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ON A CONVEX FUNCTION INEQUALITY FOR MARTINGALES[1]

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A new proof is given for the inequality

$$E(\Phi(\sum_{\nu=1}^{\infty} E(z_\nu \mid \mathscr{F}_\nu))) \leqq CE(\Phi(\sum_{\nu=1}^{\infty} z_\nu)),$$

where $z_1, z_2, \cdots, z_n, \cdots$ are nonnegative random variables on a probability space $(\Omega, \mathscr{F}, P)$, $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_n \subset \cdots \mathscr{F}$ is a sequence of σ-fields and $\Phi(u)$ is a convex function satisfying $\Phi(2u) \leqq c\Phi(u)$.

0. Introduction. Let $(\Omega, \mathscr{F}, P)$ be a probability space and

$$\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n \subset \cdots \subset \mathscr{F}$$

be a fixed sequence of σ-fields.

In a recent paper [1] Burkholder, Davis and Gundy prove the following result.

THEOREM 0.1. *Let $\Phi(u)$ be a convex function from $[0, \infty]$ to $[0, \infty)$ such that $\Phi(0) = 0$ and*

$$\Phi(2u) \leqq c\Phi(u) \qquad \forall u > 0.$$

Then, if $\{Z_n\}$ is any sequence of nonnegative $\mathscr{F}$-measurable functions on Ω

$$(0.1) \qquad E(\Phi(\sum_{n=1}^{\infty} E(Z_n \mid \mathscr{F}_n))) \leqq CE(\Phi(\sum_{n=1}^{\infty} Z_n))$$

where C depends on c.

This inequality is not only interesting in itself, but it is a key step in these authors' proof of a rather remarkable inequality for martingales (see Theorem 1.1 in [1]).

In this paper we shall present a new proof of this inequality. Indeed, we shall show that (0.1) is actually very intimately related to some martingale inequalities which may be considered "classical."

We hope that our efforts here will not only provide a quick path to (0.1) but also shed some additional light on the true nature of this very interesting inequality.

1. Preliminaries about convex functions. To avoid getting lost in technicalities, we shall assume that our convex function $\Phi(u)$ is of the type

$$\Phi(u) = \int_0^u \varphi(t)\, dt$$

with, $\varphi(t)$ *strictly* increasing and nonnegative in $[0, \infty)$.

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To such a $\Phi(u)$ we can associate a convex function $\Psi(v)$ of the same type (the conjugate of Φ in the sense of Young) such that

$$\Psi(v) = \int_0^v \psi(t)\, dt ,$$

where $\varphi(u)$ and $\psi(v)$ are inverses of each other.

The following relations are well known and easily established

$$u\varphi(u) = \Phi(u) + \Psi(\varphi(u)) , \tag{1.1}$$

$$\int_0^v t d\psi(t) = \Phi(\psi(v)) , \tag{1.2}$$

$$uv \leqq \Phi(u) + \Psi(v) \qquad \text{(Young's inequality)}, \tag{1.3}$$

$$\Phi\left(\frac{u}{\lambda}\right) \leqq \frac{1}{\lambda}\Phi(u) , \qquad \forall \lambda \geqq 1 . \tag{1.4}$$

Furthermore, when

$$\Phi(2u) \leqq c\Phi(u)$$

then, setting

$$p = \sup_{u>0} \frac{u\varphi(u)}{\Phi(u)}$$

we easily get $1 < p \leqq c - 1 < \infty$.

Finally, we also have

$$\Phi(\rho u) \leqq \rho^p \Phi(u) \qquad \forall \rho > 1 \tag{1.5}$$

$$\Psi(v) \leqq (p-1)\Phi(\psi(v)) . \tag{1.6}$$

The proofs of these assertions can be found in [3].

2. Some "classical" martingale inequalities. Let $\{f_n\}$ be a nonnegative $\{\mathscr{F}_n\}$-submartingale, i. e. $\mathscr{F}(f_n) \subset \mathscr{F}_n$ and

$$0 \leqq f_\nu \leqq E(f_n \mid \mathscr{F}_\nu) , \qquad 0 \leqq \nu \leqq n \tag{2.1}$$

assume further that

$$f_0 = 0 . \tag{2.2}$$

Furthermore, set

$$f_n^* = \max_{0 \leqq \nu \leqq n} f_\nu , \qquad f^* = \sup_\nu f_\nu . \tag{2.3}$$

THEOREM 2.1. *If $m(t)$ is any non-decreasing function in $[0, \infty)$ and $m(0) = 0$ then*

$$E\left(\int_0^{f_n^*} t\, dm(t)\right) \leqq E(f_n m(f_n^*)) . \tag{2.4}$$

PROOF. From (2.2) we get

$$\begin{aligned} E\left(\int_0^{f_n^*} t\, dm(t)\right) &= \sum_{\nu=1}^n E\left(\int_{f_{\nu-1}^*}^{f_\nu^*} t\, dm(t)\right) \\ &\leqq \sum_{\nu=1}^n E(f_\nu^*[m(f_\nu^*) - m(f_{\nu-1}^*)]) . \end{aligned}$$

Now, since $f_\nu^* = f_\nu$ when $m(f_\nu^*) > m(f_{\nu-1}^*)$, we must conclude that

$$E\left(\int_0^{f_n^*} t\, dm(t)\right) \leqq \sum_{\nu=1}^n E(f_\nu[m(f_\nu^*) - m(f_{\nu-1}^*)]) . \tag{2.5}$$

This given, (2.1) immediately gives (2.4).

If we now replace $m(t)$ in (2.4) by our function $\psi(t)$ and use (1.2) we get

$$E(\Phi(\psi(f_n^*))) \leqq E(f_n \psi(f_n^*)) .$$

So, by Young's inequality

$$E(\Phi(\psi(f_n^*))) \leqq E\left(\Phi\left(\frac{\psi(f_n^*)}{p}\right)\right) + E(\Psi(pf_n)) ,$$

and by (1.4)

$$\frac{p-1}{p} E(\Phi(\psi(f_n^*))) \leqq E(\Psi(pf_n)) .$$

Combining with (1.6) we finally obtain

$$E(\Psi(f_n^*)) \leqq pE(\Psi(pf_n)) . \tag{2.6}$$

Note. Inequalities of this type, especially in the case $\Psi(u) = u^q (q > 1)$, are classical. (See Doob [2] page 317.)

The usual method of proof consists in deriving first (2.4) in the special case

$$\begin{aligned} m(t) &= 1 \quad \text{if} \quad t \geqq \lambda \\ &= 0 \quad \text{if} \quad t < \lambda \end{aligned} \qquad (\lambda > 0) ,$$

by a stopping time argument.

This gives

$$\lambda P[f_n^* \geqq \lambda] \leqq \int_{[f_n^* \geqq \lambda]} f_n \, dP . \tag{2.7}$$

Then inequalities such as (2.4), in the case $m(t) = t^{p-1} (p \geqq 1)$, are obtained by multiplying (2.7) by λ^{p-2} and integrating for λ in $[0, \infty)$. Indeed, (2.4) itself can be obtained from (2.7) by a similar method.

The reason for our including proofs of these known things here is that we wish to point out that not only (2.7) but (2.4) as well is an almost immediate consequence of the submartingale condition.

It is also worthwhile noticing that when we put $dm(t) = (1 + t^2)^{-1} dt$ we obtain

$$E(\log (1 + f_n^{*2})) \leqq \pi E(f_n) .$$

This, in the L_1-bounded case, yields

$$E(\log (1 + f^{*2})) \leqq \pi \sup_n E(f_n) ,$$

in particular $f^* < \infty$ a.s.

3. Proof of the Burkholder-Davis-Gundy inequality. We shall show that (0.1) holds with $C = p^{2p}$. Clearly, to do this, we need only show that

$$E(\Phi(\textstyle\sum_{\nu=1}^n E(z_\nu \mid \mathscr{F}_\nu))) \leqq p^{2p} E(\Phi(\sum_{\nu=1}^n z_\nu)) . \tag{3.1}$$

To this end, set

$$\gamma = \textstyle\sum_{\nu=1}^n z_\nu , \qquad \dot{\gamma} = \sum_{\nu=1}^n E(z_\nu \mid \mathscr{F}_\nu)$$

$$f_0 = 0,\ f_\nu = E(\varphi(\dot{\gamma}) \mid \mathscr{F}_\nu) , \qquad 1 \leqq \nu \leqq n .$$

This given, we have

$$E(\dot{\gamma}\varphi(\dot{\gamma})) = \sum_{\nu=1}^{n} E(z_\nu f_\nu) \leqq E(\gamma f_n^*) .$$

Thus, from Young's inequality we obtain

$$E(\dot{\gamma}\varphi(\dot{\gamma})) \leqq E(\Phi(p^2\gamma)) + E\left(\Psi\left(\frac{f_n^*}{p^2}\right)\right),$$

and (2.6) yields

$$E(\dot{\gamma}\varphi(\dot{\gamma})) \leqq E(\Phi(p^2\gamma)) + pE\left(\Psi\left(\frac{\varphi(\dot{\gamma})}{p}\right)\right).$$

Using (1.1) and (1.4) we then get

$$E(\Phi(\dot{\gamma})) + E(\Psi(\varphi(\dot{\gamma}))) \leqq E(\Phi(p^2\gamma)) + E(\Psi(\varphi(\dot{\gamma}))) ,$$

and (3.1) finally follows from (1.5).

Note. A slight modification of the above proof in the case $\Phi(u) = u^p/p$, $\psi(v) = v^q/q(p^{-1} + q^{-1} = 1)$, yields (0.1) with $C = p^p$. It would be interesting to find out whether (3.1) is best possible in the general case.

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Note added in proof. In "Recent progress in the theory of Martingale inequalities" (seminar notes), an entirely different argument yields 0.1 with $C = p^{p+1}$.