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INEQUALITIES FOR MULTITYPE BRANCHING PROCESSES[1]

BY BRUCE W. TURNBULL[2]

Cornell University

Some results of the paper "Inequalities for Branching Processes" [*Ann. Prob.* **1** (1973)] by the same author are extended to a multitype branching process. Bounds are obtained on the probability of extinction and mean time to extinction of the process when the probability transition laws are allowed to vary from period to period and are required only to belong to some class $\mathscr{M}$.

1. Introduction. In Turnbull [4], a branching process was considered in which the conditional litter size distributions, given the past, were allowed to vary within a certain class $\mathscr{M}$ from generation to generation. Chebyshev-like bounds were then obtained for various quantities of interest. In this paper the results of [4] are extended to multitype branching processes and bounds are obtained for the probability of extinction—both with finite and infinite horizon.

2. The multitype Galton–Watson process. The multitype (or vector) Galton–Watson Process is defined in Harris [2] Chapter 2, and in Bharuchta–Reid [1] Chapter 1.8. We shall use the notation of the latter. Vectors and matrices will be denoted by boldface letters, and we shall say that a vector or matrix is "positive," "nonnegative," or "zero" if all its components have those properties. For n some positive integer, define $\mathbf{0}$ and $\mathbf{1}$ as the n-vectors with all components 0 or 1, respectively. Also let $\mathbf{e}_i$, $i = 1, 2, \cdots, n$, denote the n-vector whose ith component is 1 and whose other components are 0.

We consider a population in which each individual belongs to precisely one of n distinct types. Let $\mathbf{Z}_N = (Z_{1N}, Z_{2N}, \cdots, Z_{nN})$ represent the population at time N, where Z_{iN} is the number of individuals of type i alive at time N ($i = 1, 2, \cdots, n$; $N = 0, 1, 2, \cdots$).

Let $p_i(\alpha_1, \alpha_2, \cdots, \alpha_n)$ be the probability that an individual of type i produces α_1 individuals of type 1, α_2 individuals of type 2, $\cdots$, and α_n individuals of type n in the next generation ($i = 1, 2, \cdots, n$). All branches evolve independently of each other.

Let $F_{iN}(\mathbf{s})$ be the generating function of $\mathbf{Z}_N$ when $\mathbf{Z}_0 = \mathbf{e}_i$, where $\mathbf{s} = (s_1, s_2, \cdots, s_n)$, for $1 \leqq i \leqq n$, $N = 0, 1, 2, \cdots$. Hence:

$$F_{i0}(\mathbf{s}) = s_i ,$$

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[2] Now at Mathematical Institute, University of Oxford.

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and

$$F_{i1}(\mathbf{s}) = \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} p_i(\alpha_1, \alpha_2, \cdots, \alpha_n) s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_n^{\alpha_n} \tag{1}$$
$$= F_i(\mathbf{s}), \quad \text{say.}$$

A fundamental result ([2] Chapter 2, Theorem 3.1) is that:

$$F_{i,N+1}(\mathbf{s}) = F_i[F_{1N}(\mathbf{s}), F_{2N}(\mathbf{s}), \cdots, F_{nN}(\mathbf{s})], \qquad (i = 1, 2, \cdots, n). \tag{2}$$

Differentiating (2) we get: $E[\mathbf{Z}_N] = \mathbf{Z}_0 \cdot \mathbf{M}^N$, where $\mathbf{M} = (m_{ij})$ and $m_{ij} = E[Z_{j1} \mid \mathbf{Z}_0 = \mathbf{e}_i]$. We will need the following two definitions:

The process is called positively regular if $\mathbf{M}^N$ is positive for some positive integer N. In this case, $\mathbf{M}$ has a positive real eigen-value, λ say, that is larger in absolute value than any other eigen-value (see [2] Chapter 2, Theorem 5.1).

The process will be called singular if the generating functions $F_i(\mathbf{s})$ $(1 \leqq i \leqq n)$ are all linear with no constant term, i.e., each individual has exactly one offspring.

Define $\rho_i(N) = \Pr[\mathbf{Z}_N = \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{e}_i]$, for $1 \leqq i \leqq n$ and $N = 0, 1, 2, \cdots$. Then if $\boldsymbol{\rho}(N) = (\rho_1(N), \cdots, \rho_n(N))$ and $\mathbf{F}(\mathbf{s}) = (F_1(\mathbf{s}), \cdots, F_n(\mathbf{s}))$ we have, by (1):

$$\boldsymbol{\rho}(N+1) = \mathbf{F}(\boldsymbol{\rho}(N)). \tag{3}$$

Since the branches are independent we have the further result that:

$$\Pr[\mathbf{Z}_N = \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{z}] = [\rho_1(N)]^{z_1} \cdot [\rho_2(N)]^{z_2} \cdots [\rho_n(N)]^{z_n}, \tag{4}$$

where $\mathbf{z} = (z_1, z_2, \cdots, z_n)$.

Harris [2] Chapter 2, Theorems 7.1, 7.2, shows that if the process is positively regular and not singular then $\boldsymbol{\rho}(N) \to \boldsymbol{\rho}$ as $N \to \infty$, where $\boldsymbol{\rho} = (\rho_1, \rho_2, \cdots, \rho_n)$ and the convergence is componentwise. Also $\boldsymbol{\rho}$ satisfies the equation $\boldsymbol{\rho} = \mathbf{F}(\boldsymbol{\rho})$. Furthermore, if $\lambda \leqq 1$ then $\boldsymbol{\rho} = \mathbf{1}$ and the population will become extinct with probability one, whereas if $\lambda > 1$, then $\mathbf{0} \leqq \boldsymbol{\rho} < \mathbf{1}$.

The concept of an $\mathscr{M}$-sequence is defined in [4] Section 2. In the remaining sections of this paper we will consider multitype branching processes where the transition laws $\{p_i(\alpha_1, \cdots, \alpha_n); 1 \leqq i \leqq n\}$ are allowed to vary within a certain class from period to period and at time N $(\geqq 1)$ may depend on the past history $\{\mathbf{Z}_i, 1 \leqq i \leqq N-1\}$ of the process. These processes will be defined as $\mathscr{M}$-sequences. In Section 3, upper bounds for the probability of extinction and lower bounds for the expected time to extinction are derived and in Section 4 bounds in the reverse direction for the same quantities are obtained. We also examine the conditions under which these bounds are attained. The results will be multitype analogues of those appearing in [4] Section 3.

Throughout this paper, the set C, referred to in [4] Section 2, will be taken to be the set of all n-vectors whose components are nonnegative integers. Also, define T_e, the time to extinction by:

$$T_e = \min[N: \mathbf{Z}_N = \mathbf{0}] \quad \text{if} \quad \mathbf{Z}_N = \mathbf{0} \quad \text{for some} \quad N,$$
$$= \infty \qquad \text{otherwise.}$$

3. The probability of extinction (upper bounds) and the expected time to extinction (lower bounds).

THEOREM 1 (*Extinction in a finite time*). *Take N some positive integer and let $\{\boldsymbol{\rho}(k); k = 0, 1, 2, \cdots, N\}$ be a sequence of real n-vectors such that $\mathbf{0} \leqq \boldsymbol{\rho}(k) \leqq \mathbf{1}$, $(0 \leqq k \leqq N)$ and $\boldsymbol{\rho}(0) = \mathbf{0}$; and set*

$$\mathscr{M}(\mathbf{z}) = \{P(\mathbf{Z}_1 \mid \mathbf{Z}_0 = \mathbf{z}): \text{ where } \mathbf{Z}_1 \text{ is the population vector of a multitype Galton–Watson branching process such that } \mathbf{F}(\boldsymbol{\rho}(k)) \leqq \boldsymbol{\rho}(k+1),\ k = 0, 1, 2, \cdots, N-1\},$$

for all $\mathbf{z} \in C$.

Then for $\mathbf{Z}_0, \mathbf{Z}_1, \mathbf{Z}_2, \cdots$, an $\mathscr{M}$-sequence starting at $\mathbf{z} = (z_1, z_2, \cdots, z_n)$, we have:

$$\Pr[\mathbf{Z}_N = \mathbf{0}] \leqq [\rho_1(N)]^{z_1} \cdot [\rho_2(N)]^{z_2} \cdots [\rho_n(N)]^{z_n}. \tag{5}$$

PROOF. We apply Theorem 2.1 of [4] with C the set of nonnegative integer n-vectors, $T = N$, $r(\mathbf{z}) \equiv 0$, $f_k(\mathbf{z}) = [\rho_1(k)]^{z_1} \cdot [\rho_2(k)]^{z_2} \cdots [\rho_n(k)]^{z_n}$ for $\mathbf{z} \in C$ and $0 \leqq k \leqq N$.

Then r and f_k are nonnegative and, for $P(\mathbf{Z}) \in \mathscr{M}(\mathbf{z})$, we have:

$$\begin{aligned} r(\mathbf{z}) + Ef_k(\mathbf{Z}) &= E\{[\rho_1(k)]^{Z_1} \cdot [\rho_2(k)]^{Z_2} \cdots [\rho_n(k)]^{Z_n}\} \\ &= [F_1(\boldsymbol{\rho}(k))]^{z_1} \cdot [F_2(\boldsymbol{\rho}(k))]^{z_2} \cdots [F_n(\boldsymbol{\rho}(k))]^{z_n} \\ &\leqq [\rho_1(k+1)]^{z_1} \cdot [\rho_2(k+1)]^{z_2} \cdots [\rho_n(k+1)]^{z_n} \\ &= f_{k+1}(\mathbf{z}); \end{aligned}$$

which verifies the hypotheses of Theorem 2.1 of [4].

The result (5) now follows by noting that

$$E[\textstyle\sum_{k=0}^{T(N)-1} r(\mathbf{Z}_k) + f_{N-T(N)}(\mathbf{Z}_{T(N)})] = Ef_0(\mathbf{Z}_N) = \Pr[\mathbf{Z}_N = \mathbf{0}],$$

since $\boldsymbol{\rho}(0) = \mathbf{0}$, and we have adopted the convention that $0^0 = 1$. □

COROLLARY 1 (*The expected time to extinction*). *Suppose $\{\boldsymbol{\rho}(k); k = 0, 1, 2, \cdots\}$ is a sequence of real n-vectors such that $\mathbf{0} \leqq \boldsymbol{\rho}(k) \leqq \mathbf{1}$, for all k, and $\boldsymbol{\rho}(0) = \mathbf{0}$. Take*

$$\mathscr{M}(\mathbf{z}) = \{P(\mathbf{Z}_1 \mid \mathbf{Z}_0 = \mathbf{z}): \text{ where } \mathbf{Z}_1 \text{ is the population vector of a multitype Galton–Watson process such that } \mathbf{F}(\boldsymbol{\rho}(k)) \leqq \boldsymbol{\rho}(k+1), \text{ for all } k\},$$

for all $\mathbf{z} \in C$.

Then, for $\mathbf{Z}_0, \mathbf{Z}_1, \mathbf{Z}_2, \cdots$, an $\mathscr{M}$-sequence starting at $\mathbf{z} = (z_1, z_2, \cdots, z_n)$, we have:

$$E[T_e] \geqq \textstyle\sum_{k=0}^{\infty} \{1 - \{[\rho_1(k)]^{z_1} \cdot [\rho_2(k)]^{z_2} \cdots [\rho_n(k)]^{z_n}\}\}. \tag{6}$$

PROOF. The proof is analogous to that of Corollary 1 of Theorem 3.1 of [4] and follows by noting that: $E[T_e] = \sum_{k=0}^{\infty} (1 - \Pr[\mathbf{Z}_k = \mathbf{0}])$. □

COROLLARY 2 (*Achievement of bounds*). (A) *If there exists an $\mathscr{M}$-sequence such that at stage $N - k$ (i.e., with k periods remaining), conditional on the past, the generating function of the transition probabilities, as defined in* (1), *is $\mathbf{F}^{(k)}$ with*

$$\boldsymbol{\rho}(k) = \mathbf{F}^{(k)}(\boldsymbol{\rho}(k-1))$$

for $1 \leqq k \leqq N$, *then this* $\mathscr{M}$*-sequence achieves the bound* (5). *The proof follows by induction on* N *using* (3).

(B) *If it turns out that* $\mathbf{F}^{(k)}$ *is independent of* k ($\mathbf{F}^{(k)} = \mathbf{F}$ *say*), *then the* $\mathscr{M}$*-sequence in* (A) *is a multitype Galton–Watson process and, by* (3) *and* (4), *this process achieves the bound* (5) *simultaneously for all* N, *and thus the bound* (6) *on the expected time to extinction is also attained. This is unlike the problems studied in Section* 3 *of* [4], *in all of which the optimal policy was independent of the time horizon* N. *There always the strategy which maximized* (*minimized*) *the probability of extinction also minimized* (*maximized*) *the mean time to extinction.*

THEOREM 2 (*Eventual extinction*). *Let* $\boldsymbol{\rho}$, $\mathbf{0} \leqq \boldsymbol{\rho} \leqq \mathbf{1}$ *be a real n-vector and set*

$$\mathscr{M}(\mathbf{z}) = \{P(\mathbf{Z}_1 \mid \mathbf{Z}_0 = \mathbf{z}) : \text{where } \mathbf{Z}_1 \text{ is the population vector of a multitype Galton–Watson branching process such that } \mathbf{F}(\boldsymbol{\rho}) \leqq \boldsymbol{\rho}\},$$

for all $\mathbf{z} \in C$.

Then for $\mathbf{Z}_0, \mathbf{Z}_1, \cdots$, *an* $\mathscr{M}$*-sequence starting at* $\mathbf{z} = (z_1, \cdots, z_n)$, *we have*:

$$\Pr[\mathbf{Z}_N = \mathbf{0} \text{ for some } N] \leqq \rho_1^{z_1} \cdot \rho_2^{z_2} \cdots \rho_n^{z_n}. \tag{7}$$

PROOF. This is the multitype analogue of Theorem 3.2 of [4], and the proof follows similarly, with $f(\mathbf{z}) = \rho_1^{z_1} \cdot \rho_2^{z_2} \cdots \rho_n^{z_n}$. The details are omitted. □

COROLLARY 1 (*Achievement of bounds*). *If there exists a positively regular, non-singular multitype Galton–Watson process with* $\mathbf{F}(\boldsymbol{\rho}) = \boldsymbol{\rho}$ *then, as explained in Section* 2, *this process achieves the bound* (7).

4. The probability of extinction (lower bounds) and the expected time to extinction (upper bounds).

THEOREM 3. *Take* N *some positive integer and let* $\{\boldsymbol{\rho}(k); k = 0, 1, \cdots, N\}$ *be a sequence of real n-vectors such that* $\mathbf{0} \leqq \boldsymbol{\rho}(k) \leqq \mathbf{1}$, $(0 \leqq k \leqq N)$ *and* $\boldsymbol{\rho}(0) = \mathbf{0}$; *and set*

$$\mathscr{M}(\mathbf{z}) = \{P(\mathbf{Z}_1 \mid \mathbf{Z}_0 = \mathbf{z}) : \text{where } \mathbf{Z}_1 \text{ is the population vector of a multitype Galton–Watson process such that } \mathbf{F}(\boldsymbol{\rho}(k)) \geqq \boldsymbol{\rho}(k+1);\ k = 0, 1, 2, \cdots, N-1\}$$

for $\mathbf{z} \in C$.

Then, for $\mathbf{Z}_0, \mathbf{Z}_1, \mathbf{Z}_2, \cdots$, *an* $\mathscr{M}$*-sequence starting at* $\mathbf{z} = (z_1, z_2, \cdots, z_n)$, *we have*:

$$\Pr[\mathbf{Z}_N = \mathbf{0}] \geqq [\rho_1(N)]^{z_1} \cdot [\rho_2(N)]^{z_2} \cdots [\rho_n(N)]^{z_n}. \tag{8}$$

PROOF. This is the multitype analogue of Theorem 3.5 of [4] and the proof follows similarly, with $f_k(\mathbf{z}) = [\rho_1(k)]^{z_1} \cdot [\rho_2(k)]^{z_2} \cdots [\rho_n(k)]^{z_n}$. The details are omitted. □

COROLLARY 1 (*The expected time to extinction*). *Suppose* $\{\boldsymbol{\rho}(k); k = 0, 1, 2, \cdots\}$ *is a sequence of real n-vectors* $\mathbf{0} \leqq \boldsymbol{\rho}(k) \leqq \mathbf{1}$, *for all* k, *and* $\boldsymbol{\rho}(0) = \mathbf{0}$. *Take*

$$\mathscr{M}(\mathbf{z}) = \{P(\mathbf{Z}_1 \mid \mathbf{Z}_0 = \mathbf{z}) : \text{where } \mathbf{Z}_1 \text{ is the population vector of a multitype Galton–Watson process such that } \mathbf{F}(\boldsymbol{\rho}(k)) \geqq \boldsymbol{\rho}(k+1), \text{ for all } k\}$$

for all $\mathbf{z} \in C$.

Then, for $\mathbf{Z}_0, \mathbf{Z}_1, \mathbf{Z}_2, \cdots$, *an* $\mathscr{M}$*-sequence starting at* $\mathbf{z} = (z_0, z_1, \cdots, z_n)$, *we have*:

$$E[T_e] \leqq \sum_{k=0}^{\infty} \{1 - \{[\rho_1(k)]^{z_1} \cdot [\rho_2(k)]^{z_2} \cdots [\rho_n(k)]^{z_n}\}\} . \tag{9}$$

PROOF. The proof is the same as that for Corollary 1 of Theorem 1 but with the inequalities reversed. □

COROLLARY 2 (*Achievement of bounds*). (A) *If there exists an* $\mathscr{M}$*-sequence such that at stage* $N - k$ (*i.e., with* k *periods remaining*), *conditional on the past, the generating function of the transition probabilities, as defined in* (1), *is* $\mathbf{F}^{(k)}$ *with*

$$\boldsymbol{\rho}(k) = \mathbf{F}^{(k)}(\boldsymbol{\rho}(k-1))$$

for $1 \leqq k \leqq N$, *then this* $\mathscr{M}$*-sequence achieves the bound* (8).

(B) *If it turns out that* $\mathbf{F}^{(k)}$ *is independent of* k ($\mathbf{F}^{(k)} = \mathbf{F}$ *say*), *then the* $\mathscr{M}$*-sequence in* (A) *is a multitype Galton–Watson process and this process achieves the bound* (8) *simultaneously for all* N, *and thus the bound* (9) *on the expected time to extinction is also attained.*

This result follows as in Corollary 2 of Theorem 1.

COROLLARY 3 (*Eventual extinction*). *Suppose* $\boldsymbol{\rho}(k) \to \boldsymbol{\rho}$ *as* $k \to \infty$ *for some* $\boldsymbol{\rho}$, $\mathbf{0} \leqq \boldsymbol{\rho} \leqq \mathbf{1}$, *then, since* $\Pr[\mathbf{Z}_N = \mathbf{0}$ *for some* $N] \geqq \Pr[\mathbf{Z}_N = \mathbf{0}]$ *and the left-hand side is independent of* N, *we have*:

$$\Pr[\mathbf{Z}_N = \mathbf{0} \ \text{ for some } \ N] \geqq \rho_1^{z_1} \cdot \rho_2^{z_2} \cdots \rho_n^{z_n} . \tag{9}$$

Furthermore if there exists a multitype Galton–Watson process with $\mathbf{F}(\boldsymbol{\rho}) = \boldsymbol{\rho}$ *then, as explained in Section* 2, *this process achieves the bound* (9).

5. Remarks. The multitype branching process defined in this paper can be used to describe a stochastic age-structured population growth model. Some applications of this model are developed in Turnbull [3] Chapter 6.

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MATHEMATICAL INSTITUTE
24-29 ST. GILES
OXFORD, ENGLAND