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A NOTE ON THE CONVERGENCE OF SERIES OF STOCHASTIC PROCESSES[1]

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A tightness condition for families of Gaussian processes with continuous paths is given and applied to the question of convergence of sums of independent Gaussian processes.

According to a classical result of Paul Lévy, a series of independent real-valued random variables converges almost surely iff the partial sums converge in distribution. Various extensions of this result to random variables taking values in more general spaces are known. Tortrat [8] has established the analogue of Lévy's result for random variables taking values in certain topological groups, including separable Banach spaces. For the case of symmetrically-distributed Banach-space-valued variables, further necessary and sufficient conditions for convergence have been given by Itô and Nisio [4]. Recently, Jain and Kallianpur [5] used the Itô–Nisio conditions to prove the almost sure uniform convergence of certain series representations of Gaussian processes. Here we give a different, and somewhat simpler, proof of results analogous to those of [5]; our argument employs the Banach space version of Lévy's theorem and an inequality of Anderson [1] concerning the probability content of convex symmetric sets under normal distributions. Additional information about this topic is to be found in the paper [2] by Berman.

Let T be a compact metric space, and let C be the space of continuous real-valued functions on T. Endow C with the topology of uniform convergence, and let $\mathscr{C}$ be the corresponding Borel σ-algebra. We pause here to review some definitions and well-known results concerning weak convergence of probabilities on $(C, \mathscr{C})$ (see, for example, [3], where proofs are given for $T = [0, 1]$).

Let $\mathscr{P}$ denote the class of probabilities on $(C, \mathscr{C})$. The weak topology on $\mathscr{P}$ is that induced by the mappings $P \to Pf = \int_C f\, dP$ as f ranges over all bounded continuous real-valued functions on C. The topology of weak convergence is metrizable, and the relatively compact sets are exactly the so-called "tight" subsets of $\mathscr{P}$, i.e. families $(P_\theta)_{\theta \in \Theta}$ such that

$$\lim_{c\to\infty} \sup_\theta P_\theta\{||\cdot|| \geqq c\} = 0 \tag{1}$$

$$\lim_{\delta\to 0} \sup_\theta P_\theta\{\omega_\delta(\cdot) \geqq \varepsilon\} = 0 \qquad \text{for all } \varepsilon > 0\,; \tag{2}$$

here $||\cdot|| : x \to ||x|| = \sup_{t\in T} |x(t)|$ and $\omega_\delta(\cdot) : x \to \omega_\delta(x) = \sup\{|x(t) - x(s)| : s, t \in T;\ d(s, t) < \delta\}$. In order that probabilities $P_n (n \geqq 1)$ converge weakly to a

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probability P it is necessary and sufficient that (i) $(P_n)_{n\geqq 1}$ be tight, and (ii) the "finite-dimensional" distributions $P_n\pi_S^{-1}$ converge weakly to $P\pi_S^{-1}$ for all finite subsets $S \subset T$; here $\pi_S: x \to (x(s))_{s\in S} \in R^S$.

Now let $(\Omega, \mathscr{B})$ be a measurable space and let $X: \Omega \to C$. X is measurable between $\mathscr{B}$ and $\mathscr{C}$ iff $X = (X_t)_{t\in T}$ is a stochastic process, i.e. iff each X_t is a random variable. Given a family of such C-valued processes defined on (possibly different) probability spaces, one defines the tightness and weak convergence of the processes involved in terms of the tightness and convergence of the distributions they induce on $(C, \mathscr{C})$.

Let X_θ, $\theta \in \Theta$, be C-valued Gaussian processes with zero means and covariances $\Gamma_\theta(\Gamma_\theta(s, t) = EX_\theta(s)X_\theta(t))$. Suppose that for some $\theta_0 \in \Theta$, one has $\Gamma_\theta \leqq \Gamma_{\theta_0}$ (i.e. $\Gamma_{\theta_0} - \Gamma_\theta$ positive semi-definite) for all $\theta \in \Theta$. Then

PROPOSITION 1. *Under the above assumptions,*

$$P\{||X_\theta|| \leqq c\} \geqq P\{||X_{\theta_0}|| \leqq c\} \tag{3}$$

$$P\{\omega_\delta(X_\theta) \leqq \varepsilon\} \geqq P\{\omega_\delta(X_{\theta_0}) \leqq \varepsilon\} \tag{4}$$

for all c, δ, and ε. In particular, the family $(X_\theta)_{\theta\in\Theta}$ is tight.

PROOF. In Corollary 3 of [1], T. W. Anderson showed that for any (closed) convex symmetric set E in R^p, one has

$$P\{Z_1 \in E\} \geqq P\{Z_2 \in E\} \tag{5}$$

whenever Z_1 and Z_2 are p-variate normal random vectors having zero means and non-singular covariance matrices Σ_1 and Σ_2 satisfying $\Sigma_1 \leqq \Sigma_2$. By applying the Brunn–Minkowski theorem for a suitable subspace in the proof of Anderson's Theorem 1 [1], one finds that Anderson's Corollary 2 holds for random vectors having a symmetric unimodal density in some subspace of R^p, and thus that (5) holds even if the Z's have singular normal distributions. Inequalities (3) and (4) follow from (5) by a simple limiting argument. □

A few comments on the proposition are in order at this point. First, instead of assuming that each X_θ has continuous paths, one can assume only that each X_θ is a separable process and that X_{θ_0} has continuous paths; then (3) and (4) are in force and imply that all the X_θ's have continuous paths. Second, the proposition is not as useful as might be wished, for methods which allow one to prove that X_{θ_0} has continuous paths generally can be applied directly to prove the tightness of the X_θ's; see for example, [2]. Finally, the Gaussian hypothesis can be replaced by a weaker one, namely that the X_θ's are "elliptically contoured" processes. One says that a random vector Z in R^p is elliptically contoured with parameters f and Σ, written $Z \sim EC(f, \Sigma, p)$, if Z has density of the form $f(z) = |\Sigma|^{-\frac{1}{2}}f(z'\Sigma^{-1}z)$, where f: $[0, \infty) \to [0, \infty)$ and Σ is a positive definite symmetric matrix. That normal random vectors are elliptically contoured is seen by taking $f(t)$ to be $(1/2\pi)^{p/2}e^{-t/2}$. It has recently been shown ([7]) that (5) holds whenever $Z_i \sim EC(f, \Sigma_i, p)$ $(i = 1, 2)$ with $\Sigma_1 \leqq \Sigma_2$. Consequently if the X_θ's have elliptically contoured finite-dimensional distributions with appropriate parameters, the conclusion of the proposition still holds.

PROPOSITION 2. *Let* $X_n = (X_n(t))_{t \in T}$ *be independent C-valued Gaussian processes, all defined on the same probability space, and having zero means and covariances* Γ_n $(n \geqq 1)$. *Set* $\Gamma = \Sigma_n \Gamma_n$. *If there exists (on some probability space) a C-valued Gaussian process (say X) with covariance* Γ, *then with probability one the series* $\Sigma_n X_n(t)$ *converges uniformly in* $t \in T$.

PROOF. Put $S_n = \Sigma_{m \leqq n} X_m$. It suffices ([4] or [8]) to show that the S_n's converge in distribution to X. Convergence of finite-dimensional distributions is immediate, and tightness follows from Proposition 1. □

The simplest case covered by Proposition 2 is perhaps that in which each X_n is of the form $\mathscr{E}_n g_n$, where the $\mathscr{E}_n$'s are i.i.d. $N(0, 1)$ random variables, and the functions $g_n : T \rightarrow R$ are continuous and satisfy $\Sigma_n g_n(s) g_n(t) = \Gamma(s, t)$ for all $s, t \in T$. For such g_n's one may take any complete orthonormal system for the reproducing kernel Hilbert space associated with (the continuous) covariance kernel Γ (see, e.g. [6]). This gives another proof of the results of [5].

REFERENCES

[1] ANDERSON, T. W. (1955). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* **6** 170–176.

[2] BERMAN, S. M. (1968). Some continuity properties of Brownian motion with the parameter in Hilbert space. *Trans. Amer. Math. Soc.* **131** 182–198.

[3] BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[4] ITÔ, K. and NISIO, M. (1968). On the convergence of sums of independent Banach space valued random variables. *Osaka J. Math.* **5** 35–48.

[5] JAIN, N. C. and KALLIANPUR, G. (1970). A note on uniform convergence of stochastic processes. *Ann. Math. Statist.* **41** 1360–1362.

[6] JØRSBOE, O. G. (1968). Equivalence or singularity of Gaussian measures on function spaces. Aarhus Universitet, Matematisk Institut, Various Publications Series No. 4.

[7] DAS GUPTA, EATON, OLKIN, PERLMAN, SAVAGE, and SOBEL (1972). Inequalities on the probability content of convex regions for elliptically contoured distributions. To appear in *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **2**.

[8] TORTRAT, A. (1965). Lois de probabilité sur un espace topologique complètement régulier et produits infinis à terms indépendants dans un groupe topologique. *Ann. Inst. H. Poincaré Sect. B* **1** 217–237.

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