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MAXIMAL INEQUALITIES AND THE LAW OF THE ITERATED LOGARITHM[1]

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A supermartingale maximal inequality is derived. A maximal inequality is derived for arbitrary random variables $\{S_n, n \geqq 1\}$ (let $S_0 = 0$) satisfying $E\exp[u(S_{m+n} - S_m)] \leqq \exp(Knu^2)$ for all real u, all integers $m \geqq 0$ and $n \geqq 1$, and some constant K. These two maximal inequalities are used to derive upper half laws of the iterated logarithm for supermartingales, multiplicative random variables, and random variables not satisfying particular dependence assumptions.

1. Law of the iterated logarithm for supermartingales. Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space. Let $S_0 = 0$ and $\mathscr{F}_0 = (\phi, \Omega)$. Throughout Section 1, $\{S_n, \mathscr{F}_n, n \geqq 0\}$ denotes a supermartingale (i.e., σ-fields $\mathscr{F}_{n-1} \subset \mathscr{F}_n \subset \mathscr{F}$, $E[S_n \mid \mathscr{F}_{n-1}] \leqq S_{n-1}$ a.s., and S_{n-1} is $\mathscr{F}_{n-1}$ measurable for each $n \geqq 1$). Let $Y_i = S_i - S_{i-1}$ for $i \geqq 1$. First we derive the basic maximal supermartingale inequality.

LEMMA 1.1. *Suppose* $Y_i \leqq c$ a.s. *for each* $i \geqq 1$ *and some constant* $0 \leqq c < \infty$. *Fix* $\lambda > 0$ *such that* $\lambda c \leqq 1$, *let*

$$T_n = \exp(\lambda S_n)\exp[-(\lambda^2/2)(1 + \lambda c/2)\textstyle\sum_{i=1}^n E(Y_i^2 \mid \mathscr{F}_{i-1})]$$

for $n \geqq 1$, *and let* $T_0 = 1$. *Then* $\{T_n, \mathscr{F}_n, n \geqq 0\}$ *is a nonnegative supermartingale, thus satisfying*

$$P[\sup_{n\geqq 0} T_n > \alpha] \leqq \alpha^{-1} \qquad \textit{for each } \alpha > 0\,. \tag{1.1}$$

PROOF. We first show that $\{T_n, \mathscr{F}_n, n \geqq 0\}$ is a supermartingale. Fix $i \geqq 1$. Using $\lambda c \leqq 1$, series expansion yields

$$\exp(\lambda Y_i) \leqq 1 + \lambda Y_i + Y_i^2(\lambda^2/2)(1 + \lambda c/2)\,.$$

Thus

$$E[\exp(\lambda Y_i) \mid \mathscr{F}_{i-1}] \leqq 1 + (\lambda^2/2)(1 + \lambda c/2)E(Y_i^2 \mid \mathscr{F}_{i-1}) \quad \text{a.s.}$$

since $E(Y_i \mid \mathscr{F}_{i-1}) \leqq 0$ a.s. Hence

$$E[\exp(\lambda Y_i) \mid \mathscr{F}_{i-1}] \leqq \exp[(\lambda^2/2)(1 + \lambda c/2)E(Y_i^2 \mid \mathscr{F}_{i-1})] \quad \text{a.s.} \tag{1.2}$$

since $1 + x \leqq e^x$ for all real x. Fix $n \geqq 1$.

$$\begin{aligned} E[T_n \mid F_{n-1}] &= \exp(\lambda S_{n-1})\exp[-(\lambda^2/2)(1 + \lambda c/2) \\ &\qquad \times \textstyle\sum_{i=1}^n E(Y_i^2 \mid \mathscr{F}_{i-1})]E[\exp(\lambda Y_n) \mid \mathscr{F}_{n-1}] \\ &\leqq T_{n-1} \quad \text{a.s.}\,, \end{aligned}$$

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using (1.2). Thus $(T_n, \mathscr{F}_n, n \geqq 0)$ is a nonnegative supermartingale, as desired.

Fix $\alpha > 0$. Let t be the smallest integer $n \geqq 0$ such that $T_n > \alpha$ if such an n exists; otherwise let $t = \infty$. $\{T_{t\wedge n}, n \geqq 0\}$ is also a nonnegative supermartingale with first element equal to one. Hence, for each $n \geqq 1$,

$$1 \geqq ET_{t\wedge n} \geqq \alpha P[t \leqq n] .$$

Letting $n \to \infty$, we obtain

$$1 \geqq \alpha P[t < \infty] = \alpha P[\sup_{n\geqq 0} T_n > \alpha] ,$$

completing the proof.

P. Meyer ([5], pages 70–72) has established a result similar to Lemma 1.1. It is this result of Meyer's that provided the stimulus for Section 1.

Let $s_n^2 = \sum_{i=1}^n E[Y_i^2 \mid \mathscr{F}_{i-1}]$ and $u_n = [2 \log\log (e^2 \vee s_n^2)]^{\frac{1}{2}}$ for $n \geqq 1$.

THEOREM 1.1. *Suppose* $s_n^2 < \infty$ a.s. *for each* $n \geqq 1$ *and* $s_n^2 \to \infty$ a.s. *Let* K_i *be* $\mathscr{F}_{i-1}$ *measurable for* $i \geqq 1$. *Suppose for some constant* $0 < K \leqq \frac{1}{2}$ *that*

$$\text{(1.3)} \qquad \limsup K_i < K \quad \text{a.s.} \qquad \textit{and} \qquad Y_i \leqq K_i s_i/u_i \quad \text{a.s.}$$

for each $i \geqq 1$. *Then there exists a function* $\varepsilon(\cdot)$ *such that* $\varepsilon(K) < 1$ *and* $\varepsilon(x) \downarrow 0$ *as* $x \downarrow 0$ *for which*

$$\limsup S_n/(s_n u_n) \leqq 1 + \varepsilon(K) \quad \text{a.s.}$$

REMARKS. The form of $\varepsilon(\cdot)$, although not very important, is given in the proof below. The most important application of Theorem 1.1 is Corollary 1.1 below, in which case $K_i \to 0$ a.s. is assumed.

PROOF OF THEOREM 1.1. Let $Y_i' = Y_i I(K_i \leqq K)$ for $i \geqq 1$ and $S_n' = \sum_{i=1}^n Y_i'$ for $n \geqq 1$. Since $P[Y_i' \neq Y_i \text{ i.o.}] = 0$, it suffices to show $\limsup S_n'/(s_n u_n) \leqq 1 + \varepsilon(K)$ a.s. in order to prove the theorem. Fix a real number $p > 1$ and an integer $k \geqq 1$. Let t_k be the smallest integer $n \geqq 0$ such that $s_{n+1}^2 \geqq p^{2k}$. Since s_{n+1}^2 is $\mathscr{F}_n$ measurable for each $n \geqq 0$, it follows that t_k is a stopping rule; i.e., $(t_k = n) \in \mathscr{F}_n$ for each $n \geqq 0$. Hence

$$\begin{aligned} S_n^{(k)} &= S_n' && \text{if} \quad n \leqq t_k ; \\ &= S'_{t_k} && \text{if} \quad n > t_k \end{aligned}$$

defines a supermartingale $\{S_n^{(k)}, n \geqq 0\}$. Fix $\delta' > 0$ and choose δ such that

$$\text{(1.4)} \qquad (1 + \delta)p^{-1} > 1 + \delta' .$$

$$\begin{aligned} &P[S_n' > (1 + \delta)s_n u_n \text{ i.o.}] \\ &\qquad \leqq P[\sup_{t_k \geqq n \geqq 0} S_n' > (1 + \delta)s_{t_{k-1}+1} u_{t_{k-1}+1} \text{ i.o. in } k] \\ \text{(1.5)} \qquad &\qquad = P[\sup_{n\geqq 0} S_n^{(k)} > (1 + \delta)s_{t_{k-1}+1} u_{t_{k-1}+1} \text{ i.o. in } k] . \end{aligned}$$

$$\begin{aligned} &s_{t_{k-1}+1}^2 u_{t_{k-1}+1}^2/[2p^{2k} \log\log (e^2 \vee p^{2k})] \\ &\qquad \geqq p^{-2} \log\log (e^2 \vee p^{2(k-1)})/\log\log (e^2 \vee p^{2k}) \\ &\qquad \approx p^{-2} . \end{aligned}$$

Thus

$$(1.6)\qquad P[\sup_{n\geq 0} S_n^{(k)} > (1+\delta)s_{t_{k-1}+1}u_{t_{k-1}+1} \text{ i.o. in } k] \leqq P\{\sup_{n\geq 0} S_n^{(k)} > (1+\delta')[2p^{2k}\log\log(e^2 \vee p^{2k})]^{\frac{1}{2}}, \text{ i.o. in } k\}.$$

We will produce $1 > \varepsilon(x) \downarrow 0$ as $x \downarrow 0$ such that

$$(1.7)\qquad \sum_{k=1}^{\infty} P\{\sup_{n\geq 0} S_n^{(k)} > (1+\delta')[2p^{2k}\log\log(e^2 \vee p^{2k})]^{\frac{1}{2}}\} < \infty$$

for all $\delta' > \varepsilon(K)$. It will then follow by the Borel–Cantelli lemma and (1.4) through (1.7) that $P[S_n' > (1+\delta)s_n u_n \text{ i.o.}] = 0$ for all $\delta > p[1+\varepsilon(K)] - 1$. Since $p > 1$ is arbitrary, the result will then follow for all $\delta > \varepsilon(K)$. Thus $\limsup S_n'/(s_n u_n) \leqq 1 + \varepsilon(K)$ a.s., the desired result, will follow from (1.7).

For each $n \geqq 0$ and k sufficiently large

$$S_{n+1}^{(k)} - S_n^{(k)} \leqq Ks_{t_k}/u_{t_k} \leqq Kp^k/[2\log\log(e^2 \vee p^{2k})]^{\frac{1}{2}} \quad \text{a.s.}$$

by (1.3), the definition of t_k, and the fact that s_n/u_n is non-decreasing in n. Let $(s_n^{(k)})^2 = \sum_{i=1}^{n} E[(S_i^{(k)} - S_{i-1}^{(k)})^2 \mid \mathscr{F}_{i-1}]$ for $n \geqq 1$. We apply Lemma 1.1 to $\{S_n^{(k)}, \mathscr{F}_n, n \geqq 0\}$ for k sufficiently large, taking

$$c = Kp^k/[2\log\log(e^2 \vee p^{2k})]^{\frac{1}{2}} \qquad \text{and} \qquad \lambda = (1+\delta')[2\log\log(e^2 \vee p^{2k})]^{\frac{1}{2}}/p^k$$

with $T_n = \exp(\lambda S_n^{(k)})\exp[-(\lambda^2/2)(1+\lambda c/2)(s_n^{(k)})^2]$ for $n \geqq 1$ and $T_0 = 1$. $\lambda c = (1+\delta')K$. Choose $\delta' \leqq 1$, thus implying $\lambda c \leqq 1$ as required in order to apply Lemma 1.1. Note that $\sup_{n\geq 1}(s_n^{(k)})^2 \leqq s_{t_k}^2$ a.s.

$$\begin{aligned} P\{\sup_{n\geq 0} S_n^{(k)} &> (1+\delta')[2p^{2k}\log\log(e^2 \vee p^{2k})]^{\frac{1}{2}}\} \\ &= P[\sup_{n\geq 0} S_n^{(k)} > \lambda p^{2k}] = P[\sup_{n\geq 0}\exp(\lambda S_n^{(k)}) > \exp(\lambda^2 p^{2k})] \\ &\leqq P\{\sup_{n\geq 0} T_n > \exp[\lambda^2 p^{2k} - (\lambda^2/2)(1+\lambda c/2)s_{t_k}^2]\} \\ &\leqq P\{\sup_{n\geq 0} T_n > \exp[\lambda^2 p^{2k} - (\lambda^2/2)(1+\lambda c/2)p^{2k}]\} \\ &\leqq \exp[-\lambda^2 p^{2k} + (\lambda^2/2)(1+\lambda c/2)p^{2k}] \qquad \text{by Lemma 1.1.} \end{aligned}$$

Substituting for λ and c,

$$-\lambda^2 p^{2k} + (\lambda^2/2)(1+\lambda c/2)p^{2k} = -(1+\delta')^2[1 - K(1+\delta')/2]\log\log(e^2 \vee p^{2k}).$$

Let $g_K(x) = (1+x)^2[1 - K(1+x)/2] - 1$ for $0 < x \leqq 1$, $0 < K \leqq \frac{1}{2}$. $g_K(\cdot)$ is increasing. Moreover $g_K(0) < 0$, $g_K(1) > 0$ for each $0 < K \leqq \frac{1}{2}$. Let $\varepsilon(K)$ be the zero of $g_K(\cdot)$ for each $0 < K \leqq \frac{1}{2}$. $g_K(x)$ increases to a strictly positive number as $K \downarrow 0$ for each fixed x. Thus $1 > \varepsilon(K)$ for each $0 < K \leqq \frac{1}{2}$ and $\varepsilon(K) \downarrow 0$ as $K \downarrow 0$. Also $(1+\delta')^2[1 - K(1+\delta')/2] - 1 > 0$ for all $1 \geqq \delta' > \varepsilon(K)$. Choose such a δ'. Then there exists $\beta > 1$ such that

$$P\{\sup_{n\geq 0} S_n^{(k)} > (1+\delta')[2p^{2k}\log\log(e^2 \vee p^{2k})]^{\frac{1}{2}}\} \leqq \exp[-\beta\log\log(e^2 \vee p^{2k})] = (2k\log p)^{-\beta}$$

for k sufficiently large. $\sum_{k=1}^{\infty}(2k\log p)^{-\beta} < \infty$, establishing (1.7) and thus completing the proof.

COROLLARY 1.1. *Suppose* $s_n^2 < \infty$ a.s. *for each* $n \geqq 1$ *and* $s_n^2 \to \infty$ a.s. *Let* K_i *be* F_{i-1} *measurable for each* $i \geqq 1$ *with* $K_i \to 0$ a.s. *Suppose*

$$Y_i \leqq K_i s_i / u_i \quad \text{a.s.} \tag{1.8}$$

for $i \geqq 1$. *Then* $\limsup S_n/(s_n u_n) \leqq 1$ a.s.

PROOF. $\limsup K_i = 0$ a.s. Thus Corollary 1.1 follows immediately from Theorem 1.1.

Corollary 1.1 is essentially the generalization of the upper half of Kolmogorov's law of the iterated logarithm given in [9]. The method of proof given here is different and considerably simpler than that given in [9]. Note that Theorem 1.1 is a generalization of Corollary 1.1 since the hypothesis $K_i \to 0$ a.s. is replaced by the weaker hypothesis $\limsup K_i < K \leqq \frac{1}{2}$ a.s. A similar generalization for independent random variables is given by Feller in [4].

2. Law of the iterated logarithm for generalized Gaussian random variables. In Section 2 we use the maximal inequality approach of Serfling [7] to derive an upper half law of the iterated logarithm for generalized Gaussian random variables. According to Chow [2], a random variable X is generalized Gaussian with parameter A if there exists a positive number A such that $E\exp(uX) \leqq \exp(u^2A/2)$ for all real u. Normal and uniformly bounded random variables each with mean zero provide two examples of generalized Gaussian random variables. It is easy to see that X generalized Gaussian implies $EX = 0$. Throughout Section 2 $\{X_i, i \geqq 1\}$ will denote an arbitrary sequence of random variables. Let $S_{m,n} = \sum_{i=m+1}^{m+n} X_i$ for $m \geqq 0$ and $n \geqq 1$ and $S_n = S_{0,n}$ for $n \geqq 1$. First we derive the basic maximal inequality for generalized Gaussian random variables.

LEMMA 2.1. *Let* $S_{m,n}$ *be generalized Gaussian with parameter* An *for some positive number* A *and all* $m \geqq 0$ *and* $n \geqq 1$. *Then for each* $\nu > 2$, *there exists a positive constant* K_ν *such that*

$$E\max_{i\leqq n} |S_{m,i}|^\nu \leqq K_\nu (An)^{\nu/2} \quad \textit{for all } m \geqq 0 \textit{ and } n \geqq 1. \tag{2.1}$$

PROOF. Fix $\nu > 2$, $m \geqq 0$, and $n \geqq 1$.

$$E|S_{m,n}|^\nu = \nu \textstyle\int_0^\infty x^{\nu-1} P[|S_{m,n}| > x]\, dx. \tag{2.2}$$

$$\begin{aligned} P[|S_{m,n}| > x] &\leqq [E\exp(uS_{m,n}) + E\exp(-uS_{m,n})]\exp(-ux) \\ &\leqq 2\exp(u^2An/2)\exp(-ux) \end{aligned}$$

for all real u and all positive x by hypothesis. Taking $u = x/(An)$, it follows that

$$P[|S_{m,n}| > x] \leqq 2\exp[-x^2/(2An)] \tag{2.3}$$

for all positive x. Combining this inequality with (2.2), it follows that

$$E|S_{m,n}|^\nu \leqq 2\nu \textstyle\int_0^\infty x^{\nu-1}\exp[-x^2/(2An)]\, dx = J_\nu (An)^{\nu/2} \tag{2.4}$$

where $J_\nu = \nu 2^{\nu/2}\Gamma(\nu/2)$.

Using the maximal inequality approach of Serfling ([7]) it is easy to complete the proof. For, according to Serfling,

$$E|S_{m,n}|^\nu \leqq g^{\nu/2}(n) \tag{2.5}$$

for some $\nu > 2$, all $m \geqq 0$ and $n \geqq 1$, some function $g(\cdot)$ non-decreasing with $2g(n) \leqq g(2n)$ for all $n \geqq 1$ and $g(n)/g(n+1) \to 1$ as $n \to \infty$ implies that there exists a positive constant I_ν for which $E \max_{i \leqq n} |S_{m,i}|^\nu \leqq I_\nu g^{\nu/2}(n)$ for all $m \geqq 0$ and $n \geqq 1$. By (2.4), $g(n) = J_\nu^{2/\nu} An$ satisfies (2.5). Thus $E \max_{i \leqq n} |S_{m,i}|^\nu \leqq K_\nu (An)^{\nu/2}$ for each $m \geqq 0$ and $n \geqq 1$ as desired, taking $K_\nu = I_\nu J_\nu$.

It is now an easy matter to state a law of the iterated logarithm for generalized Gaussian random variables.

THEOREM 2.1. *Let $S_{m,n}$ be generalized Gaussian with parameter An for some positive number A and all $m \geqq 0$ and $n \geqq 1$. Then*

$$\limsup |S_n|/(2nA \log\log n)^{\frac{1}{2}} \leqq 1 \quad \text{a.s.} \tag{2.6}$$

PROOF. The proof is standard and consists in showing

$$\limsup |S_{n_k}|/(2n_k A \log\log n_k)^{\frac{1}{2}} \leqq 1 \quad \text{a.s.}$$

for an appropriate integer subsequence $\{n_k, k \geqq 1\}$ and then showing that

$$\max_{n_k \leqq n < n_{k+1}} |S_n - S_{n_k}|/(2n_k A \log\log n_k)^{\frac{1}{2}} \to 0 \quad \text{a.s.} \qquad \text{as } k \to \infty .$$

For details one can consult the proof of Serfling's Theorem 4.1 ([8]) which is virtually identical to the proof of Theorem 2.1.

The essential characteristic of Theorem 2.1 is that the X_i's are not assumed to have a specific dependence structure. That $S_{m,n}$ is generalized Gaussian with parameter An for $m \geqq 0$ and $n \geqq 1$ is the *only* restriction. A variety of dependence structures satisfy this assumption. For example, X_i's independent with mean zero and $|X_i| \leqq A^{\frac{1}{2}}$ a.s. or more generally X_i's martingale differences with $|X_i| \leqq A^{\frac{1}{2}}$ a.s. satisfy the assumption. X_i's independent normal random variables with mean zero and variance A also satisfy the assumption. Note that the conclusion of Theorem 2.1 is sharp when the X_i's are independent normal random variables with mean zero and variance A. An essentially weaker version of Theorem 2.1 is proved by Csaki [3] where $\{S_n, n \geqq 1\}$ is in addition assumed to be a submartingale. A result due to Serfling (Theorem 4.1, [8]) is closely related too: Let $|X_i| \leqq A^{\frac{1}{2}}$ a.s. and $EX_i = 0$ for $i \geqq 1$ and (2.3) and (2.4) hold for $m \geqq 0$ and $n \geqq 1$. Then (2.6) holds.

Theorem 2.1 yields a known law of the iterated logarithm for multiplicative random variables.

DEFINITION 2.1. Random variables $\{X_i, i \geqq 1\}$ are multiplicative if $E(X_{i_1} X_{i_2} \cdots X_{i_n}) = 0$ for all $1 \leqq i_1 < i_2 < \cdots < i_n$ and all $n \geqq 1$. Note that a martingale difference sequence $\{X_i, i \geqq 1\}$ with $EX_1 = 0$ and all moments finite is a multiplicative sequence. It is easy to prove (see Azuma [1]) that if the X_i's are multiplicative with $|X_i| \leqq A^{\frac{1}{2}}$ a.s. for all $i \geqq 1$, then $S_{m,n}$ is generalized Gaussian with parameter nA for all $m \geqq 0$ and $n \geqq 1$.

COROLLARY 2.1. (Takahashi [10] or Serfling [8]). *Let* $\{X_i, i \geqq 1\}$ *be multiplicative with* $|X_i| \leqq A^{\frac{1}{2}}$ a.s. *for some constant* A *and all* $i \geqq 1$. *Then*

$$\limsup |S_n|/(2An \log\log n)^{\frac{1}{2}} \leqq 1 \quad \text{a.s.}$$

PROOF. Immediate from the remark preceding Corollary 2.1 and from Theorem 2.1.

The basic Lemma 2.1 together with the maximal inequality approach of Serfling [7] can be used to give a relatively simple proof of an important and sharp law of the iterated logarithm for uniformly bounded equinormed strongly multiplicative random variables (Theorem 2.2 below) due to Takahashi [10].

DEFINITION 2.2. A multiplicative sequence $\{X_i, i \geqq 1\}$ is said to be equinormed strongly multiplicative (ESMS) if $EX_i^2 = 1$ for all $i \geqq 1$ and $E(X_{i_1}^{r_1} X_{i_2}^{r_2} \cdots X_{i_n}^{r_n}) = \prod_{j=1}^n EX_{i_j}^{r_j}$ for all $1 \leqq i_1 \leqq i_2 < \cdots < i_n$, all r_j's such that $r_j = 1$ or 2 for $j = 1, 2, \cdots, n$, and all $n \geqq 1$. The lemma below, whose proof we give for completeness, is due to Serfling [6].

LEMMA 2.2. *Let* $\{X_i, i \geqq 1\}$ *be a uniformly bounded* ESMS *sequence. Then, given* $\delta > 0$,

$$E\exp(uS_n) \leqq \exp[(1+\delta)u^2n/2] \qquad \textit{for all} \quad n \geqq 1$$

and $|u|$ *sufficiently small.*

PROOF. Series expansion shows that $e^x \leqq 1 + x + (1+\delta)x^2/2$ for $|x|$ sufficiently small. Fix $n \geqq 1$. Thus

$$\begin{aligned} E\exp(uS_n) &= E\textstyle\prod_{i=1}^n \exp(uX_i) \\ &\leqq E\textstyle\prod_{i=1}^n [1 + uX_i + (1+\delta)u^2X_i^2/2] \end{aligned}$$

for u sufficiently small, using the uniform boundedness of the X_i's. Using the ESMS assumption,

$$\begin{aligned} E\exp(uS_n) &\leqq [1 + (1+\delta)u^2/2]^n \\ &\leqq \exp[(1+\delta)u^2n/2]\,. \end{aligned}$$

THEOREM 2.2. (Takahashi [10]). *Let* $\{X_i, i \geqq 1\}$ *be a uniformly bounded* ESMS *sequence. Then*

$$\limsup |S_n|/(2n \log\log n)^{\frac{1}{2}} \leqq 1 \quad \text{a.s.}$$

PROOF. Fix $n \geqq 1$ and $\delta > 0$. By Lemma 2.2, $E\exp(uS_n) \leqq \exp[(1+\delta)u^2n/2]$ for $|u|$ sufficiently small. Let $u_n = (2\log\log n/n)^{\frac{1}{2}}$. Thus $E\exp(u_nS_n) \leqq \exp[(1+\delta)u_n^2n/2]$ for n sufficiently large. Taking $n_k = [\exp k^a]$ for $k \geqq 1$ where a satisfies $(1+\delta)^{-1} < a < 1$, it follows that $\sum_{k=1}^\infty E\exp[u_{n_k}S_{n_k} - 2(1+\delta)\log\log n_k] < \infty$. Thus, with probability one, $\sum_{k=1}^\infty \exp[u_{n_k}S_{n_k} - 2(1+\delta) \times \log\log n_k] < \infty$ and therefore $u_{n_k}S_{n_k} - 2(1+\delta)\log\log n_k \leqq 0$ for k sufficiently large. Thus with probability one, $S_{n_k}/(2n_k \log\log n_k)^{\frac{1}{2}} \leqq 1 + \delta$ for k sufficiently large. Replacing X_i by $-X_i$ for $i \geqq 1$, with probability one

$$|S_{n_k}|/(2n_k \log\log n_k)^{\frac{1}{2}} \leqq (1 + \delta) \tag{2.7}$$

for k sufficiently large. As previously remarked, $S_{m,n}$ is generalized Gaussian with parameter An for some constant A, all $m \geqq 0$ and $n \geqq 1$. Thus, (2.1) holds by Lemma 2.1. Proceeding in the standard way (again see the proof of Theorem 4.1 in [8]),

$$\max_{n_k \leqq n < n_{k+1}} |S_n - S_{n_k}|/(2n_k \log\log n_k) \to 0 \quad \text{a.s.}$$

follows easily. Combining this with (2.7), it follows that

$$\limsup |S_n|/(2n \log\log n)^{\frac{1}{2}} \leqq 1 \quad \text{a.s.}$$

as desired.

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