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A CONTINUUM OF COLLISION PROCESS LIMIT THEOREMS[1]

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Let $\{x_i(t): i = \cdots -1, 0, 1, \cdots\}$ be a collection of one-dimensional symmetric stable processes of order $\gamma \in (0, 1]$ with the property that the starting positions $\cdots < x_{-1}(0) < x_0(0) = 0 < x_1(0) < \cdots$ form a Poisson system with rate one. By generalizing the order preserving property of elastic collision, these can be used to define a set of collision processes $\{y_i(t)\}$. It is shown in this paper that for large values of A, the finite dimensional distributions of $y_0(At)/A^{1/2\gamma}$ approach the Gaussian distribution with mean zero and covariance $r(t, s) = c(t^{1/\gamma} + s^{1/\gamma} - |t - s|^{1/\gamma})$.

0. Introduction. A collision path $y_0(t)$ is a function generated by a collection of functions $\{x_i(t)\}$ which are thought of as colliding elastically. A collision process is formed if $\{x_i(t)\}$ is a collection of stochastic processes. The limit theorem presented in this paper (Theorem 2.2) gives a continuum of limit results for collision processes which interpolates between the limit results for two cases previously described in the literature [3], [6]. Section 1 contains the analysis necessary for the definition of collision processes. Section 3 contains the proof of the result stated in Section 2.

1. Definition and existence of collision processes. If two identical point masses traveling in one dimension collide elastically, they simply exchange trajectories. If, instead of two, there are $2n + 1$ point masses, the effect of elastic collisions will be to keep the middle particle on the middle trajectory. The following definition is the obvious way to generalize this notion to the case where there are an infinite number of paths $x_i(t)$ and the theorem from Harris [3] assures that the definition is non-vacuous.

DEFINITION 1.1. Let $\{x_i(t): i = 0, \pm 1, \cdots\}$ be a collection of real-valued functions of a nonnegative real variable. The set $\{y_i(t): i = 0, \pm 1, \cdots\}$ of collision paths generated by $\{x_i(t)\}$ is defined for nonnegative t by

$$y_i(t) = \lim_{n\to\infty} \text{med}\,(x_{i-n}(t), \cdots, x_{i+n}(t))\,.$$

THEOREM 1.2. *Let $\{x_i(t)\}$ and $\{y_i(t)\}$ be as in the above definition. Let $\{x_i(t)\}$ satisfy:*

(A1) $x_i(0) \leqq x_{i+1}(0)$ *for* $i = \cdots -1, 0, 1, \cdots$ *and* $x_0(0) = 0$.

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(A2) *For each i and t,* $\inf_{0\leq\tau\leq t} x_i(\tau)$ *and* $\sup_{0\leq\tau\leq t} x_i(\tau)$ *exist and satisfy*

$$\lim_{i\to\infty} \inf_{0\leq\tau\leq t} x_i(\tau) = \infty \qquad and \qquad \lim_{i\to-\infty} \sup_{0\leq\tau\leq t} xi(t) = -\infty .$$

(A3°) *If* $i \neq j$, *then* $\{t\colon x_i(t) = x_j(t)\}$ *does not contain a nondegenerate interval.*

(A4°) *For each* i, $x_i(t)$ *is continuous for* $t \geqq 0$.

Then the collection $\{y_i(t)\}$ is well defined and is the unique set of functions with the following properties:

(B1) $y_i(0) = x_i(0)$ for $i = \cdots, -1, 0, \cdots$.

(B2) $y_i(t) \leqq y_{i+1}(t)$ for $i = \cdots, -1, 0, 1, \cdots$ and for $t \geqq 0$.

(B3) For each $t \geqq 0$ $\lim_{i\to\infty} \inf_{0\leq\tau\leq t} y_i(\tau) = \infty$ and $\lim_{i\to-\infty} \sup_{0\leq\tau\leq t} y_i(\tau) = -\infty$.

(B4) The union of the graphs of the y_i is identical to the union of the graphs of the x_i.

(B5°) For $i \neq j$, $\{t\colon y_i(t) = y_j(t)\}$ does not contain a nondegenerate interval.

(B6°) The $y_i(t)$ are continuous in t.

In order to apply this machinery to a wider class of initial functions $x_i(t)$, we wish to drop the continuity requirement (A4°). Obviously (B6°) goes with it. However, the following example indicates that uniqueness is lost if these two conditions are simply discarded. Define $x_i(t) \equiv i$ and $z_i(t) = i + [t]$ where [ ] denotes the greater integer function. Let $y_i(t)$ be defined by Definition 1.1, i.e., $y_i(t) \equiv i$. Now both $\{y_i(t)\}$ and $\{z_i(t)\}$ satisfy (B1) through (B5°) above. We approach the problem with the following definition.

DEFINITION 1.3. Let $x_i(t)$ be a collection of functions. The discontinuities of $\{x_i(t)\}$ are said to be semi-isolated in i if for any given t_0 only a finite number of the x_i are discontinuous at t_0.

The condition that the discontinuities are semi-isolated in i will be denoted by (A4) when referring to the initial trajectories $\{x_i(t)\}$ and (B6) when referring to the collision paths $\{y_i(t)\}$. Condition (A3°) will be replaced by

(A3) for $i \neq j$, $\{t\colon x_i(t) = x_j(t)\}$ is nowhere dense.

The analogous condition for the collision paths will be denoted (B5). Now the desired theorem reads

THEOREM 1.4. *Let* $\{x_i(t)\}$ *satisfy* (A1)—(A4). *Then the collection* $\{y_i(t)\}$ *is well defined and satisfies* (B1)—(B6). *Further, if* $\{z_i(t)\}$ *is any other set of functions satisfying* (B1)—(B6), *then* $\{t\colon y_i(t) \neq z_i(t)\}$ *is nowhere dense for each* i.

The proof that the definition is meaningful and that properties (B1)—(B4) hold is unchanged from Harris'. We state below two needed facts and give proofs of the remaining properties. For all positive n and for $t \in [0, T]$

$$\text{(1.1a)} \qquad \text{med}\,(x_{i-n}(t), \cdots, x_{i+n}(t)) \geqq \min_{j\geq i} \inf_{0\leq t\leq T} x_j(t) = \alpha_i(T) > -\infty$$

and

$$\text{(1.1b)} \qquad \text{med}\,(x_{i-n}(t), \cdots, x_{i+n}(t)) \leqq \max_{j\leq i} \sup_{0\leq t\leq T} x_j(T) = \beta_i(T) < \infty .$$

Further, there exists an integer k such that

$$(1.2) \qquad \operatorname{med}(x_{i-k}(t), \cdots, x_{i+k}(t)) = \operatorname{med}(x_{i-k-n}(t), \cdots, x_{i+k+n}(t))$$

where t and n are as before. Harris actually proved a more general version of (1.2) based on the $k + l + 1$-tuple $(x_{i-k}(t), \cdots, x_{i+l}(t))$ where k and l are allowed to increase independently.

For (B5) it suffices to show that for a fixed value of i, $E = \{t : t \leqq T, y_i(t) = y_{i+1}(t)\}$ is nowhere dense. By (1.1), (1.2) and (A2) it is possible to find a k such that for $t \leqq T$

$$y_i(t) = \operatorname{med}(x_{i-k}(t), \cdots, x_{i+k}(t))$$
$$y_{i+1}(t) = \operatorname{med}(x_{i-k+1}(t), \cdots, x_{i+k+1}(t))$$
$$x_{i+k+1}(t) > \beta_{i+1}(T)$$

and

$$x_{i-k}(t) < \alpha_i(T) .$$

One then has that

$$E \subset \bigcup_{n=i-k}^{i+k} \bigcup_{m=n+1}^{i+k+1} \{t : t \leqq T, x_n(t) = x_m(t)\} .$$

Thus E is contained in the union of a finite number of nowhere dense sets, and is itself nowhere dense.

We next prove that (B6) holds. Let t_0 be arbitrary. Suppose first that all the x_i are continuous at t_0. Then for any j, it is possible to choose a k by (1.2) such that in a neighborhood of t_0, y_j will be the median of a finite collection of functions $(x_{i-k}(t), \cdots, x_{i+k}(t))$. Since each x_j in this collection is continuous at t_0, y_i also is.

If, on the other hand, a finite number of the x_i say $x_{i_1}, \cdots, x_{i_n}$, are discontinuous at t_0, then set

$$k = \lim \max_{\varepsilon \to 0} \{j : y_j(t) = x_{i_r}(t) \text{ for some }$$
$$t \in (t_0 - \varepsilon, t_0 + \varepsilon) \text{ and } r = 1, 2, \cdots, n\}$$

and

$$l = \lim \min_{\varepsilon \to 0} \{j : y_j(t) = x_{i_r}(t) \text{ for some }$$
$$t \in (t_0 - \varepsilon, t_0 + \varepsilon) \text{ and } r = 1, 2, \cdots, n\} .$$

Since the discontinuities of the x_{i_r} at t_0 are finite by (A2) and since all but a finite number of the y_j will be outside a bounded set by (B3), we see that k and l are finite integers. Let i be an integer not in $[l, k]$. In some ε-neighborhood of t_0, y_i can be expressed as the median of an odd number of functions. But all discontinuities at t_0 in this set of functions will either be above $y_i(t)$ if $i < l$ or below $y_i(t)$ if $i > k$. Therefore, y_i will be continuous at t_0.

To show uniqueness, suppose that $\{z_i(t)\}$ is another set of functions satisfying (B1)—(B6). Let $G =$ interior (closure $\{t : z_i(t) \neq y_i(t)\}$) and assume $G \neq \emptyset$. Let $t_0 = \inf G$. Because of the semi-isolated condition on the discontinuities and because the union of the graphs of the z_i is the same as the union of the graphs of the y_i, it is possible to choose integers k and k' such that the following conditions hold:

(a) $y_k(t_0) = z_{k'}(t_0)$
(b) $y_k(t_0) > \max\{y_i(t_0), z_i(t_0)\}$
(c) the functions y_i, y_{k+1}, $z_{k'}$, $z_{k'+1}$ are all continuous at t_0
(d) $y_k(t_0) < y_{k+1}(t_0)$ and $z_{k'}(t_0) < z_{k+1}(t_0)$.

Similarly, we may choose l and l' such that y_l, $z_{l'}$, y_{l-1}, and $z_{l'-1}$ satisfy conditions (a)—(d) with the inequality signs reversed and with "max" replaced by "min."

Choose $\varepsilon > 0$ such that

$$\varepsilon < \tfrac{1}{3}\max\{|y_k(t_0) - y_{k+1}(t_0)|, |z_{k'}(t_0) - z_{k'+1}(t_0)|, \\ |y_l(t_0) - y_{l-1}(t_0)|, |z_{l'}(t_0) - z_{l'-1}(t_0)|\}.$$

Choose δ such that $0 < |t - t_0| < \delta$ implies that the value at t of each of the above functions (i.e., $y_k, \cdots, z_{l'-1}$) is within ε of its value at t_0. We now restrict our attention to the rectangle

$$R = \{(x, t)\colon x \in (y_k(t_0) + \varepsilon, y_l(t_0) - \varepsilon) \text{ and } t \in (t_0 - \delta, t_0 + \delta) \cap [0, \infty)\}.$$

in which the y-indices run from l to k and the z-indices from l' to k'.

Define

$$\Gamma = \{t\colon t \in (t_0 - \delta, t_0 + \delta) \cap [0, \infty), y_n(t) \neq y_m(t) \text{ for } n \neq m;\ n, m \in \{l, \cdots, k\} \\ \text{and } z_n(t) \neq z_m(t) \text{ for } n \neq m;\ n, m \in \{l', \cdots, k'\}\}.$$

Note that, by (A1) and (B1), zero will be in Γ if $t_0 - \delta < 0$ and that by (B5) Γ is the complement of a nowhere dense set. Choose $t_1 \in \Gamma \cap \{t\colon t \leqq t_0 \text{ and } y_i(t) = z_i(t)\}$. Such a point exists because, if $t_0 > 0$ then $\{t\colon t \leqq t_0 \text{ and } z_i(t) = y_i(t)\}$ is also the complement of a nowhere dense set and, if $t_0 = 0$, then $t_1 = t_0$.

Now in the cross section of R given by $R \cap \{(x, t_1)\}$, the indices of y_j and z_j are determined by (B2) and (B4). This gives $k = k'$ and $l = l'$. Let $t_2 \in \Gamma \cap \{t\colon t \geqq t_0\}$. Because of (B4), the point (a, t_2) where

$$a = \sup\{x\colon (x, t_2) \in R \text{ and } x = x_i(t_2) \text{ for some } i\},$$

is equal to $(y_j(t_2), t_2)$ and $(z_{j'}(t_2), t_2)$ for some j and j'. But j (respectively j') must be the maximum y index (z index) in the rectangle R. So $a = y_k(t_2) = z_k(t_2)$. Similarly $y_l(t_2) = z_l(t_2)$. As we previously counted out from $y_i(t_1) = z_i(t_1)$ at t_1, we can now count in from $y_k(t_2) = z_k(t_2)$ and $y_l(t_2) = z_l(t_2)$. There are $k - l - 1$ indices to be assigned to $k - l - 1$ distinct points $y_j(t_2)$ and to $k - l - 1$ distinct points $z_j(t_2)$. If this assignment is to satisfy (B2), we must have $y_i(t_2) = z_i(t_2)$.

Since $\Gamma \cap \{t\colon t \geqq t_0\}$ is the complement of a nowhere dense set in $(t_0, t_0 + \delta)$ this contradicts the original assumption that $G \neq \varnothing$ and completes the proof.

It is easy to see that complete uniqueness is not given without a further restriction. For an appropriate example, suppose that $\{x_i(t)\}$ is a collection of continuous paths, exactly two of which cross at t_0, i.e., $x_{i_1}(t_0) = x_{i_2}(t_0)$ for $i_1 \neq i_2$.

There are exactly two subscripts j and $j+1$ such that $y_j(t_0) = y_{j+1}(t_0) = x_{i_1}(t_0)$. We may redefine $y_{j+1}(t_0)$ by setting it equal to $y_{j+2}(t_0)$ without violating (B1)—(B6). However, the following obvious corollary gives complete uniqueness in the case which will be of interest to us in the next section.

COROLLARY 1.5. *Let $\{x_i(t)\}$, $\{y_i(t)\}$ and $\{z_i(t)\}$ be as in the above theorem. If each x_i is right-continuous, then each y_i is right-continuous. If further each z_i is right-continuous, then $y_i(t) \equiv z_i(t)$ for each i.*

2. Statement of results concerning collision processes. In applying the above theorem we shall assume that the starting positions $x_i(0)$ are arranged such that $x_0(0) = 0$ and such that $x_i(0)$ and $x_{-i}(0)$ for $i = 1, 2, \cdots$ form independent Poisson systems with rate one on $(0, \infty)$ and $(-\infty, 0)$ respectively. This assumption has two advantages besides the fact that it seems natural. First, it allows certain computations to be done as integrals instead of sums as would be the case with non-random initial conditions. Also, according to a theorem of Doob ([1] page 405), if each increment $x_i(t) - x_i(0)$ for $i = \pm 1, \pm 2, \cdots$ has a distribution which is independent of i, of the other increments and of the starting positions $\{x_j(0) : j = \pm 1, \pm 2, \cdots\}$, then the points $\{x_i(t) : i = \pm 1, \pm 2, \cdots\}$ again form a Poisson process with rate one.

There are two continuous collision processes which have been described in the literature. Because it will make sense later, we will identify these as the $\gamma = 2$ and $\gamma = 1$ cases.

In the $\gamma = 2$ case, the $\{x_i(t)\}$ is a collection of Wiener processes, stable processes of order 2. This situation was first described by Harris [3]. He showed that for large t, $y_0(t)$ was normally distributed with mean zero and variance $O(t^{\frac{1}{2}})$ In a private communication, Spitzer showed that this fact and independent increments in the limit indicated a limiting covariance for $y_0(At)/A^{\frac{1}{4}}$ of the form $r(t, s) = c(t^{\frac{1}{2}} + s^{\frac{1}{2}} - |t-s|^{\frac{1}{2}})$.

For the case $\gamma = 1$, $\{x_i(t)\}$ is a collection of straight line processes, $x_i(t) = x_i(0) + v_i t$ where the v_i are independent random variables with mean zero. Since $P[x_0(ct) \leqq \alpha] = P[x_0(t) \leqq \alpha/c] = P[v_0 \leqq \alpha/(ct)]$, the distribution function F_t of $x_0(t)$ satisfies $F_{ct}(x) = F_t(xc^{-1})$. Thus $\{x_i(t)\}$ is, in a trivial sense, a collection of stable processes of order one. This situation was investigated by Spitzer in [6] where he showed that $\lim_{A\to\infty} y_0(At)/A^{\frac{1}{2}}$ was a Wiener process. The convergence involved was weak convergence on the function space $C[0, 1]$.

The extension of these results that we have in mind has now been clearly hinted at. We proceed as follows.

THEOREM 2.1. *Let the set of initial positions $\{x_i(0)\}$ be Poisson distributed as outlined above. Let $\{x_i(t) - x_i(0)\}$ be a set of independent, identically distributed, symmetric stable processes of order $\gamma \in (1, 2]$, which are taken to be separable and right-continuous. Then with probability one the set of paths $\{x_i(t)\}$ satisfies* (A1)—(A4).

PROOF. Note that (A1) is trivial. The assertion that $\inf_{0 \leqq \tau \leqq t} x_i(\tau)$ and

$\sup_{0\leq\tau\leq t} x_i(\tau)$ exist holds because $x_i(t)$ is a separable martingale ([1] page 361). To show the second half of (A2), assume first that $\lim_{i\to\infty} \inf_{0\leq\tau\leq t} x_i(\tau) = m < \infty$. By translation, we may assume $m < 0$. Let B_k be the event that $\inf_{0\leq\tau\leq t} x_i(\tau) < 0$. Clearly, $\lim_{i\to\infty} \inf_{0\leq\tau\leq t} x_i(\tau) = m < 0$ implies that B_k happens for infinitely many k. By the Borel–Cantelli lemma $\sum_{k=1}^{\infty} \Pr[B_k] = \infty$.

Because the paths $x_i(t)$ satisfy the strong Markov property and have symmetrically distributed increments, the paths satisfy a reflection principle, i.e.,

$$\Pr[B_k] \leqq 2\Pr[x_k(t) < 0] .$$

This gives that $\sum_{k=1}^{\infty} \Pr[x_k(t) < 0] = \infty$.

Since the initial points $x_k(0)$ form a Poisson process with rate one, and since the paths have identical distributions F_t

$$\sum_{k=1}^{\infty} \Pr[x_k(t) < 0] = \sum_{k=1}^{\infty} F_t(-x_k(0)) .$$

This is a random variable, call it X, which depends on the Poisson process $x_k(0)$. By Campbell's theorem ([2] page 176),

$$E[X] = \int_0^{\infty} F_t(-s)\, ds < \infty$$

because the stable distribution of order γ has finite absolute first moment when $\gamma > 1$. Thus X is almost surely finite, a contradiction which completes the proof of (A2).

Because the sum of two independent stable processes $x_i(t) - x_j(t)$ is, after centering, distributed the same as $x_0(2t)$, (A3) will hold if the zero set Γ of a stable process is nowhere dense. By right continuity, if t is the limit point, from the right, of points in Γ, t will also be Γ. Therefore if Γ is dense in any open set O, Γ will contain O and have positive measure. But by Fubini's theorem

$$\begin{aligned}\mathscr{E}[\lambda(\Gamma \cap [0, M])] &= \mathscr{E}[\textstyle\int_0^M \chi_{[x(t)=0]}\, d\lambda(t)] \\ &= \textstyle\int_0^M \mathscr{E}[\chi_{[x(t)=0]}\, d\lambda(t) \\ &= \textstyle\int_0^M \Pr[x(t) = 0]\, d\lambda(t) = 0\end{aligned}$$

where λ denotes Lebesgue measure. A positive random variable with zero expectation is zero with probability one, therefore Γ is nowhere dense.

For a proof of (A4), let $J_{i,n}(t)$ be the number of discontinuities of magnitude larger than $1/n$ in the path $x_i(s)$ for $s \in [0, t]$. Following Doob ([1] page 423), $J_{i,n}(t)$ is a Poisson process with some parameter $\lambda(n)$. Accordingly, when $i \neq j$, $J_{i,n}(t) + J_{j,n}(t)$ is a Poisson process with parameter $2\lambda(n)$. The event that $x_i(t)$ and $x_j(t)$ both have discontinuities at the point t_0 is the event that, for some n, $J_{i,n}(t) + J_{j,n}(t)$ has a jump of two units, an event of probability zero ([1] page 399). Thus the probability that any two paths will have discontinuities at the same value of t is bounded above by

$$\sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} \sum_{j=i+1}^{\infty} \Pr[J_{i,n}(t) + J_{j,n}(t) \text{ has a two unit jump}] = 0 .$$

Thus, with probability one, the discontinuities in $\{x_i(t)\}$ are "isolated in i" and consequently semi-isolated in i.

THEOREM 2.2. *If $\{x_i(t)\}$ is a collection of stable processes as in Theorem* 2.1, *then*

$$\lim_{A\to\infty} \Pr [y_0(At_i)/A^{1/2\gamma} \leqq \alpha_i, i = 1, 2, \cdots, n]$$

is a Gaussian distribution with mean 0 *and covariance*

$$r(t, s) = C_\gamma(2\pi)^{-\frac{1}{2}}(t^{1/\gamma} + s^{1/\gamma} - |t - s|^{1/\gamma}) \tag{2.1}$$

where C_γ is a constant which will be defined in the proof.

This theorem will be proved in the next section using a generalization of Spitzer's proof for the uniform velocity case [6]. It should be noted that the Gaussian process with covariance given by (2.1) can be realized with continuous sample paths ([5] page 98). Further, formula (2.1) with $\gamma = 1$ is the correct covariance for the uniform velocity case. However, when $\{x_i(t)\}$ is a family of nontrivial stable processes of order one, i.e., Cauchy processes, the proof of (A2) in Theorem 2.1 fails and the constant C_γ in (2.1) is infinite. The establishment of tightness and weak convergence is still missing from this limit theorem. Spitzer was able to establish tightness by using the fact that uniform velocity processes $x_i(t) = x_i(0) + v_i t$ cross a straight line $x = M$ only once. Stable processes, of course, can cross a straight line many times.

3. Proof of Theorem 2.2. Let $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$ and $\mathbf{t} = (t_1, \cdots, t_n) \in [0, \infty)^n$ be given position and time vectors. The vector $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ will also be used below. The symbol $R(\mathbf{t})$ will denote an $n \times n$ matrix function of $\mathbf{t}$ defined by

$$R(\mathbf{t}) = \frac{C_\gamma}{(2\pi)^{\frac{1}{2}}} (t_i^{1/\gamma} + t_j^{1/\gamma} - |t_i - t_j|^{1/\gamma}) .$$

We will use $V(\mathbf{t}, \boldsymbol{\alpha})$ and $W(\mathbf{t}, \boldsymbol{\alpha})$ to denote the vector-valued function defined by

$$\begin{aligned} V(\mathbf{t}, \boldsymbol{\alpha}) &= (v(t_i, \alpha_i)) = (\textstyle\sum_{k=1}^{\infty} \chi_{[x_k(t_i) \leq \alpha_i]}) \\ W(\mathbf{t}, \boldsymbol{\alpha}) &= (w(t_i, \alpha_i)) = (-\textstyle\sum_{k=-\infty}^{-1} \chi_{[x_k(t) > \alpha]}) . \end{aligned} \tag{3.1}$$

These two functions are related to the distribution of $y_0(t)$ through the equality of the following two events: $\{y_0(t_i) \leqq \alpha_i\}$ and $\{v(t_i, \alpha_i) + w(t_i, \alpha_i) + \chi_{[x_0(t_i) \leq \alpha_i]} \geqq 1\}$ (see Harris [3] Theorem 5.1).

Next we note that by the technique of subordination ([2] page 336), we may take

$$F_t(y) = \textstyle\int_0^\infty N_s(y) U_t(ds) \tag{3.2}$$

where $F_t(\cdot)$ is the distribution function for the increments $x_i(t + s) - x_i(s)$ of our stable processes, where $N_s(\cdot)$ is the normal distribution function with mean zero and variance s, and where $U_t(\cdot)$ is the distribution function for a positive stable process of order $\gamma/2$. By using (3.2) the $\gamma = 2$ case will be the key to our calculations. For $\gamma = 2$, let U_t denote the point mass at t.

The proof will consist of evaluating the characteristic function

$$\mathscr{E}[\exp\{iA^{-\frac{1}{2}}\boldsymbol{\lambda} \cdot (V(A\mathbf{t}, A^{1/2\gamma}\boldsymbol{\alpha}) + W(A\mathbf{t}, A^{1/2\gamma}\boldsymbol{\alpha}))\}] .$$

Following Spitzer ([6] formula 2.13), we have

$$
\begin{aligned}
&-\log \mathscr{E}[\exp\{iA^{-1/2\gamma}\boldsymbol{\lambda}(V+W)\}] \\
&\quad = \textstyle\int_0^\infty \int_\Omega 1 + (i \sum_{j=1}^n \lambda_j A^{-1/2\gamma} I_j) + \frac{1}{2}(i \sum_{j=1}^n \lambda_j A^{-1/2\gamma} I_j)^2 \\
&\qquad - \exp\{i \textstyle\sum_{j=1}^n \lambda_j A^{-1/2\gamma} I_j\}\, dP\, dx \\
&\qquad + \textstyle\int_0^\infty \int_\Omega 1 + (-i \sum_{j=1}^n \lambda_j A^{-1/2\gamma} J_j) + \frac{1}{2}(-i \sum_{j=1}^n \lambda_j A^{-1/2\gamma} J_j)^2 \qquad (3.3)\\
&\qquad - \exp\{-i \textstyle\sum_{j=1}^n \lambda_j A^{-1/2\gamma} J_j\}\, dP\, dx \\
&\qquad + (-i) \textstyle\int_0^\infty \int_\Omega \sum_{j=1}^n \lambda_j A^{-1/2\gamma}(I_j - J_j)\, dP\, dx \\
&\qquad + \frac{1}{2} \textstyle\int_0^\infty \int_\Omega (\sum_{j=1}^n \lambda_j A^{-1/2\gamma} I_j)^2 + (\sum_{j=1}^n \lambda_j A^{-1/2\gamma} J_j)^2\, dP\, dx
\end{aligned}
$$

where $\int_\Omega \cdots dP$ denotes the integral over all sample paths with respect to the measure determined by the stable distributions F_t. The symbols I_j and J_j denote functions defined on this space of sample paths by

$$I_j = \chi_{[x(At_j) - x(0) \leqq \alpha_j A^{1/2\gamma} - x]}$$

and

$$J_j = \chi_{[x(At_j) - x(0) \geqq \alpha_j A^{1/2\gamma} + x]} \,.$$

The proof that the first, second, and third integrals on the right side of (3.3) are given by $O(A^{-1/2\gamma})$, $O(A^{-1/2\gamma})$ and $-i\boldsymbol{\lambda} \cdot \boldsymbol{\alpha} + O(A^{-1/2\gamma})$ respectively for large A is the same as the proof in Spitzer. The integral $\int_\Omega \cdots dP$ in this case involves only the (one-dimensional) distribution $\int_R \cdots F_t(dy)$ and the result depends on the fact that F_t has a finite first moment.

The final integral in (3.3) can be expressed as a sum involving terms of the form

$$\lambda_j \lambda_k A^{-1/\gamma} \textstyle\int_0^\infty \int_\Omega I_j I_k\, dP\, dx \,. \tag{3.4}$$

Assume that $t_j \geqq t_k$, let $t = t_k$ and let $s = t_j - t_k$. Now (3.4) becomes

$$
\begin{aligned}
&\lambda_j \lambda_k A^{-1/\gamma} \textstyle\int_0^\infty P[x(A(t+s)) - x(0) \leqq \alpha_j A^{1/2\gamma} - x \text{ and} \\
&\qquad x(At) - x(0) \leqq \alpha_k A^{1/2\gamma} - x \,|\, x(0) = x]\, dx \\
&\qquad = \lambda_j \lambda_k A^{-1/\gamma} \textstyle\int_0^\infty dx \int_{-\infty}^{(\alpha_k A^{-1/2\gamma} - x)} F_{At}(dy) F_{As}(\alpha_j A^{-1/2\gamma} - x - y) \\
&\qquad = \lambda_j \lambda_k A^{-1/\gamma} \textstyle\int_0^\infty dx \int_{-\infty}^{-x} F_{At}(dy) F_{As}(-x - y) + O(A^{-1/2\gamma}) \,.
\end{aligned}
$$

The error term in the last line is bounded by

$$
\begin{aligned}
&\lambda_j \lambda_k A^{-1/\gamma} \textstyle\int_0^\infty P[-x \leqq x(At) - x(0) \leqq \alpha_k A^{1/2\gamma} - x]\, dx \\
&\qquad = \lambda_j \lambda_k \{\textstyle\int_{-\infty}^{\alpha_k t^{-1/\gamma} A^{-1/2\gamma}} F_1(y)\, dy - \int_{-\infty}^0 F_1(y)\, dy\} = O(A^{-1/2\gamma}) \,.
\end{aligned}
$$

Upon application of (3.2) and Fubini's theorem and after ignoring the $O(A^{-1/2\gamma})$ error term (3.4) may be written as

$$
\begin{aligned}
&\lambda_j \lambda_k \textstyle\int_0^\infty U_{As}(du) \int_0^\infty U_t(dv) \{\int_0^\infty dx \int_{-\infty}^{-x} A^{-1/\gamma} N_u(-x - y) N_v(dy)\} \\
&\qquad = \lambda_j \lambda_k \textstyle\int_0^\infty U_s(du) \int_0^\infty U_t(dv) \{\int_0^\infty dx \int_{-\infty}^{-x} N_u(-x - y) N_v(dy)\} \,.
\end{aligned}
$$

Looking first at the part in brackets, denote this by $\varphi(u)$. We have

$$
\begin{aligned}
\varphi(u) &= \textstyle\int_0^\infty dx \int_{-\infty}^{-x} N_u(-x - y) N_v(dy) \\
&= \textstyle\int_0^\infty N_v(dy) \int_0^y N_u(x)\, dx
\end{aligned}
$$

after some rearranging. Further, since the normal density satisfies the heat equation,

$$\varphi'(u) = 1/(4(2\pi)^{\frac{1}{2}})[(u + v)^{-\frac{1}{2}} - u^{-\frac{1}{2}}] .$$

Thus $\varphi(u) = 1/(2(2\pi)^{\frac{1}{2}})((u + v)^{\frac{1}{2}} - u^{\frac{1}{2}} + C)$. The constant is evaluated by noting that, for $u = 0$,

$$\varphi(0) = \int_0^\infty N_v(dy) \cdot y = v^{\frac{1}{2}}/(2\pi)^{\frac{1}{2}} .$$

Therefore $\varphi(u) = 1/(2(2\pi)^{\frac{1}{2}})(v^{\frac{1}{2}} + (u + v)^{\frac{1}{2}} - u^{\frac{1}{2}})$.

Applying this in the computation for (3.4) given that

$$\begin{aligned}\lambda_j \lambda_k A^{-1/\gamma} \int_0^\infty \int_\Omega I_i I_k \, dP \, dx \\ &= O(A^{-1/2\gamma}) + \lambda_j \lambda_k \int_0^\infty U_s(du) \int_0^\infty U_t(dv)[1/(2(2\pi)^{\frac{1}{2}})(v^{\frac{1}{2}} + (u + v)^{\frac{1}{2}} - u^{\frac{1}{2}})] \\ &= O(A^{-1/2\gamma}) + \lambda_j \lambda_k (C_\gamma/(2(2\pi)^{\frac{1}{2}}))(t^{1/\gamma} + (t + s)^{1/\gamma} - s^{1/\gamma})\end{aligned}$$

by the stable property.

The value of the constant C_γ is given by

$$C_\gamma = \int_0^\infty w^{\frac{1}{2}} U_1(dw) .$$

A calculation for

$$\lambda_j \lambda_k A^{-1/\gamma} \int_0^\infty \int_\Omega J_j J_k \, dP \, dx$$

proceeds similarly to the same result. After replacing t and s by t_k and $t_j - t_k$ respectively, one sees that the final integral in (3.2) has as a limit when A approaches infinity the value

$$\tfrac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \lambda_j \lambda_k (C_\gamma/(2\pi)^{\frac{1}{2}})(t_k^{1/\gamma} + t_j^{1/\gamma} - |t_k - t_j|^{1/\gamma}) = \tfrac{1}{2} \boldsymbol{\lambda} R(\mathbf{t}) \boldsymbol{\lambda}^T .$$

The proof that this calculation implies the desired result is the same as in Spitzer [5].

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