The Annals of Probability
1973, Vol. 1, No. 3, 488–496

AN ALMOST SURE INVARIANCE PRINCIPLE FOR MUTIVARIATE KOLMOGOROV–SMIRNOV STATISTICS[1]

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An almost sure invariance principle for Kolmogorov–Smirnov statistics for vector chance variables is established along the lines of Theorems 1.4 and 4.9 of Strassen [*Proc. Fifth Berkeley Symp. Math. Statist. Prob.* (1967) **2** 315–343]. This strengthens certain asymptotic expressions on the probability of moderate deviations for Kolmogorov–Smirnov statistics, obtained earlier by Gnedenko, Karoluk and Skorokhod, and by Kiefer and Wolfowitz, among others.

1. Introduction. Let $\{\mathbf{X}_i = (X_{i1}, \cdots, X_{ip})'; i \geqq 1\}$ be a sequence of independent and identically distributed random vectors (i.i.d. rv) defined on a probability space $(\Omega, \mathscr{A}, P)$, where $\mathbf{X}_i$ has a continuous distribution function (df) $F(\mathbf{x})$, $\mathbf{x} \in R^p$, the $p\,(\geqq 1)$-dimensional Euclidean space. Define the empirical df's by $F_n(\mathbf{x}) = n^{-1} \sum_{i=1}^n c(\mathbf{x} - \mathbf{X}_i)$, $\mathbf{x} \in R^p$, $n \geqq 1$, where $c(\mathbf{u}) = 1$, if all the p components of $\mathbf{u}$ are $\geqq 0$; otherwise $c(\mathbf{u}) = 0$. Consider then the general p-variate Kolmogorov–Smirnov statistics: $D_n^+ = \sup\{F_n(\mathbf{x}) - F(\mathbf{x}) : \mathbf{x} \in R^p\}$, $D_n^- = \sup\{F(\mathbf{x}) - F_n(\mathbf{x}) : \mathbf{x} \in R^p\}$ and $D_n = \sup\{|F_n(\mathbf{x}) - F(\mathbf{x})| : \mathbf{x} \in R^p\} = \max\{D_n^+, D_n^-\}$, $n \geqq 1$; all of these are nonnegative random variables.

Gnedenko, Koroluk and Skorokhod (1961, pages 154–155) reported for $p = 1$ the results of Karplevskaia of Li-Tsian that if $\{\lambda_n\}$ be a sequence of positive numbers such that $n\lambda_n^3 = O(1)$, then as $n \to \infty$,

$$(1.1) \qquad P\{D_n^+ \geqq \lambda_n\} = P\{D_n^- \geqq \lambda_n\} = \tfrac{1}{2}P\{D_n \geqq \lambda_n\}[1 + o(1)] = \exp\{-2n\lambda_n^2\}[1 + o(1)]\,.$$

Kiefer and Wolfowitz (1958), and later on, Kiefer (1961) have shown that for every $\varepsilon > 0$ and $p \geqq 1$, there exists a positive $c(p, \varepsilon)$, such that for every $n \geqq 1$, $\lambda_n \geqq 0$, $P\{D_n \geqq \lambda_n\} \leqq c(p, \varepsilon) \exp\{-(2 - \varepsilon)n\lambda_n^2\}$; but, in general, it is not possible to make $\varepsilon = 0$. From these, it readily follows that if $n\lambda_n^2 \to \infty$ as $n \to \infty$, then for every $p \geqq 1$,

$$(1.2) \qquad (n\lambda_n^2)^{-1} \log P\{D_n \geqq \lambda_n\} \to -2 \quad \text{as} \quad n \to \infty\,, \quad \text{whenever} \quad n\lambda_n^3 = O(1)\,,$$

and the same result holds for $\{D_n^-\}$ and $\{D_n^+\}$.

Received February 3, 1972; revised August 25, 1972.

[1] Work sponsored by the Aerospace Research Laboratories, Air Force Systems Command, U.S. Air Force, Contract No. F33615-71-C-1927. Reproduction in whole or in part permitted for any purpose of the U.S. Government.

AMS 1970 *subject classifications.* Primary 60B10, 60F15; Secondary 62E20.

Key words and phrases. Almost sure invariance principle, multivariate Kolmogorov–Smirnov statistics, probability of moderate deviations, reverse sub-martingales.

The object of the present investigation is to show that an *almost sure invariance principle* similar to the one in Theorems 1.4 and 4.9 of Strassen (1967) holds for the Kolmogorov–Smirnov statistics defined above. The main theorems along with the basic regularity conditions are formulated in Section 2. The proofs of the theorems, based on a reverse martingale property of D_n^+ (and D_n^-) and Theorem 1 of Kiefer (1961), are considered in Section 3. A few remarks are included in Section 2.

2. The main theorems. Let $\phi = \{\phi(t): 0 \leqq t < \infty\}$ be a positive function, defined on $[0, \infty)$, with a continuous derivative $\phi'(t)$, such that (i)

$$\psi(t) = t^{-\frac{1}{2}}\phi(t) \text{ is } \uparrow \text{ but } t^{-h}\phi(t) \text{ is } \downarrow \text{ in } t, \quad \tfrac{1}{2} < h < \tfrac{3}{5}, \tag{2.1}$$

(ii) as $t \to \infty$, with $s/t \to 1$,

$$\phi'(s)/\phi'(t) \to 1, \qquad (\Rightarrow \psi'(s)/\psi'(t) \to 1), \tag{2.2}$$

and (iii) the Kolmogorov–Petrovski–Erdös criterion holds for 2ϕ, i.e.,

$$J_n(k\phi) = (2\pi)^{-\frac{1}{2}}(k/2) \textstyle\int_n^\infty t^{-\frac{3}{2}}\phi(t)\exp\{-\tfrac{1}{2}k^2t^{-1}\phi^2(t)\}\,dt < \infty, \tag{2.3}$$

for every $k \geqq 2$, $n \geqq 1$. Note that by (2.1)

$$\psi'(t) = t^{-\frac{1}{2}}\phi'(t) - \tfrac{1}{2}t^{-\frac{3}{2}}\phi(t) \quad (> 0) \text{ is continuous in } t; \tag{2.4}$$

$$0 < (2t)^{-1}\phi(t) < \phi'(t) < 3(5t)^{-1}\phi(t). \tag{2.5}$$

Since the integrand in (2.3) decreases for large t, there exists a positive t_0 ($< \infty$), such that for all $t \geqq t_0$ and $k \geqq 2$,

$$\nu_k(t) = \tfrac{1}{2}k^2\psi^2(t) - \log\psi(t) - \log\log t \text{ is } \uparrow \text{ to } \infty \text{ as } t \uparrow \infty. \tag{2.6}$$

Thus, for large t, $\psi^2(t) > \frac{1}{2}\log\log t$ when (2.3) holds. We further assume that uniformly in n, $\nu_k(n)/\nu_2(n)$ is a continuous function of $k \in [2, 2+\delta]$, for some $\delta > 0$. Thus for every δ' ($0 < \delta' \leqq \delta$), there exists an $\eta > 0$, such that for $n \geqq n_0$,

$$|\nu_k(n)/\nu_2(n) - 1| < \eta \qquad \text{whenever} \quad |k-2| < \delta'. \tag{2.7}$$

In fact, if for some $\varepsilon > 0$, $\limsup_{t\to\infty}[(\log\log t)/\psi^2(t)] \leqq 2 - \varepsilon$, then (2.1)—(2.3) imply (2.7). A counterexample where (2.1)—(2.3) hold but not (2.7) is $\psi^2(t) = \frac{1}{2}\log\log t + \lambda\log\log\log t$ where λ (> 0) is a positive number. Let us now define

$$P_n(\psi) = P\{mD_m \geqq \psi(m) \text{ for some } m \geqq n\}, \tag{2.8}$$

and in (2.8) on replacing D_m by D_m^+ and D_m^-, we define $P_n^+(\psi)$ and $P_n^-(\psi)$, respectively. Then, we have the following.

THEOREM 1. *Under* (2.1), (2.2), (2.3) *and* (2.7),

$$\lim_{n\to\infty}\{[\log P_n(\psi)]/\nu_2(n)\} = -1, \tag{2.9}$$

and the same result holds for $\{P_n^+(\psi)\}$ *and* $\{P_n^-(\psi)\}$.

We may remark that whenever $\psi^2(n)$ increases with n in such a way that as

$n \to \infty$, $(\log \log n)/\psi^2(n) \to 0$, we do not require (2.2), (2,3) and (2.7), and we may also extend the range of $\psi(n)$ to $O(n^{\frac{1}{2}})$. We have the following.

THEOREM 2. *If for some* C: $0 < C < \infty$, $\psi(n) < Cn^{\frac{1}{2}}$ *and* $\lim_{n\to\infty} (\log \log n)/\psi^2(n) = 0$, *then*

$$\lim_{n\to\infty} \{[\log P_n(\psi)]/\psi^2(n)\} = -2 , \tag{2.10}$$

and the same result holds for $\{P_n^+(\psi)\}$ *and* $\{P_n^-(\psi)\}$.

We postpone the proof of the theorems to Section 3.

REMARKS. (I) If we let $\psi^2(t) = (\frac{1}{2} + \varepsilon) \log \log t$, $\varepsilon > 0$, it follows that (2.1), (2.2), (2.3) and (2.7) hold, where $\nu_2(t) = 2\varepsilon \log \log t - (\log \log \log t)/2 - \frac{1}{2} \log (\frac{1}{2} + \varepsilon)$ ($\to \infty$ as $t \to \infty$). Therefore, by (2.9), $P_n(\psi) \to 0$ as $n \to \infty$, i.e.,

$$P\{\limsup_n (2_n/\log \log n)^{\frac{1}{2}} D_n \leqq 1\} = 1 . \tag{2.11}$$

On the other hand, comparing D_n with any of its univariate version (which is always smaller or equal) and using the law of iterated logarithm for such a statistic [viz. Chung (1949)], we obtain that

$$P\{\limsup_n (2n/\log \log n)^{\frac{1}{2}} D_n \geqq 1\} = 1 . \tag{2.12}$$

Hence,

$$P\{\limsup_n (2n/\log \log n)^{\frac{1}{2}} D_n = 1\} = 1 , \tag{2.13}$$

and the same result holds for $\{D_n^+\}$ and $\{D_n^-\}$. (2.13) was proved with a different approach by Kiefer (1961), and we may also refer to Wichura (1973) and Kiefer (1972) for certain related results in comparatively more general setups.

(II) If we let $\psi(n) = n^{\frac{1}{2}}\lambda_n$, we observe that under the condition that $(\log \log n)/\psi^2(n) \to 0$ with $n \to \infty$, (2.10) extends (1.2) in the sense that $[D_n \geqq \lambda_n]$ is replaced by $[D_m \geqq \lambda_m$ for some $m \geqq n]$. This extension is comparable to Theorem 1.4 of Strassen (1967) which provides similar extension of the probability of moderate deviations for sample cumulative sums, studied earlier by Cramér (1938), Linnik (1961), Rubin and Sethuraman (1965), and others.

3. The proof of Theorems 1 and 2. For a positive k, we define

$$I_n(k\psi) = (2\pi)^{-\frac{1}{2}} \int_n^\infty k\psi'(t)t^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}k^2t^{-1}\psi^2(t)\}\, dt , \qquad n \geqq 1 . \tag{3.1}$$

Note that by (2.3), (2.5) and (3.1), for every $n \geqq 1$, $k \geqq 2$,

$$0 < \tfrac{1}{2}J_n(k\psi) < I_n(k\psi) < \tfrac{3}{5}J_n(k\psi) . \tag{3.2}$$

We consider the following.

LEMMA 3.1. *Under* (2.1), (2.2), (2.3) *and* (2.7), *as* $n \to \infty$,

$$\{\log I_n(k\psi)\}/\nu_k(n) \to -1 , \qquad \textit{for every } k \geqq 2 . \tag{3.3}$$

PROOF. By virtue of (3.2), it suffices to show that as $n \to \infty$

$$\{\log J_n(k\psi)\}/\nu_k(n) \to -1 , \qquad \text{for every } k \geqq 2 . \tag{3.4}$$

Since $u\exp\{-\frac{1}{2}u^2\}$ is $\downarrow$ in u ($\geqq 1$), on denoting t_0 by $\psi^2(t_0) = k^{-2}$, we obtain that for all $n \geqq t_0$,

$$(3.5)\qquad \sum_{m=n}^{\infty} km^{-1}\psi(m+1)\exp\{-\tfrac{1}{2}k^2\psi^2(m+1)\} \leqq 2(2\pi)^{\frac{1}{2}}J_n(k\psi) \leqq \sum_{m=n}^{\infty} km^{-1}\psi(m)\exp\{-\tfrac{1}{2}k^2\psi^2(m)\}\,; \qquad k \geqq 2\,.$$

Let us now define a set of points $\{n_s, s = 0, 1, \cdots\}$ by $n_0 = n$, and

$$(3.6)\qquad n_s = [\exp\{(\log n)^{1+\varepsilon s}\}] + 1\,, \qquad s = 0, 1, \cdots, \varepsilon > 0\,,$$

where $[k]$ denotes the integral part of k ($\geqq 0$), and ε is arbitrarily small. Then, the right-hand side of (3.5) is bounded from above by

$$(3.7)\qquad \begin{aligned} &\sum_{s=0}^{\infty} k\psi(n_s)\exp\{-\tfrac{1}{2}k^2\psi^2(n_s)\}\sum_{m=n_s}^{n_{s+1}-1} m^{-1} \\ &\leqq [(\log n)^{\varepsilon} - 1]\sum_{s=0}^{\infty} k\psi(n_s)\exp\{-\tfrac{1}{2}k^2\psi^2(n_s)\}\log n_s \\ &\leqq (\log n)^{\varepsilon}k[\exp\{-\nu_k(n)\}][\textstyle\sum_{s=0}^{\infty}\chi_n(s)]\,; \end{aligned}$$

$$(3.8)\qquad \begin{aligned} \chi_n(s) &= \exp\{-\tfrac{1}{2}k^2[\psi^2(n_s) - \psi^2(n)] + \log[\psi(n_s)/\psi(n)] \\ &\qquad + \log\log n_s - \log\log n\} \\ &= \exp\{-[\nu_k(n_s) - \nu_k(n)]\}\,, \qquad s = 0, 1, 2, \cdots. \end{aligned}$$

We prove the lemma first for $k = 2 + \delta$, $\delta > 0$. Since $\nu_k(n) - \nu_2(n) = (k^2/2 - 2)\psi^2(n) > 2\delta\psi^2(n)$, by (2.6) and the remark made thereafter, for large n and every $s \geqq 1$,

$$(3.9)\qquad \nu_{2+\delta}(n_s) - \nu_{2+\delta}(n) > \delta(\log\log n_s - \log\log n) = \delta\varepsilon s\log\log n\,.$$

Hence, for every $\varepsilon > 0$, $\delta > 0$, for large n,

$$(3.10)\qquad \sum_{s=0}^{\infty}\chi_n(s) \leqq \sum_{s=0}^{\infty}[(\log n)^{-\delta\varepsilon}]^s = \{1 - (\log n)^{-\delta\varepsilon}\}^{-1} < K_{\varepsilon,\delta} < \infty\,.$$

Therefore, for $k = 2 + \delta$, $\delta > 0$, we have by (3.7) and (3.8) that

$$(3.11)\qquad \limsup_n\{[\log J_n(k\psi)]/\nu_k(n)\} \leqq -1 + \varepsilon[\limsup_n\{(\log\log n)/\nu_k(n)\}]\,.$$

Now, for $k = 2 + \delta$, $\nu_k(n) > \nu_2(n) + 2\delta\psi^2(n) > 2\delta\psi^2(n) > \delta\log\log n$, so that the right-hand side of (3.11) is bounded above by $-1 + \varepsilon/\delta$. Thus, for every $\delta > 0$ and $\eta > 0$, there exists an $\varepsilon > 0$, such that $\varepsilon/\delta < \eta$ and

$$(3.12)\qquad \limsup_n\{[\log J_n(k\psi)]/\nu_k(n)\} \leqq -1 + \eta \qquad \text{for } k = 2 + \delta,\ \delta > 0\,.$$

Similarly, by working with the left hand-side of (3.7), it follows that for every $\eta > 0$,

$$(3.13)\qquad \liminf_n\{[\log J_n(k\psi)]/\nu_k(n)\} \geqq -1 - \eta \qquad \text{for every } k = 2 + \delta,\ \delta > 0\,.$$

Thus, for every $k > 2$,

$$(3.14)\qquad \lim_{n\to\infty}\{[\log J_n(k\psi)]/\nu_k(n)\} = -1\,.$$

Now to consider the case of $k = 2$, we note that, by definition in (2.3),

$J_n(k\phi)$ is $\downarrow$ in k, and hence, for every $\delta > 0$,

$$[\log J_n(2\phi)]/\nu_2(n) \geqq [\nu_{2+\delta}(n)/\nu_2(n)]\{[\log J_n((2+\delta)\phi)]/\nu_{2+\delta}(n)\} . \tag{3.15}$$

Thus, from (2.7), (3.13) and (3.15), it follows that for every $\eta > 0$,

$$\liminf_n \{[\log J_n(2\phi)]/\nu_2(n)\} \geqq -1 - \eta . \tag{3.16}$$

Also, by (2.3) and a few simple steps, we obtain on using (2.7) that for every $\eta > 0$, there exists a $\delta > 0$, such that as $n \to \infty$,

$$\begin{aligned} J_n(2\phi) &= (2\pi)^{-\frac{1}{2}} \textstyle\int_{t=n}^{\infty} \exp\{-\nu_2(t)\}(\log t)^{-1}\, d(\log t) \\ &= (2\pi)^{-\frac{1}{2}} \textstyle\int_{t=n}^{\infty} \exp\{-\nu_{2+\delta}(t)[\nu_2(t)/\nu_{2+\delta}(t)]\}(\log t)^{-1}\, d(\log t) \\ &\leqq (2\pi)^{-\frac{1}{2}} \textstyle\int_{t=n}^{\infty} \exp\{-\nu_{2+\delta}(t)(1-\eta)\}(\log t)^{-1}\, d(\log t) \\ &= (2\pi)^{-\frac{1}{2}} \textstyle\int_{t=n}^{\infty} \exp\{-\frac{1}{2}(1-\eta)(2+\delta)^2\psi^2(t) \\ &\qquad + (1-\eta)\log\psi(t)\}(\log t)^{-\eta} t^{-1}\, dt \\ &\leqq (2\pi)^{-\frac{1}{2}}(\log n)^{-\eta} \textstyle\int_n^{\infty} \exp\{-\frac{1}{2}(1-\eta)(2+\delta)^2\psi^2(t)\}[\psi(t)]^{1-\eta} t^{-1}\, dt . \end{aligned} \tag{3.17}$$

Therefore, by the same technique as in (3.7) through (3.12), we obtain on choosing δ (> 0) sufficiently small that

$$\limsup_n \{[\log J_n(2\phi)]/\nu_2(n)\} \leqq -1 + \eta , \qquad \eta > 0 . \tag{3.18}$$

By letting η in (3.16) and (3.18) be arbitrarily small, we conclude that

$$\lim_{n\to\infty} \{[\log J_n(2\phi)]/\nu_2(n)\} = -1 . \tag{3.19}$$

□

LEMMA 3.2. *Under* (2.1), (2.2), (2.3) *and* (2.7) *for every* $\mathbf{x} \in R^p$,

$$\begin{aligned} \lim_{n\to\infty} \{[\log P\{m^{\frac{1}{2}}[F_m(\mathbf{x}) - F(\mathbf{x})] \geqq \psi(m) \text{ for some } m \geqq n\}]/\nu_{k(\mathbf{x})}(n)\} \\ = -1 , \qquad \text{where } k(\mathbf{x}) = \{2F(\mathbf{x})[1 - F(\mathbf{x})]\}^{-1} (\geqq 2) . \end{aligned} \tag{3.20}$$

PROOF. Since $F_m(\mathbf{x})$ involves an average of i.i.d. rv's which assume only the values 0 and 1, the existence of the moment generating function is insured, and $mV[F_m(\mathbf{x}) - F(\mathbf{x})] = F(\mathbf{x})[1 - F(\mathbf{x})]$ ($< \frac{1}{4}$, for all $\mathbf{x} \in R^p$). Thus, by Theorems 1.4 and 4.9 of Strassen (1967),

$$P\{m^{\frac{1}{2}}[F_m(\mathbf{x}) - F(\mathbf{x})] \geqq \psi(m) \text{ for some } m \geqq n\} \sim I_n(k(\mathbf{x})\phi) , \quad \text{as } n \to \infty , \tag{3.21}$$

where $I_n(k\phi)$ is defined by (3.1) and $k(\mathbf{x})$ by (3.20), and $\sim$ indicates that the ratio of the two sides converges to 1 as $n \to \infty$. The rest of the proof follows from Lemma 3.1 and (3.21). □

LEMMA 3.3. *Under* (2.1), (2.2), (2.3) *and* (2.7)

$$\liminf_n \{[\log P_n(\psi)]/\nu_2(n)\} \geqq -1 . \tag{3.22}$$

PROOF. For every $n \geqq 1$, by definition of D_n^+,

$$\begin{aligned} P_n^+(\psi) &\geqq \sup_{\mathbf{x}} [P\{m^{\frac{1}{2}}[F_m(\mathbf{x}) - F(\mathbf{x})] \geqq \psi(m) \text{ for some } m \geqq n\}] \\ &\geqq P\{m^{\frac{1}{2}}[F_m(\mathbf{x}^0) - F(\mathbf{x}^0)] \geqq \psi(m) \text{ for some } m \geqq n\} , \end{aligned} \tag{3.23}$$

where $\mathbf{x}^0$ is a point (in R^p) for which $F(\mathbf{x}^0) = \frac{1}{2}$. Then, by Lemma 3.2, we obtain that

$$\begin{aligned}\liminf_n \{[\log P_n^+(\psi)]/\nu_2(n)\} &\geq \lim_{n\to\infty} \{[\log P\{m^{\frac{1}{2}}[F_m(\mathbf{x}^0) - F(\mathbf{x}^0)] \geq \psi(m) \\ &\qquad \text{for some } m \geq n]/\nu_2(n)\} \\ &= -1\,, \end{aligned} \tag{3.24}$$

as $k(\mathbf{x}^0) = 2$. Similarly, it follows that (3.24) holds for $\{P_n^-(\psi)\}$, replacing $\{P_n^+(\psi)\}$, and hence the lemma follows.

For every $n \geq 1$, let $\mathscr{C}_n$ be the σ-field generated by the collection of $\mathbf{X}_1, \cdots, \mathbf{X}_n$ (without any regard to the order of the sequence) and by $\mathbf{X}_{n+1}, \mathbf{X}_{n+2}, \cdots$; $\mathscr{C}_n$ is $\downarrow$ in n. Then, we have the following.

LEMMA 3.4. *$\{D_n^+, \mathscr{C}_n; n \geq 1\}$ and $\{D_n^-, \mathscr{C}_n; n \geq 1\}$ are both nonnegative reverse submartingales for every $p \geq 1$.*

PROOF. For every $n \geq 1$, let $\mathbf{Z}_n$ (a random vector) be a point in R^p where $F_n(\mathbf{x}) - F(\mathbf{x})$ attains a maximum, i.e.,

$$D_n^+ = \sup\{F_n(\mathbf{x}) - F(\mathbf{x}) : \mathbf{x} \in R^p\} = F_n(\mathbf{Z}_n) - F(\mathbf{Z}_n)\,; \tag{3.25}$$

$\mathbf{Z}_n$ need not be unique. Therefore, using the fact that $D_n^+ = \sup\{F_n(\mathbf{x}) - F(\mathbf{x}) : \mathbf{x} \in R^p\} \geq F_n(\mathbf{Z}_{n+1}) - F(\mathbf{Z}_{n+1})$, for every $n \geq 1$, we obtain that

$$\begin{aligned} E(D_n^+ \mid \mathscr{C}_{n+1}) &\geq E\{F_n(\mathbf{Z}_{n+1}) - F(\mathbf{Z}_{n+1}) \mid \mathscr{C}_{n+1}\} \\ &= n^{-1} \textstyle\sum_{i=1}^n E\{[c(\mathbf{Z}_{n+1} - \mathbf{X}_i) - F(\mathbf{Z}_{n+1})] \mid \mathscr{C}_{n+1}\} \\ &= E\{[c(\mathbf{Z}_{n+1} - \mathbf{X}_1) - F(\mathbf{Z}_{n+1})] \mid \mathscr{C}_{n+1}\} \\ &= (n+1)^{-1} \textstyle\sum_{j=1}^{n+1} [c(\mathbf{Z}_{n+1} - \mathbf{X}_j) - F(\mathbf{Z}_{n+1})] \\ &= F_{n+1}(\mathbf{Z}_{n+1}) - F(\mathbf{Z}_{n+1}) \\ &= D_{n+1}^+ \,(\geq 0)\,, \qquad \text{for every } n \geq 1\,. \end{aligned} \tag{3.26}$$

The case of $\{D_n^-\}$ follows similarly, and hence the lemma follows.

LEMMA 3.5. *For each $p\,(\geq 1)$ and every $\varepsilon > 0$, there exists a positive $c(p, \varepsilon)$ $(< \infty)$, such that for every $k \geq 0$,*

$$E[(n^{\frac{1}{2}} D_n^+)^k] \leq c(p, \varepsilon)(2 - \varepsilon)^{-k/2}\Gamma(\tfrac{1}{2}k + 1)\,, \qquad \textit{for all } F\,, \tag{3.27}$$

and the same result holds for $\{D_n^-\}$ and $\{D_n\}$.

PROOF. It follows from Theorem 1 of Kiefer (1961) that for each $p\,(\geq 1)$ and every $\varepsilon > 0$, there exists a positive $c(p, \varepsilon)$ $(\leq \infty)$, such that for every $n \geq 1$, $r \geq 0$ and F,

$$P\{n^{\frac{1}{2}} D_n^+ > r\} \leq P\{n^{\frac{1}{2}} D_n^+ \geq r\} \leq c(p, \varepsilon) \exp\{-(2 - \varepsilon) r^2\}\,. \tag{3.28}$$

Therefore, by routine steps,

$$\begin{aligned} E[(n^{\frac{1}{2}} D_n^+)^k] &= \textstyle\int_0^\infty x^k \, dP\{n^{\frac{1}{2}} D_n^+ \leq x\} = k \int_0^\infty x^{k-1} P\{n^{\frac{1}{2}} D_n^+ > x\}\, dx \\ &\leq kc(p, \varepsilon) \textstyle\int_0^\infty x^{k-1} \exp\{-(2 - \varepsilon) x^2 dx \\ &= c(p, \varepsilon)\Gamma(\tfrac{1}{2}k + 1)(2 - \varepsilon)^{-k/2}\,. \end{aligned} \tag{3.29}$$

The other two cases follow similarly. □

LEMMA 3.6. *If* $\{c_m\}$ *be non-decreasing, then for every* $N \geqq n \geqq 1$, $t > 0$,

$$P\{\max_{n \leqq m \leqq N} c_m D_m^+ \geqq t\} \leqq t^{-k} c(p, \varepsilon)\Gamma(\tfrac{1}{2}k + 1)(2 - \varepsilon)^{-k/2} \times \{n^{-k/2}c_n^k + \textstyle\sum_{s=n+1}^N (c_s^k - c_{s-1}^k)s^{-k/2}\} . \tag{3.30}$$

The proof is a direct consequence of Lemmas 3.4 and 3.5 and of Theorem 1 of Chow (1960) which extends the Hájek–Rényi inequality for sub-martingales.

LEMMA 3.7. *Under* (2.3) *and* (2.7), *for all non-decreasing* $\{\psi(t)\}$ *such that* $\psi(t) < ct^{\frac{1}{2}}$, $c < \infty$,

$$\limsup_n \{[\log P_n^+(\psi)]/\nu_2(n)\} \leqq -1 , \tag{3.31}$$

and the same result holds for $\{P_n^-(\psi)\}$ *and* $\{P_n(\psi)\}$.

PROOF. We only consider the proof for $\{P_n^+(\psi)\}$; the case of $\{P_n^-(\psi)\}$ follows on identical lines. Also, noting that $P_n^+(\psi) \leqq P_n(\psi) \leqq P_n^+(\psi) + P_n^-(\psi)$, the case of $\{P_n(\psi)\}$ follows trivially. We consider the events

$$A_n(\lambda\psi) = \{m^{\frac{1}{2}}D_m^+ \geqq \lambda\psi(m) \text{ for some } m \geqq n\} , \qquad n \geqq 1, \lambda \geqq 0 , \tag{3.32}$$

$$A_n^0(\lambda\psi) = \{\textstyle\bigcup_{s=0}^\infty [m^{\frac{1}{2}}D_m^+ \geqq \lambda\psi(n_s) \text{ for some } n_s \leqq m < n_{s+1}]\} , \qquad n \geqq 1 , \tag{3.33}$$

where $n_0 = n$ and $\{n_s\}$ is an increasing sequence of positive integers to be chosen later on. Since $\psi(t)$ is $\uparrow$ in t, we have

$$P(A_n(\lambda\psi)) = P_n^+(\lambda\psi) \leqq P(A_n^0(\lambda\psi)) \qquad \text{for all } \lambda \geqq 0, n \geqq 1 . \tag{3.34}$$

On letting $k_{n,s} = 2\lambda^2(2 - \varepsilon)\psi^2(n_s)$, $s = 0, 1, \cdots$, we obtain by Lemma 3.6 and a few steps that

$$\begin{aligned} P(A_n^0(\lambda\psi)) &\leqq \textstyle\sum_{s=0}^\infty P\{\max_{n_s \leqq m < n_{s+1}} m^{\frac{1}{2}}D_m^+ > \psi(n_s)\} \\ &\leqq c(p, \varepsilon) \textstyle\sum_{s=0}^\infty [\lambda\psi(n_s)]^{-k_{n,s}}(2 - \varepsilon)^{-k_{n,s}/2}\Gamma(\tfrac{1}{2}k_{n,s} + 1) \\ &\qquad \times \{1 + \textstyle\sum_{j=n_s}^{n_{s+1}-1} [1 - (1 - j^{-1})^{\frac{1}{2}k_{n,s}}]\} . \end{aligned} \tag{3.35}$$

For every $s = 0, 1, \cdots$, by Sterling's approximations, for large n,

$$\begin{aligned} &\log \Gamma(\tfrac{1}{2}k_{n,s} + 1) \\ &\quad = -\tfrac{1}{2}\log(2\pi) + \tfrac{1}{2}(k_{n,s} + 1)\log(\tfrac{1}{2}k_{n,s}) - \tfrac{1}{2}k_{n,s} + o(1) \\ &\quad = -\tfrac{1}{2}\log(2\pi) + [(2 - \varepsilon)\lambda^2\psi^2(n_s) + \tfrac{1}{2}]\log[(2 - \varepsilon)\lambda^2\psi^2(n_s)] \\ &\qquad - (2 - \varepsilon)\lambda^2\psi^2(n_s) + o(1) . \end{aligned} \tag{3.36}$$

Also, note that

$$1 - (1 - j^{-1})^{\frac{1}{2}k_{n,s}} = \tfrac{1}{2}k_{n,s}j^{-1} + O([k_{n,s}/j]^2) , \tag{3.37}$$

so that for every $s \leqq 0$,

$$\begin{aligned} &\textstyle\sum_{j=n_s}^{n_{s+1}-1} [1 - (1 - j^{-1})^{\frac{1}{2}k_{n,s}}] \\ &\quad = \tfrac{1}{2}k_{n,s}\left(\frac{1}{n_s + 1} + \cdots + \frac{1}{n_{s+1} - 1}\right) + O(k_{n,s}^2 n_s^{-1}) \\ &\quad \leqq \tfrac{1}{2}k_{n,s}(\log n_{s+1} - \log n_s) + O(1) . \end{aligned} \tag{3.38}$$

Let us now set $\lambda = 1 + \frac{1}{2}\varepsilon$, so that $\frac{1}{2}k_{n,s} = (2 + \varepsilon')\psi^2(n_s)$ where $\varepsilon' = \varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{4}\varepsilon^4 > 0$ for every $0 < \varepsilon < \varepsilon_0$ $(> \frac{1}{2})$. We consider first the case of $\psi(t)$ satisfying (2.1)—(2.3) and (2.7), such that $\psi^2(t) < C \log t$, $0 < C < \infty$. We set then $\log n_s \simeq (\log n)^{1+\varepsilon s/3}$, $s = 0, 1, \cdots$, where $\simeq$ indicates that n_s is the least positive integer for which the left-hand side is $\geqq$ the right-hand side. Then, for every $s \geqq 0$,

$$\log n_{s+1} - \log n_s = (\log n_s)((\log n)^{\varepsilon/3} - 1) \leqq (\log n)^{\varepsilon/3}(\log n_s)\,, \tag{3.40}$$

so that by (3.35) and (3.40), we have for large n,

$$\begin{aligned} &P(A_n^0((1+\varepsilon)\psi)) \\ &\quad \leqq K_\varepsilon[\textstyle\sum_{s=0}^\infty \psi^2(n_s) \exp\{-\nu_2(n_s) - \varepsilon'\psi^2(n_s) + (\varepsilon/3)\log\log n_s\}] \\ &\quad = K_\varepsilon \psi^2(n) \exp\{-\nu_2(n) - \varepsilon'\psi^2(n) + (\varepsilon/3)\log\log n\} \textstyle\sum_{s=0}^\infty \chi_n(s)\,, \end{aligned} \tag{3.41}$$

where K_ε $(< \infty)$ depends only on ε (> 0), $\nu_2(n)$ is defined by (2.6), and

$$\begin{aligned} \chi_n(s) = \exp\{&-[\nu_2(n_s) - \nu_2(n)] - \varepsilon'[\psi^2(n_s) - \psi^2(n)] \\ &+ (\varepsilon/3)[\log\log n_s - \log\log n] + 2\log[\psi(n_s)/\psi(n)]\}\,, \quad s \geqq 0\,. \end{aligned} \tag{3.42}$$

By the remark made after (2.6) and (3.40), it follows that for large n,

$$\begin{aligned} \chi_n(s) &\leqq \exp\{-[\tfrac{1}{4}\varepsilon\varepsilon' s \log\log n - (\varepsilon^2 s/9)\log\log n]\,, \\ &\qquad [-(\varepsilon'/12)(\psi^2(n_s) - \psi^2(n)) - \log(\psi^2(n_s)/\psi^2(n))]\} \\ &\leqq \exp\{-(\varepsilon^2 s/18)\log\log n\} \\ &= (\log n)^{-(\varepsilon^2 s/18)}\,, \qquad \text{for } 0 < \varepsilon < \varepsilon_0 (< \tfrac{1}{2})\,, \end{aligned} \tag{3.43}$$

as $\varepsilon' = \varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{4}\varepsilon^2 > 23\varepsilon/32 > (\frac{2}{3})\varepsilon$ for all $0 < \varepsilon < \varepsilon_0$ $(< \frac{1}{2})$, and for large n, $(\varepsilon'/12)[\psi^2(n_s) - \psi^2(n)]$ can be made larger than $\log[\psi^2(n_s)/\psi^2(n)]$. Therefore, for n sufficiently large,

$$1 \leqq \textstyle\sum_{s=0}^\infty \chi_n(s) \leqq \{1 - (\log n)^{-\varepsilon^2/18}\} \to 1 \qquad \text{as } n \to \infty\,. \tag{3.44}$$

In a similar way, it follows that for every $\varepsilon > 0$, as $n \to \infty$,

$$\varepsilon'\psi^2(n) \geqq (\varepsilon'/3)\log\log n + 2\log\psi(n)\,, \qquad \text{when (2.3) holds.} \tag{3.45}$$

Therefore from (3.41) through (3.45), it follows that for every ε $(0 < \varepsilon < \varepsilon_0 < \frac{1}{2})$,

$$\limsup_n \{[\log P(A_n^0((1 + \tfrac{1}{2}\varepsilon)\psi))]/\nu_2(n)\} \leqq -1\,, \tag{3.46}$$

and (3.32) follows from (2.7), (3.34) and (3.46) by letting ε (> 0) to be arbitrarily small.

We next consider the case when $\psi(t)$ is $\uparrow$ in t such that $(\log t)/\psi^2(t) \to 0$ and $t^{-\frac{1}{2}}\psi^2(t) \leqq c < \infty$ as $t \to \infty$. In this case, we let $n_s = [n(1 + \varepsilon)^s]$, $s = 0, 1, \cdots;$ $\varepsilon > 0$, and virtually repeat the steps (3.40)—(3.46), with some further simplifications; for brevity, the details are omitted.

Returning now to the proofs of the theorems, we note that (2.9) follows directly from Lemmas 3.3 and 3.7. For (2.10), we define by $\tilde{D}_n^+$, $\tilde{D}_n^-$ and $\tilde{D}_n$ the Kolmogorov–Smirnov statistics for the univariate observations $X_{11}, \cdots, X_{n1}$, so that, by definition, $D_n^+ \geqq \tilde{D}_n^+$, $D_n^- \geqq \tilde{D}_n^-$ and $D_n \geqq \tilde{D}_n$ for all $n \geqq 1$. Also, by (1.1), for every $0 \leqq x < cn^{\frac{1}{6}}$, $P\{n^{\frac{1}{2}}\tilde{D}_n^+ \geqq x\} = P(n^{\frac{1}{2}}\tilde{D}_n^- \geqq x\} = \exp\{-2x^2\} \times \{1 + 2x/3n^{\frac{1}{2}} + O(1/n)\}$; $P\{n^{\frac{1}{2}}\tilde{D}_n \geqq x\} = 2\exp\{-2x^2\}\{1 + O(n^{-\frac{1}{2}})\}$. Therefore, by (1.2) and the above, we have

$$2\exp\{-2\psi^2(n)\}\{1 + O(n^{-\frac{1}{3}})\} \leqq P\{\tilde{D}_n \geqq n^{-\frac{1}{2}}\psi(n)\} \leqq c(p, \varepsilon)\exp\{-(2-\varepsilon)\psi^2(n)\}, \tag{3.47}$$

for all $\psi^2(n) = O(n^{\frac{1}{3}})$. Further, by (2.8), $P_n(\psi) \geqq P\{D_n \geqq n^{\frac{1}{2}}\psi(n)\}$ for every $n \geqq 1$, and hence, by (3.47), $\liminf_n \{\log P_n(\psi)]/\psi^2(n)\} \geqq -2$. On the other hand, by Lemma 3.7 and the fact that $\nu_2(n)/\psi^2(n) \to 2$ as $n \to \infty$, we have

$$\limsup_n \{[\log P_n(\psi)]/\psi^2(n)\} \leqq -2,$$

which completes the proof for $\{P_n(\psi)\}$; the cases of $\{P_n^+(\psi)\}$ and $\{P_n^-(\psi)\}$ follow similarly.

Acknowledgments. The author is grateful to the Editor and the referees for their useful comments on the paper.

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