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A NOTE ON STRONG-MIXING GAUSSIAN SEQUENCES

BY CHANDRAKANT M. DEO

University of California, Davis

This note extends a theorem of Welsch (1971) on the joint asymptotic distribution of some order statistics of a strong-mixing, stationary, Gaussian sequence.

Introduction. Let $\{X_n: 1 \leqq n < \infty\}$ be a stationary, Gaussian sequence of random variables with $E(X_1) = 0$, $E(X_1^2) = 1$ and $E(X_1 X_{n+1}) = r_n$. We assume further that $\{X_n\}$ is strong-mixing, i.e.,

$$\sup |P(AB) - P(A)P(B)| = \alpha(k) \to 0 \qquad \text{as } k \to \infty$$

where the supremum is taken over all sets A, B such that A is in the σ-field generated by the random variables $X_1, \cdots, X_m$ and B is in the σ-field generated by $X_{m+k}, X_{m+k+1}, \cdots$ for some m. Denote by M_n, S_n respectively the maximum and the second maximum of the random variables $X_1, X_2, \cdots, X_n$ Let Λ be the extreme-value distribution function

$$\Lambda(x) = \exp(-e^{-x}), \qquad -\infty < x < \infty$$

and let H denote the two-dimensional distribution function

$$\begin{aligned} H(x, y) &= \Lambda(y)\{1 + \log[\Lambda(x)/\Lambda(y)]\} && \text{if } y < x \\ &= \Lambda(x) && \text{if } y \geqq x . \end{aligned}$$

Define constants a_n, b_n by

$$a_n = (2 \log n)^{-\frac{1}{2}}, \qquad b_n = (2 \log n)^{\frac{1}{2}} - \tfrac{1}{2}(2 \log n)^{-\frac{1}{2}}(\log \log n + \log 4\pi) .$$

Then,

Welsch (1971) proved the following theorem under the additional assumption $\sup_n |r_n \log n| < \infty$ on the covariance function r_n of the sequence $\{X_n\}$. The object of this note is to show that the additional assumption is unnecessary. Thus,

THEOREM. *If $\{X_n\}$ is a strong-mixing, stationary, Gaussian sequence with $E(X_1) = 0$, $E(X_1^2) = 1$ then the joint distribution of $a_n^{-1}(M_n - b_n)$, $a_n^{-1}(S_n - b_n)$ converges to H as $n \to \infty$.*

PROOF. We simply indicate the modifications to be made in the proof of Theorem 3 of Welsch (1971).

Ibragimov and Rozanov (1970), on page 250, show that a strong-mixing, stationary, Gaussian sequence has a spectral density which, although possibly

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discontinuous, is in $L_p(-\pi, \pi]$ for all $p > 0$. We need only the fact that the spectral density is square-integrable whence it follows that $\sum r_n^2 < \infty$.

Now from the proof of Theorem 3 of Welsch (1971) it is clear that (using the same notation) it suffices to show that the expression

$$n \sum_{j=[p_n^\alpha]}^{p_n} |r_j| n^{-2/(1+|r_j|)} (\log n)^{1/(1+|r_j|)}$$

goes to zero as $n \to \infty$. But this expression is dominated by

$$n \log n n^{-2/(1+\delta[p_n^\alpha])} \sum_{j=1}^{p_n} |r_j|$$

which, by Schwarz's inequality is less than

$$n \log n n^{-2/(1+\delta[p_n^\alpha])} [\sum_{j=1}^{p_n} |r_j|^2]^{\frac{1}{2}} p_n^{\frac{1}{2}} .$$

But $p_n^{\frac{1}{2}} n^{-\frac{1}{2}} \to 0$. Also $\sum_{j=1}^{\infty} |r_j|^2 < \infty$. Thus we need only show that

$$n^{\frac{3}{2}} \log n n^{-2/(1+\delta[p_n^\alpha])} \to 0 .$$

Since $\sum r_n^2 < \infty$, we must have $r_n \to 0$ from which it follows that $\delta[p_n^\alpha] \to 0$. Clearly then $n^{\frac{3}{2}} \log n n^{-2/(1+\delta[p_n^\alpha])} \to 0$ and the proof is complete.

Note that Theorem 2 of Welsch (1971) also is valid for strong-mixing, stationary, Gaussian sequences without the assumption $\sup_n |r_n \log n| < \infty$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CALIFORNIA 95616