\frac{3}{2} - \frac{32}{9\pi}\right) \\ \frac{1}{2}\left(\frac{3}{2} - \frac{32}{9\pi}\right) & \frac{1}{x}\left(\frac{1}{2} - \frac{8}{9\pi}\right) \end{bmatrix}\right).
\end{aligned}$$

The final distribution we will examine is that of $K_0'(x_0)$, where $0 < x_0 < 1$. Remember, $K_0'(x_0)$ is almost surely the derivative of the process $\{K_0(t) : 0 \leq t \leq 1\}$, evaluated at x_0. Groeneboom (1983) derives the density of $K_0'(x_0)$. We state the density of $K_0'(x_0)$ in the following theorem and then analyze some of the properties of this distribution.

THEOREM 3.14. *For $0 < x_0 < 1$, the distribution function and density function of $K_0'(x_0)$ are given respectively by*

$$\begin{aligned}
&F_{K_0'(x_0)}(a) \\
&= P(K_0'(x_0) \leq a) \\
&= \begin{cases} 2\big((1-x_0)a^2 + x_0\big)\overline{\Phi}\left(|a|\frac{1}{c_0}\right) - 2\sqrt{x_0(1-x_0)}|a|\phi\left(|a|\frac{1}{c_0}\right), & a < 0, \\ 1 - 2\big(x_0 a^2 + 1 - x_0\big)\overline{\Phi}(|a|c_0) + 2\sqrt{x_0(1-x_0)}|a|\phi(|a|c_0), & a \geq 0, \end{cases}
\end{aligned}$$

where $c_0 = \sqrt{\frac{x_0}{1-x_0}}$ and

$$f_{K_0'(x_0)}(a) = \begin{cases} \displaystyle\int_{-a(1-x_0)}^{\infty} 4\overline{\Phi}\left(\frac{y}{\sqrt{x_0(1-x_0)}}\right)dy, & a < 0, \\ \displaystyle\int_{ax_0}^{\infty} 4\overline{\Phi}\left(\frac{y}{\sqrt{x_0(1-x_0)}}\right)dy, & a \geq 0, \end{cases}$$

$$= \begin{cases} 4\sqrt{x_0(1-x_0)}\phi\left(|a|\sqrt{\dfrac{1-x_0}{x_0}}\right) - 4(1-x_0)|a|\overline{\Phi}\left(|a|\sqrt{\dfrac{1-x_0}{x_0}}\right), & a<0, \\ 4\sqrt{x_0(1-x_0)}\phi\left(|a|\sqrt{\dfrac{x_0}{1-x_0}}\right) - 4x_0|a|\overline{\Phi}\left(|a|\sqrt{\dfrac{x_0}{1-x_0}}\right), & a\geq 0, \end{cases}$$

where $\overline{\Phi}$ *is the standard normal survival function and* ϕ *is the standard normal pdf.*

Clearly, the density function of $K_0'(x_0)$ has mode zero. We now give the following remark which relates the distribution of $K_0'(x_0)$ to the distribution of $K'(1)$.

REMARK 3.15. Given $0 < x_0 < 1$,

$$K_0'(x_0) \stackrel{d}{=} \begin{cases} -\sqrt{\dfrac{x_0}{1-x_0}}K'(1), & \text{with probability } x_0, \\ \sqrt{\dfrac{1-x_0}{x_0}}K'(1), & \text{with probability } 1-x_0. \end{cases}$$

The equivalence (in distribution) found in the above remark makes obvious that the density of $K_0'(x_0)$ is continuous and forms a cusp at 0 because the density of $K'(1)$ is convex over the positive reals. It also makes clear that the moments of $K_0'(x_0)$ are simple functions of the moments of $K'(1)$. We give the moments of $K_0'(x_0)$ in the following remark.

REMARK 3.16. The moments of $K_0'(x_0)$ are given by

$$E\left[K_0'(x_0)^r\right]$$
$$= \begin{cases} (r-1)(r-3)ts1\left[(1-x_0)\left(\dfrac{1-x_0}{x_0}\right)^{r/2} + x_0\left(\dfrac{x_0}{1-x_0}\right)^{r/2}\right]\left[\dfrac{2}{r+2}\right], \\ \qquad \text{if } r\geq 0, \text{even}, \\ (r-1)(r-3)ts2\left[(1-x_0)\left(\dfrac{1-x_0}{x_0}\right)^{r/2} - x_0\left(\dfrac{x_0}{1-x_0}\right)^{r/2}\right]\left[\dfrac{4}{\sqrt{2\pi}}\dfrac{1}{r+2}\right], \\ \qquad \text{if } r\geq 1, \text{odd}, \end{cases}$$

with mean and variance given respectively by $\frac{4(1-2x_0)}{3\sqrt{2\pi x_0(1-x_0)}}$ and $\frac{x_0^3+(1-x_0)^3-\frac{16}{9\pi}(1-2x_0)^2}{2x_0(1-x_0)}$.

The process $\{K_0(t) : 0 \leq t \leq 1\}$ can be thought of as being centered around the function

$$h(t) = E[K_0(t)] = \frac{8}{3\sqrt{2\pi}}\sqrt{t(1-t)}, \qquad 0 \leq t \leq 1$$

both in terms of its heights and its slopes [note that $\sqrt{t(1-t)}$ forms a semicircle centered at the point $\left(\frac{1}{2}, 0\right)$ over the unit interval]. For example, if we let $K_0^{(1)}$, $K_0^{(2)}$, ..., $K_0^{(n)}$ be a sample of independent least concave majorant of Brownian bridge paths, and define $\overline{K_0} = \frac{1}{n}\sum_{i=1}^{n} K_0^{(i)}$, then for x_0 in $(0, 1)$, $\overline{K_0}(x_0)$ converges almost surely to $h(x_0)$ as $n \to \infty$. In addition, $\overline{K_0}'(x_0)$ converges almost surely to $E\left[K_0'(x_0)\right]$, which amazingly happens to be $h'(x_0)$, as $n \to \infty$. Finally, by the multivariate central limit theorem,

$$\sqrt{n}\left(\begin{pmatrix} \overline{K_0}(x_0) \\ \overline{K_0}'(x_0) \end{pmatrix} - \begin{pmatrix} h(x_0) \\ h'(x_0) \end{pmatrix}\right)$$

$$\xrightarrow{d} N\left(\underset{\sim}{0}, \begin{bmatrix} \operatorname{Var}(K_0(x_0)) & \operatorname{Cov}(K_0(x_0), K_0'(x_0)) \\ \operatorname{Cov}(K_0(x_0), K_0'(x_0)) & \operatorname{Var}(K_0'(x_0)) \end{bmatrix}\right)$$

$$\stackrel{d}{=} N\left(\underset{\sim}{0}, \begin{bmatrix} x_0(1-x_0)\left(\frac{3}{2} - \frac{32}{9\pi}\right) & \frac{(1-2x_0)}{2}\left(\frac{3}{2} - \frac{32}{9\pi}\right) \\ \frac{(1-2x_0)}{2}\left(\frac{3}{2} - \frac{32}{9\pi}\right) & \frac{x_0^3 + (1-x_0)^3 - \frac{16}{9\pi}(1-2x_0)^2}{2x_0(1-x_0)} \end{bmatrix}\right).$$

4. Distributions of locations and hitting times. In this section, we will analyze the distribution of locations (or times) where the least concave majorant processes attain various attributes. For example, we might ask ourselves how long it takes for the process $\{K(t) : t \geq 0\}$ to first attain a certain height. Fix $y \geq 0$ and define the random variable $K^{(-1)}(y) \equiv \inf\{x \geq 0 : K(x) \geq y\}$. Hence, $K^{(-1)}(y)$ is the first location for which the process $\{K(t) : t \geq 0\}$ exceeds y. Since the process $\{K(t) : t \geq 0\}$ is strictly increasing, continuous, and unbounded from above almost surely, $K^{(-1)}(y)$ is well defined and is actually an inverse almost surely. By Theorem 2.1, $K^{(-1)}(y) \stackrel{d}{=} y^2 K^{(-1)}(1)$ and we give the distribution of $K^{(-1)}(1)$ in the following theorem.

THEOREM 4.1. *Define* $K^{(-1)}(1) \equiv \inf\{x \geq 0 : K(x) \geq 1\}$. *Then*

$$K^{(-1)}(1) \stackrel{d}{=} \left(\frac{1}{K(1)}\right)^2$$

and thus has a density given by

$$f_{K^{(-1)}(1)}(x) = \frac{2}{x^2}\overline{\Phi}\left(\frac{1}{\sqrt{x}}\right), \qquad x \geq 0$$

where $\overline{\Phi}$ *is the standard normal survival function.*

PROOF.

$$F_{K^{(-1)}(1)}(x) = P\left(K^{(-1)}(1) \leq x\right)$$
$$= P(K(x) \geq 1)$$

$$= P\left(K(1) \geq \frac{1}{\sqrt{x}}\right)$$

$$= P\left(\left(\frac{1}{K(1)}\right)^2 \leq x\right).$$

Therefore, $K^{(-1)}(1) \stackrel{d}{=} (\frac{1}{K(1)})^2$. □

Interestingly, the expected time for the process $\{K(t) : t \geq 0\}$ to reach a height of 1 is infinity. So, even though the taking of the least concave majorant tends to result in a path substantially larger than the Brownian motion path (we know the marginal distributions of the process $\{K(t) - W(t) : t \geq 0\}$ by Theorem 3.4), it still may a take a long time for the least concave majorant process $\{K(t) : t \geq 0\}$ to reach any height.

We might also ask ourselves how long the process $\{K(t) : t \geq 0\}$ can maintain left-hand slopes which are greater than or equal to a certain value. Fix $a > 0$ and define the random variable

$$X_a \equiv \sup\{x \geq 0 : K'(x) \geq a\}.$$

Then X_a is the location where the left-hand slope of $\{K(t) : t \geq 0\}$ changes from being greater than or equal to a to being strictly less than a. Since the process $\{K(t) : t \geq 0\}$ is concave, almost surely possessing arbitrarily large slopes and slopes arbitrarily close to zero, X_a will be well defined almost surely. By Theorem 2.1, $X_a \stackrel{d}{=} \frac{X_1}{a^2}$ and we give the distribution of X_1 in the following theorem.

THEOREM 4.2. *Define* $X_a \equiv \sup\{x \geq 0 : K'(x) \geq a\}$. *Then*

$$X_1 \stackrel{d}{=} \left[K'(1)\right]^2$$

and thus has density given by

$$f_{X_1}(x) = \frac{2}{\sqrt{x}}\phi\left(\sqrt{x}\right) - 2\overline{\Phi}\left(\sqrt{x}\right), \qquad x \geq 0$$

where ϕ *is the standard normal density and* $\overline{\Phi}$ *is the standard normal survival function.*

PROOF.

$$\begin{aligned} F_{X_1}(x) &= P(X_1 \leq x) \\ &= P(K'(x) \leq 1) \\ &= P\left(K'(1) \leq \sqrt{x}\right) \\ &= P\left(\left[K'(1)\right]^2 \leq x\right). \end{aligned}$$

Therefore $X_1 \stackrel{d}{=} [K'(1)]^2$.

We now turn our attention to the process $\{K_0(t) : 0 \le t \le 1\}$. It would be of interest to determine how long the process $\{K_0(t) : 0 \le t \le 1\}$ can maintain left-hand slopes which are greater than or equal to a certain value. Fix $a \in \Re$ and define the random variable

$$X_a^0 \equiv \sup\{0 < x \le 1 : K_0'(x) \ge a\}.$$

Then X_a^0 is the location where the left-hand slope of $\{K_0(t) : 0 \le t \le 1\}$ changes from being greater than or equal to a to being strictly less than a. Since the process $\{K_0(t) : 0 \le t \le 1\}$ is concave, almost surely possessing arbitrarily large slopes and arbitrarily small slopes, X_a^0 will be well defined almost surely. We give the distribution of X_a^0 in the following theorem.

THEOREM 4.3. *Define the random variable* $X_a^0 \equiv \sup\{0 < x_0 \le 1 : K_0'(x_0) \ge a\}$. *The distribution function of* X_a^0 *is given by*

$$\begin{aligned} F_{X_a^0}(x_0) &= P\left(X_a^0 < x_0\right) \\ &= P\left(K_0'(x_0) < a\right) \\ &= F_{K_0'(x_0)}(a) \end{aligned}$$

where the distribution function of $K_0'(x_0)$ *is given in Theorem* 3.17. *Thus, the density function of* X_a^0 *is given by*

$$f_{X_a^0}(x_0)$$
$$= 2\begin{cases} (1-a^2)\,\overline{\Phi}\left(|a|\sqrt{\frac{1-x_0}{x_0}}\right) + |a|\sqrt{\frac{x_0}{1-x_0}}\,\phi\left(|a|\sqrt{\frac{1-x_0}{x_0}}\right), & 0 < x_0 < 1, a < 0, \\ (1-a^2)\,\overline{\Phi}\left(|a|\sqrt{\frac{x_0}{1-x_0}}\right) + |a|\sqrt{\frac{1-x_0}{x_0}}\,\phi\left(|a|\sqrt{\frac{x_0}{1-x_0}}\right), & 0 < x_0 < 1, a \ge 0. \end{cases}$$

where $\overline{\Phi}$ *is the standard normal survival function, and* ϕ *is the standard normal pdf.*

Notice that $X_0^0 = \arg\max\{0 < z \le 1 : W_0(z)\} \stackrel{d}{=} U(0,1)$.

5. Estimation under stochastic orderings. Suppose that $X_1, X_2, \ldots, X_m$ is a random sample of size m from the continuous distribution function F and let H denote the distribution function corresponding to a standard uniform distribution. We will also assume that F has support which is a subset of the unit interval. We now consider the estimation of F subject to F satisfying a stochastic ordering with H. The stochastic orderings we will consider are those of simple stochastic ordering (SO) and likelihood ratio ordering (LR).

We will denote the nonparametric maximum likelihood estimator of F subject to $F <_{SO} H$ by $\widetilde{F}_m$. This ordering implies that $F(t) \ge H(t)$ for all $t \in \Re$. Dykstra (1982) derives this estimator and Praestgaard and Huang

(1996) demonstrated that the limiting process under the assumption $F = H$ is given by

$$\sqrt{m}\left(\widetilde{F}_m - F\right) \overset{d}{\to} K_0(F) - W_0(F)$$

as $m \to \infty$. Praestgaard and Huang (1996) state that the marginal distributions of this limiting process are unknown. However, we have derived the marginal distributions in Section 3.

We will denote the maximum likelihood estimator of F subject to $F <_{LR} H$ by F_m^{***}. Of course, this ordering implies that $f(t)/h(t)$ is a monotone decreasing function of t, where f and h are the densities associated with F and H respectively. Since we are assuming that H is the distribution function of a standard uniform distribution, then the restriction $F <_{LR} H$ implies that F has a monotone density. Grenander (1956) showed that F_m^{***} is given by the least concave majorant of the empirical cdf. Thus, the limiting process of F_m^{***} under $F = H$ is given by

$$\sqrt{m}\left(F_m^{***} - F\right) \overset{d}{\to} K_0(F)$$

as $m \to \infty$.

What is interesting and surprising is that even though likelihood ratio ordering is a substantially more restrictive ordering than simple stochastic ordering, these two limiting processes under the equality of F and H have the same marginal distributions. However, they behave very differently as processes. We list some of these differences:

(i) K_0 almost surely has two zeros whereas $K_0 - W_0$ almost surely has infinitely many zeros.

(ii) K_0 is almost surely piecewise linear whereas $K_0 - W_0$ is almost surely undifferentiable everywhere in the unit interval.

(iii) $\left|\left|K_0\right|\right| \le_{SO} \left|\left|K_0 - W_0\right|\right|$. This follows from the sequence of inequalities

$$\left|\left|K_0 - W_0\right|\right| \ge \left|\left|W_0^-\right|\right| \overset{d}{=} \left|\left|W_0^+\right|\right| = \left|\left|K_0\right|\right|$$

where $||g|| \equiv \sup_{t\in\Re}\left\{\left|g(t)\right|\right\}$.

We finally note that the limiting process of $\widetilde{F}_m$ would be better described by $W_0(F) - J_0(F)$ where J_0 is the greatest convex minorant over the unit interval of the Brownian bridge W_0. Hence we can say that $\left|\left|K_0\right|\right| \le \left|\left|W_0 - J_0\right|\right|$ with strict inequality holding almost surely.

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