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THE MAXIMAL OSCILLATION PROBLEM FOR REGENERATIVE PHENOMENA[1]

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An account of known upper and lower bounds for standard diagonal Markov transition functions and standard p-functions taking on a given value at time t is presented, and some new results are derived.

1. Introduction. Let $p(t)$ be a standard diagonal Markov transition function of continuous time parameter t, in the sense of Chung [4]. If we know that $p(s) = M$, what upper and lower bounds can we set on $p(t)$ for $0 < t < s$? More precisely, letting $\mathscr{P}\mathscr{M}$ denote the class of all standard diagonal Markov transition functions, we wish to determine for each fixed $t \in (0, 1)$

$$\bar{\pi}_M(t) := \sup_{p \in \mathscr{P}\mathscr{M}} \{p(t) \mid p(1) = M\}$$

and

$$\underline{\pi}_M(t) := \inf_{p \in \mathscr{P}\mathscr{M}} \{p(t) \mid p(1) = M\}$$

(where $:=$ designates a defining notational equality). The choice of $s = 1$ here is only a matter of notational convenience, since $\mathscr{P}\mathscr{M}$ is closed under constant dilations or contractions of the time scale.

Determining $\bar{\pi}$ and $\underline{\pi}$ shall be known as the maximal oscillation problem. The proper setting for the study of this problem is J. F. C. Kingman's elegant theory of regenerative phenomena [12]. Just as many aspects of discrete Markov chains are most easily treated within the framework of Feller's classical theory of recurrent events, so too is Kingman's generalization of recurrent events most appropriate for the study of continuous time problems such as ours. The analogues of discrete renewal sequences in the theory of regenerative phenomena are the so-called p-functions. For present purposes, the key fact is that $\mathscr{P}\mathscr{M}$ is dense in $\mathscr{P}$, the class of all standard p-functions, in the topology of pointwise convergence. Since there is a sequence of Markov functions converging to any standard p-function, we may equally well define $\bar{\pi}$ and $\underline{\pi}$ over $\mathscr{P}$, and as it turns out, this is the more natural approach.

In this paper we shall outline those aspects of the theory of regenerative phenomena necessary for the study of the maximal oscillation problem, summarize results previously obtained, remark on general properties of $\bar{\pi}$ and $\underline{\pi}$, and derive some new bounds. It will become apparent that this problem is far from solved;

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the author hopes that the numerous conjectures offered in the course of the discussion may encourage further research.

2. Regenerative phenomena and p-functions. According to the formulation of Kingman, let $Z := \{Z(t);\ t > 0\}$ be a stochastic process taking on values in the two point set $\{0, 1\}$. If there exists a function p defined on $(0, \infty)$ and such that whenever $0 < t_1 < t_2 < \cdots < t_n$, it follows that

$$\Pr\{Z(t_1) = Z(t_2) = \cdots = Z(t_n) = 1\} = \prod_{k=1}^{n} p(t_k - t_{k-1}) \qquad (t_0 := 0)\,, \tag{1}$$

then Z is a *regenerative phenomenon*, and p is called the *p-function* of Z. If in addition

$$\lim_{t\to 0} p(t) = 1\,,$$

then p is *standard*, and for convenience we may extend the definition of p to $[0, \infty)$ by taking $p(0) = 1$, and setting $Z(0) = 1$ a.s. The class of all standard p-functions is denoted by $\mathscr{P}$.

Useful properties of $p \in \mathscr{P}$ are the following:

(a) $p(t) > 0$ on $[0, \infty)$;
(b) p is uniformly continuous on $[0, \infty)$;
(c) $q := \lim_{t\to 0}[1 - p(t)]/t$ exists; and if $q < \infty$, then $p(t) \geqq e^{-qt}$;
(d) $p(\infty) := \lim_{t\to\infty} p(t)$ exists.

Proofs will be found in [12].

Also, if we let $\mathscr{PM}$ be the class of functions which arise as diagonal transition functions for standard Markov chains, then $\mathscr{PM} \subset \mathscr{P}$. While this inclusion is definitely strict, Kingman has shown that $\mathscr{PM}$ is dense in $\mathscr{P}$ in the topology of pointwise convergence. The precise characterization of $\mathscr{PM}$ as a subclass of $\mathscr{P}$ has been one of the central problems in the theory of regenerative phenomena; for its solution, see [12].

Consider next a sequence $\{t_k;\ 1 \leqq k \leqq n\}$ of increasing times in $(0, \infty)$, and define

$$f(t_n) := \Pr\{Z(t_k) = 0 \ \text{ for } \ 1 \leqq k \leqq n-1;\ Z(t_n) = 1\}$$

and

$$g(t_n) := \Pr\{Z(t_k) = 0 \ \text{ for } \ 1 \leqq k \leqq n\}\,.$$

We may use (1) to express $f(t_n)$ and $g(t_n)$ as polynomials in terms $p(t_k - t_j)$ $(0 \leqq j < k \leqq n;\ t_0 := 0)$, and since f and g are probabilities,

$$\begin{aligned} f(t_n) &\geqq 0 \\ g(t_n) &\geqq 0\,. \end{aligned} \tag{2}$$

The inequalities (2) are called the *Kingman inequalities of order n*. Clearly, any p-function must satisfy these inequalities for all n, and for arbitrary sequences $\{t_k\}$ of increasing times. The first order inequalities simply give $0 \leqq p(t_1) \leqq 1$, the second order ones are

$$\begin{aligned} f(t_2) &= p(t_2) - p(t_1)p(t_2 - t_1) \geqq 0 \\ g(t_2) &= 1 - p(t_2) - p(t_1) + p(t_1)p(t_2 - t_1) \geqq 0\,, \end{aligned}$$

and higher order inequalities are generated recursively by

$$(3)\qquad \begin{aligned} f(t_n) &= p(t_n) - \textstyle\sum_{k=1}^{n-1} f(t_k)p(t_n - t_k) \geqq 0 \\ g(t_n) &= 1 - \textstyle\sum_{k=1}^{n} f(t_k) \geqq 0 \,. \end{aligned}$$

If $p \in \mathscr{P}$, then (3) are defined for $t_{k+1} - t_k \geqq 0$. By only requiring that $\{t_k\}$ have nonnegative increments, we see that the nth order pair of Kingman inequalities contain all lower order ones as special cases. Thus we may consider the classes $\mathscr{K}_n$ of all real-valued functions defined on $[0, \infty)$, with $p(0) = \lim_{t\to 0} p(t) = 1$, and satisfying the nth order inequalities. In this way a useful characterization of $\mathscr{P}$ is obtained:

$$(4)\qquad \mathscr{K}_1 \supset \mathscr{K}_2 \supset \cdots \supset \mathscr{P}, \qquad \text{and} \qquad \mathscr{P} = \lim_{n\to\infty} \mathscr{K}_n \,.$$

A remark of Kingman shows that each inclusion in (4) is strict. Other more constructive characterizations of $\mathscr{P}$ may be found in [12], but this will be the most useful in our discussion of the maximal oscillation problem.

We note that $p \in \mathscr{K}_2$ is necessarily continuous on $[0, \infty)$, as the proof of property (b) for $p \in \mathscr{P}$ uses only second order Kingman inequalities and the standardness condition. Thus the requirement that p satisfy the inequalities (2) for arbitrary sequences $\{t_k\}$ imposes quite strong conditions, and makes the problem an analytic rather than algebraic one. The full implications of inequalities as low as the third order are not yet understood.

3. The maximal oscillation problem: Known results. Define for each fixed t in $(0, 1)$

$$\bar{\pi}_M(t) := \sup_{p\in\mathscr{P}} \{p(t) \,|\, p(1) = M\}$$

and

$$\underline{\pi}_M(t) := \inf_{p\in\mathscr{P}} \{p(t) \,|\, p(1) = M\} \,.$$

Also, let $\bar{\pi}_M(0) := \limsup_{t\to 0} \bar{\pi}_M(t)$, and define $\underline{\pi}_M(0)$, $\bar{\pi}_M(1)$ and $\underline{\pi}_M(1)$ analogously.

The curves $\bar{\pi}_M$ and $\underline{\pi}_M$ bound a set D_M in which the graph of any standard p-function $p(t)$ or Markov transition function $p_{ii}(t)$ must remain for $t \in (0, 1)$ given that it passes through $p(1) = M$. Whether there is a $p \in \mathscr{P}$ with $p(1) = M$ and $p(t) = c$ for any $(t, c) \in D_M$ is another matter. This would clearly be the case if $\mathscr{P}$ were additively convex, but unfortunately it is not, as Davidson [6] has shown. We *can* find such a p for $M \leqq c \leqq M^t$, since $\mathscr{P}$ contains all functions of the form $p(t) = e^{-x(t)}$, where x is nonnegative, continuous and concave on $[0, \infty)$, with $x(0) = 0$ [10]. The remainder of D_M is much more difficult to analyse, as we shall see in Sections 5 and 6, but I conjecture that all points (t, c) *inside* D_M are in fact accessible. The matter of accessibility of the boundary is more delicate; here, for instance, we must distinguish between $\mathscr{P}$ and $\mathscr{PM}$.

It is also reasonable to conjecture that the extension of $\bar{\pi}$ and $\underline{\pi}$ to $(1, \infty)$ is totally determined by their nature on $[0, 1]$ and that $\bar{\pi}_M(1/t) = \underline{\pi}_M{}^{-1}(t)$ and $\underline{\pi}_M(1/t) = \bar{\pi}_M{}^{-1}(t)$, where the inverse operation is applied to π as a function of M. But to show this we need at least that $\bar{\pi}$ and $\underline{\pi}$ are continuous and monotone as functions of M, and even this appears nontrivial.

Interest in the maximal oscillation problem until now has largely centered about

$$I(M) := \inf\{\underline{\pi}_M(t);\ 0 \leqq t \leqq 1\}$$

and

$$\nu_0 := \inf\{M \mid I(M) > 0\}\,.$$

Davidson [6] first studied the relationship between M and $I(M)$. Denoting $m(p) = \min\{p(t);\ 0 \leqq t \leqq 1\}$, he remarked that if for a given $p \in \mathscr{P}$ $m(p) < \hat{M} < M = p(1)$, then there exists a standard p-function $\hat{p}$ such that $m(\hat{p}) = m(p)$ and $\hat{p}(1) = \hat{M}$. Thus it is natural to seek the boundary between accessible and inaccessible pairs (M, m) in what Davidson referred to as the "$M - m$ diagram" (Fig. 1). Lower bounds for $I(M)$ give inaccessible regions, and families of known p-functions provide accessible regions. While it is most likely that $I(M)$ marks the boundary between these regions, this seems difficult to prove.

Davidson produced accessible points of large oscillation by considering the well-known standard "jump" p-functions (indeed the *only* well-known family of standard p-functions which are not monotone):

$$p(s) = \sum_{k=0}^{[s/b]} \frac{q^k(s - bk)^k}{k!}\, e^{-q(s-bk)}\,, \tag{5}$$

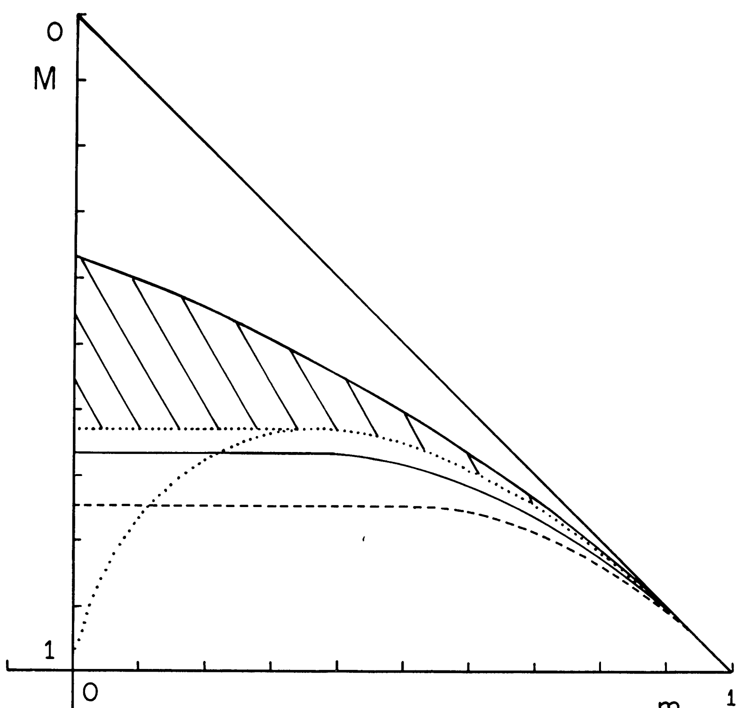

FIG. 1. The $M - m$ diagram. The area between $m = M$ and (6) (the upper solid curve) is accessible. Regions successively proved inaccessible are bounded by (8) (the dashed curve), Proposition 1 (the lower solid curve), and (9) and (15) (the dotted curves). The shaded portion of the diagram is still uncharted.

where $[s]$ denotes the integer part of s. It is worth mentioning that these functions are not Markov; a celebrated theorem of Ornstein states that $p_{ii} \in \mathscr{P}\mathscr{M}$ is continuously differentiable on $(0, \infty)$, whereas p as given by (5) has discontinuous derivative at $s = b$. In [9] Freedman constructs a family $\{p_{ii}^{(n)}\}$ of Markovian transition functions converging to $p(s)$ uniformly in s, an illustration of the density of $\mathscr{P}\mathscr{M}$ in $\mathscr{P}$. By maximizing $p(1) = M$ for $p(b) = m$ in (5), Davidson produced a sub-family of standard p-functions with $m = 1 + \log M$ $(e^{-1} \leqq M \leqq 1)$. These special p-functions are exponential on $[0, b]$ with $p'(0+) = -q$, where $b = \log m/(\log m + \log M)$ and $q = 1 - m - \log m$. For M near 1 they are exponential just beyond $t = \frac{1}{2}$, and as M decreases they stay exponential for longer and longer until at $M = e^{-1} + \varepsilon$ p is exponential on virtually all of $[0, 1)$, spiking from arbitrarily small values up to M (see Fig. 2). This example therefore gives

$$I(M) \leqq 1 + \log M \quad (e^{-1} \leqq M \leqq 1)\,, \tag{6}$$

and consequently

$$\nu_0 \geqq e^{-1}\,. \tag{7}$$

Davidson also found the first lower bound for $I(M)$, a result discovered independently by Blackwell and Freedman [1] in the Markov case. Namely, he

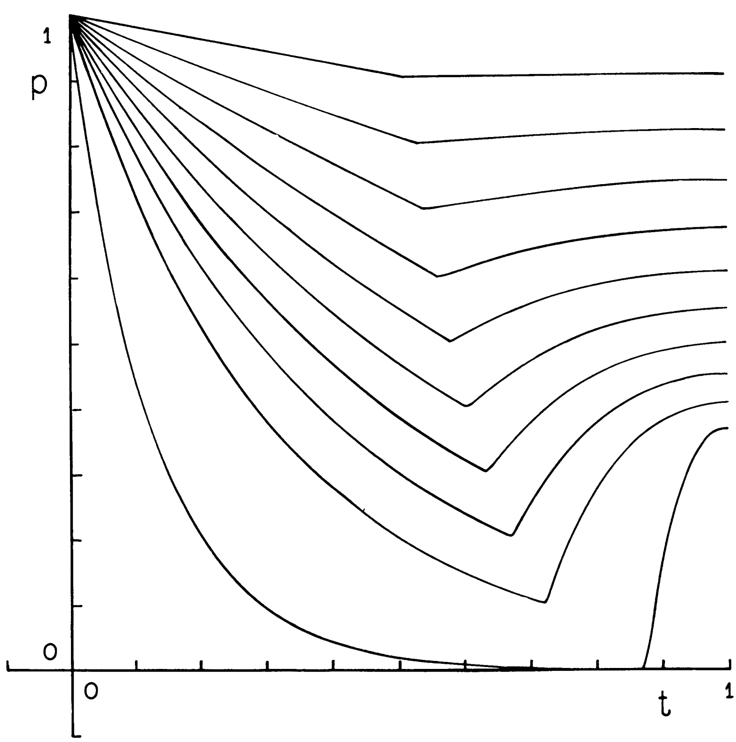

FIG. 2. The Davidson p-functions. Examples yielding (6) are shown for $m = .9, .8, \cdots, .1$ and for $m = .001$. Note that in this last case $M = .3682$, exceeding e^{-1} by less than .0004, while b is still only about .87.

showed that for $p \in \mathscr{K}_2$,

$$\text{(8)} \qquad \text{If} \quad M > \tfrac{3}{4}\,, \qquad \text{then} \quad I(M) \geqq \frac{1 + (4M - 3)^{\frac{1}{2}}}{2}\,.$$

It is not hard to prove that (8) is sharp for $p \in \mathscr{K}_2$. An involved but routine check shows that for $M > \frac{3}{4}$, if we define p to be exponential on $[0, \frac{1}{2}]$ with $p(\frac{1}{2}) = m = \frac{1}{2}[1 + (4M - 3)^{\frac{1}{2}}]$, take $p(t) = 1 - (1 - M)/p(1 - t)$ on $[\frac{1}{2}, 1]$ and define p appropriately on $(1, \infty)$, then $p \in \mathscr{K}_2$. Moreover, if $M = \frac{3}{4}$, then p defined by

$$\begin{aligned} p(t) &= e^{-qt} && \text{for} \quad 0 \leqq t \leqq b \\ &= e^{-qt} + 1 - e^{-q(t-b)} && \text{for} \quad b \leqq t \leqq 1\,, \end{aligned}$$

where $m = e^{-qb}$ and $b = q^{-1} \log [1 + e^q/4]$ defines for $q \in [\log 4, \infty)$ a family of $\mathscr{K}_2$ functions achieving as minima all values in $(0, \frac{1}{2}]$, and all spiking to $\frac{3}{4}$ at time 1. Thus these are extreme functions for the second order inequalities.

A much more extensive inaccessible region for the (M, m) diagram was determined later by Bloomfield [3] and Davidson [8], when they proved that

$$\text{(9)} \qquad M \leqq 1 + m \log m\,.$$

This bound is extremely good for large M, as the loci (6) and (9) have the same curvature at $M = 1$. For $m < e^{-1}$, however, (9) deteriorates, so that until recently it has been necessary to fall back on (8) for m near 0. New results to be described in Sections 4 and 7 have improved this situation.

A final proposition along these lines was obtained by Davidson [6], when he was able to show that for $p \in \mathscr{K}_3$,

$$\nu_0 \leqq \tfrac{2}{3}\,.$$

Unfortunately this last argument made no real contribution to the (M, m) diagram. The bounds for the diagram mentioned here and below are shown in Fig. 1.

4. A new result for the $M - m$ diagram. We record here a new bound for $I(M)$ which places Davidson's $\frac{2}{3}$ result in a more suitable light. Note that the argument, like his, is based on third order Kingman inequalities.

PROPOSITION 1. *If $M > \frac{2}{3}$, then*

$$I(M) \geqq \frac{1 + 2(3M - 2)^{\frac{1}{2}}}{3}\,.$$

PROOF. For $p \in \mathscr{K}_3$, the Kingman inequality $g(t_3) \geqq 0$ may be rewritten (taking $s = t_1$, $t = t_2 - t_1$, $u = t_3 - t_2$, and $s + t + u = t_3 = 1$)

$$1 - M \geqq p(s)[1 - p(t + u) - p(t)\{1 - p(u)\}] + p(s + t)[1 - p(u)]\,.$$

We make use of the lower order Kingman inequalities $p(t) \leqq p(t + u)/p(u)$ and $p(s + t) \geqq 1 - (1 - M)/p(u)$ to obtain

$$\text{(10)} \qquad 1 - M \geqq p(s)\left[1 - \frac{p(t + u)}{p(u)}\right] + \left[1 - \frac{1 - M}{p(u)}\right][1 - p(u)]\,.$$

Taking $p(t + u) = m$, $p(u) = (1 - M + m^2)^{\frac{1}{2}}$ (this can always be done by continuity of p because $p(0) = 1 \geqq p(u) \geqq m = p(t + u)$), and since $p(s) \geqq m$, we find that

$$M \leqq \frac{3m^2 - 2m + 3}{4} . \tag{11}$$

Next assume $m \leqq \frac{1}{3}$, and choose u to be the first time that $p(u) = \frac{2}{3}$. Substituting this value for $p(u)$ in (10), we obtain

$$1 - M \geqq p(s)[\tfrac{2}{3} - p(t + u)] + \tfrac{2}{9} . \tag{12}$$

Now choose v to be the first time that $p(v) = \frac{1}{3}$. If $p(1 - v) \leqq \frac{1}{3}$, then we must have $1 - v \geqq v$, or $v \leqq \frac{1}{2}$. Since $u \leqq v$, it follows that $u + v \leqq 1$, so we may take $s = v$ and $t = 1 - u - v$ in (12). If, on the other hand, $p(1 - v) \geqq \frac{1}{3}$, we take $s = 1 - v$ and $t = v - u$ in (12). In this manner we guarantee

$$1 - M \geqq \tfrac{1}{3}[\tfrac{2}{3} - \tfrac{1}{3}] + \tfrac{2}{9} ,$$

or

$$M \leqq \tfrac{2}{3} . \tag{13}$$

In combination, (11) and (13) are equivalent to the proposition.

Whether Proposition 1 is best possible for $p \in \mathscr{K}_3$ is not known, but I conjecture that a considerably better bound exists. A means of proving $\nu_0 \leqq \frac{1}{2}$ would be especially valuable, as Williams [14] has shown that this would have an important application to the problem of Markov groups. Namely, we could conclude that for any Markov semigroup P:

$$\text{If} \quad \inf_i \{p_{ii}(t)\} > \tfrac{1}{2} \qquad \text{for some} \quad t\ (> 0), \quad \text{then} \quad \lim_{t \to 0} \|P_t - I\| = 0 .$$

Actually, proving the weaker property: $\underline{\pi}_{\frac{1}{2}}(t) > 0$ for all $t \in (0, t_0)$ for some positive t_0 would be sufficient. In the notation of [10], either of these would yield $(F) \Rightarrow (U)$, one of the two implications needed for a complete characterization of those Markov semigroups which may be extended to strongly continuous groups on the entire real line.

5. Remarks on $\underline{\pi}_M$. Very little attention has been paid to the more general description of $\underline{\pi}_M$ and $\bar{\pi}_M$ as functions of t for fixed M. In this section we set down a few remarks on the lower boundary for standard p-functions through $p(1) = M$, and in the next we discuss the upper boundary.

Results such as Proposition 1 establish uniform lower bounds for $\underline{\pi}_M$, but as to the shape of this curve, almost nothing is known. Indeed, we do not even know in what sense it may be described as a "curve." $\underline{\pi}_M$ is easily seen to be upper semi-continuous, with $\underline{\pi}_M(s + t) \geqq \underline{\pi}_M(s)\underline{\pi}_M(t)$, but is $\underline{\pi}_M(t)$ continuous for $t \in (0, 1)$? I conjecture that it is, though this is by no means obvious.

Davidson's results show that for $M \leqq e^{-1}$, $\underline{\pi}_M(t) = 0$ on $[0, 1]$. For no value of t in $[0, 1]$ is $\underline{\pi}_M(t)$ yet known when $M > e^{-1}$. We can, however, determine an upper bound for $\underline{\pi}_M(0)$:

PROPOSITION 2. $\underline{\pi}_M(0) \leqq e^{1-(1/M)}$.

PROOF. Let $\{t_n\}$ be a sequence of times decreasing to 0. By taking $\{p_n(s)\}$ to be of the form (5) with $q_n = (1 - M)/Mt_n$ and $b_n = t_n$, we find that $\lim_{n\to\infty} p_n(t_n) = e^{1-(1/M)}$ and $\lim_{n\to\infty} p_n(1) = M$. These p-functions establish the proposition.

The proposition may also be regarded as a localized version of a result due to Bloomfield [2]:

$$p(1) \geqq \exp\left\{1 - \frac{1}{p(\infty)}\right\}, \tag{14}$$

where (14) is in fact a sharp uniform lower bound for $p(1)$ in terms of $p(\infty)$.

The following corollary is immediate:

COROLLARY. *$\mathscr{P}_M := \{p \in \mathscr{P} | p(1) = M\}$ is not compact for any $M < 1$.* (We can exhibit a sequence in $\mathscr{P}_M$ with limit function p^* such that $p^*(0) \neq 1$.)

I conjecture that for $I(M) > 0$, $\underline{\pi}_M(0)$ is in fact equal to $e^{1-(1/M)}$, and that $\underline{\pi}_M(1) = M$.

In much the same way that Davidson found the upper bound (6) for $I(M)$, we can find upper bounds for $\underline{\pi}_M(t)$ by minimizing (5) at t subject to the constraint that $p(1) = M$. This appears unmanageable by analytic techniques, but is well within the scope of computer solution. The bound $\underline{h}_M(t)$ which we obtain has, of course, $\underline{h}_M(0) = e^{1-(1/M)}$, appears to decrease continuously but not monotonically to a minimum at $\underline{h}_M(\log(1 + \log M)/(\log M + \log(1 + \log M)) = 1 + \log M$ in accordance with (6), and then increases continuously to $\underline{h}_M(1) = M$. This curve is shown for $M = .7$ in Fig. 3, along with the uniform lower bound obtained from Proposition 1 and other bounds to be described below.

6. Remarks on $\bar{\pi}_M$. The problem of upper bounds for $p(t)$ seems to have been virtually ignored. The nearest kind of result to be found in the literature is one of Blackwell and Freedman [1], derived for $p \in \mathscr{K}_2$, which states that

$$\textstyle\int_0^1 p(t)\,dt < \frac{1 + M}{2} \quad (M < 1)\,.$$

Since p is supermultiplicative we have $p(1/k) \leqq M^{1/k}$ for $k = 1, 2, \cdots$, a bound which is obviously sharp since $p(t) = M^t$ is in $\mathscr{P}$. Hence $\bar{\pi}_M(1/k) = M^{1/k}$. For no other values of t is $\bar{\pi}_M(t)$ known, however, and the same remarks as for $\underline{\pi}$ apply regarding such matters as continuity. Trivially, $\bar{\pi}_M(0) = 1$; the companion conjecture to that of the last section is that $\bar{\pi}_M(1) = M$.

By considering $\bar{h}_M(t)$, the maximum value of (6) at t given that $p(1) = M$, we are able to obtain lower bounds for $\bar{\pi}_M(t)$. Unfortunately, the computer simulations here are much more difficult than for $\underline{\pi}$, but results suggest a striking transition in the nature of $\bar{h}$. Namely, for large M it appears that $\bar{h}_M(t) = M^t$ (see Fig. 3). For M sufficiently small, on the other hand, we seem to have $\bar{h}_M(t) > M^t$ for all t in $(\varepsilon, 1)$ and not of the form $1/k$. Thus for small M $\bar{\pi}_M(t)$ is necessarily a highly complex curve, constrained to be exponential at all times

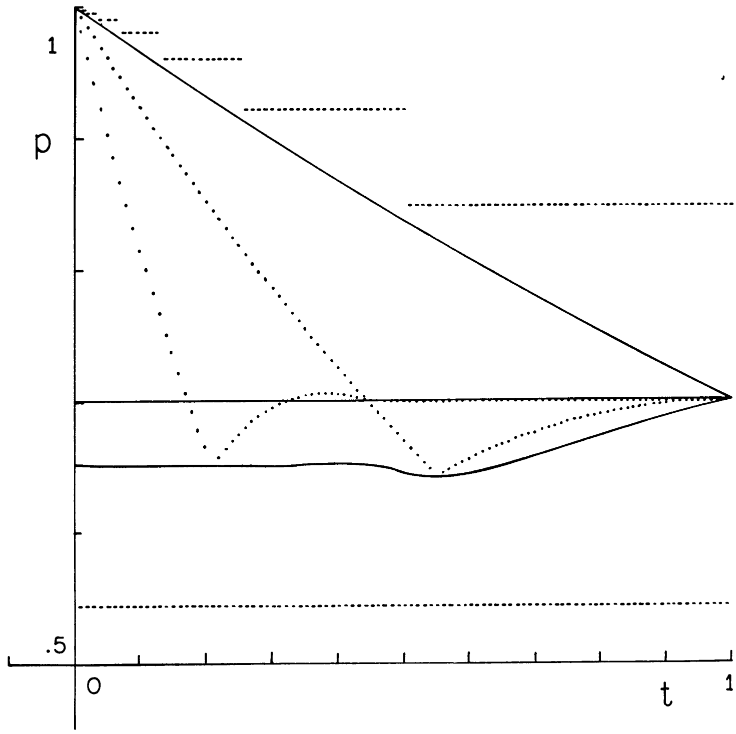

FIG. 3. $\bar{\pi}$ and $\underline{\pi}$ for $M = .7$. The uppermost and lowermost solid curves are computer simulations of $\bar{h}$ and $\underline{h}$. For $M = .7$, $\bar{h} = M^t$ while $\underline{h}$ is more complex. The line $p = .7$ is also shown. Two dotted p-functions of type (6) are given attaining minima on $\underline{h}$. Dashed bounds for $\underline{\pi}$ and $\bar{\pi}$ given by Propositions 1 and 3 are indicated.

of the form $1/k$, but able to hop above M^t during the whole of each of the intervals $(1/k + 1, 1/k)$. This does not mean that there are standard p-functions which exceed M^t in each such interval, but only that there exist $p \in \mathscr{P}$ which exceed M^t at any point in any one of these intervals and also have $p(1) = M$. A typical jump p-function exceeding M^t is shown in Fig. 4.

Computer simulations also seem to indicate that the portion of D_M lying between $\bar{h}_M(t)$ and $\underline{h}_M(t)$ is totally accessible. As a final conjecture I suggest that we may well have $\bar{h}_M = \bar{\pi}_M$ and $\underline{h}_M = \underline{\pi}_M$ for $I(M) > 0$.

We conclude this section by deriving an upper bound for $\bar{\pi}_M(t)$ based on Proposition 1, and holding for all $p \in \mathscr{K}_3$.

PROPOSITION 3. *For* $1 \geqq \alpha > \frac{2}{3}$, *if* $M < \alpha[(1 + 2(3\alpha - 2)^{\frac{1}{2}})/3]$, *then*

$$\bar{\pi}_M(t) \leqq \alpha^{1/k} \qquad \textit{for} \;\; 2^{-k} \leqq t \leqq 2^{1-k};\; k = 1, 2, \cdots .$$

PROOF. It suffices to show that $\pi_M(t) \leqq \alpha$ on $[\frac{1}{2}, 1]$, as the more general result follows from the supermultiplicity of p-functions. For $t \in [\frac{1}{2}, 1]$ we have

$$M \geqq p(t)p(1 - t)\,.$$

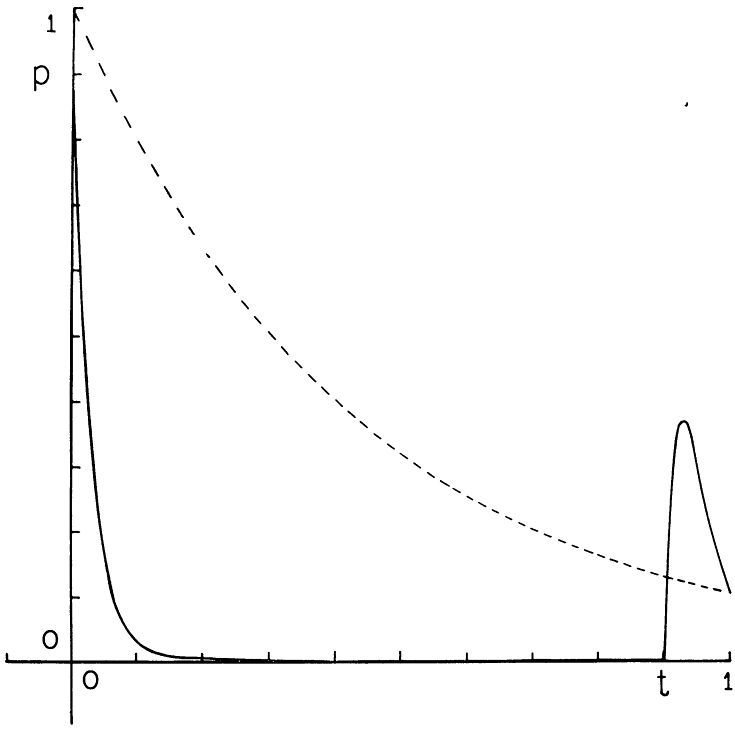

FIG. 4. A jump p-function with $p(1) = .1$ is shown exceeding $(.1)^t$ from just beyond .9 to 1.

If $p(t) > \frac{2}{3}$, then since $1 - t \leqq t$, by Proposition 1 we find that

$$M \geqq p(t)\left(\frac{1 + 2[3p(t) - 2]^{\frac{1}{2}}}{3}\right),$$

from which the proposition easily follows.

The upper bound for $\bar{\pi}$ given by Proposition 3 is shown for $M = .7$ in Fig. 3.

7. Recent developments. In this last section we discuss two new results which have been derived simultaneously with those described above.

Cornish [5] has obtained:

$$(15) \qquad I(M) \geqq \rho(M) \quad \text{for} \quad M > 1 - e^{-1},$$

where ρ is the largest root in (0, 1] of $M = 1 + \rho \log \rho$. He has thus succeeded in extending a horizontal tangent from the maximum point $(1 - e^{-1}, e^{-1})$ of the Bloomfield locus (9) to $(1 - e^{-1}, 0)$ in the $M - m$ diagram (see Fig. 1). While (15) is sharper than Proposition 1, the Cornish argument involves Kingman inequalities of arbitrarily large order. It is interesting to note that for $p \in \mathscr{K}_3$ his method gives $\nu_0 \leqq \frac{19}{27}$ whereas Proposition 1 implies $\nu_0 \leqq \frac{2}{3}$, a better value. It is therefore not too much to hope that third order inequalities may eventually yield a better bound than (15).

Cornish's work suggests one method of checking: non-linear programming. His technique is to reduce high-order inequalities to linear maximization problems, but in doing so a good deal of precision is probably lost. Perhaps computer programming will be able to help us identify the extreme $\mathscr{K}_3$ functions, which seem to hold the key to the entire problem.

This is especially true in light of a result of Kingman [13]. He has shown that if $p(t) = e^{-qt}$ on $[0, \tau]$, then $p(t) \leqq e^{-q\tau} + e^{-1}$ on $[\pi, \infty)$. This has an intriguing corollary for the maximal oscillation problem. Suppose that $\{p_n(t)\}$ is a sequence of p-functions with $m(p)$ tending to 0 and $M(p)$ tending to ν_0, and suppose further that the p_n are *all* exponential for at least some *fixed* time τ. Then since $q_n \rightarrow \infty$ and τ is bounded away from 0, the Kingman result gives $\nu_0 \leqq e^{-1}$ so that by (7), $\nu_0 = e^{-1}$.

Now the Davidson examples which give (6) are all exponential on $[0, \frac{1}{2}]$ (see Fig. 2), as are the extremal $\mathscr{K}_2$ functions. Thus, if the extremal $\mathscr{K}_3$ functions are also exponential on $[0, \frac{1}{2}]$ and appear to be tending toward Davidson's, then we have convincing evidence that these latter functions are indeed the extremal standard p-functions. If, on the other hand, the $\mathscr{K}_3$ extreme functions only stay exponential until short of $t = \frac{1}{2}$ or do not have exponential starts at all, this would seem to indicate that ν_0 may well exceed e^{-1}.

One last remark will support these claims. Namely, the second order inequality $g(t_2) \geqq 0$ yields

$$p'(t+) = \lim_{h\to 0} \frac{p(t+h) - p(t)}{h} \leqq \lim_{h\to 0}\left[\frac{1-p(h)}{h}\right][1 - p(t)] = q[1 - p(t)] .$$

In particular, for $p(t) = m$, we see that

$$p'(t+) \leqq q[1 - m] , \tag{16}$$

so that for $q < \infty$ the rate at which p can increase is regulated by q and m. Note that (16) is satisfied with equality by the extreme $\mathscr{K}_2$ functions, and also by the Davidson p-functions which we conjecture are extremal for $\mathscr{P}$. It is most plausible that the extreme p-functions are exponential on some interval $[0, b]$, the heuristics being that p should probably attain m as quickly ("efficiently") as possible. Similarly, it is reasonable that after attaining m the function should climb as quickly as possible. According to these principles, the second order inequality above suggests the differential equation

$$p'(x) = q[1 - e^{-qt}] ,$$

which the extreme $\mathscr{K}_2$ functions satisfy on $[b, 1]$. A third order inequality $g(t_3) \geqq 0$ may be written

$$[1 - p(b + kh)] - p(h)[1 - p(b + (k-1)h)] \geqq [p(b+h) - p(b)p(h)][1 - p((k-1)h)]$$

to suggest the differential equation

$$-p'(x) + q[1 - p(x)] = q[1 - e^{-q(x-b)}] . \tag{17}$$

The surprising result of this approach is that the Davidson p-functions satisfy (17). This observation reinforces our conviction that the entire maximum oscillation problem *may* be solved using third order inequalities, and that the use of higher order ones is probably misguided. Thus we consider it not at all unlikely that the extreme functions of $\mathscr{K}_3$, and so *a fortiori* of $\mathscr{P}$, are the Davidson p-functions.

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