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HIGH LEVEL OCCUPATION TIMES FOR CONTINUOUS GAUSSIAN PROCESSES

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Let $\{y(\tau), 0 \leqq \tau \leqq 1\}$ be a sample continuous Gaussian process, and let $T[y, \alpha]$ denote the time that $y(\cdot)$ spends above level α:

$$T[y, \alpha] = \int_0^1 V(y(\tau) - \alpha)\, d\tau ,$$

where $V(x) = 0$ or 1 according as $x \leqq 0$ or $x > 0$. In this paper it is proved that, as $\alpha \to \infty$,

$$P\{T[y, \alpha] > \beta\} = \exp\{-(\alpha^2/2)k_\beta(1 + o(1))\}$$

where k_β is a particular functional of the covariance function of the process.

1. Introduction. Previous studies of the distribution of the occupation time T for high levels have been made by Volkonskii and Rozanov [8], and Berman [1], [2], [3]. For a zero-mean stationary Gaussian process $\{y(\tau), \tau \geqq 0\}$ which satisfies a strong mixing condition, and whose covariance function satisfies a given smoothness condition at the origin, it is shown in [8] that

$$\alpha \int_0^t V(y(\tau) - \alpha)\, d\tau$$

has a limiting distribution as α and $t = t(\alpha)$ become infinite. A similar result under more general conditions is given in [2]. In [1] and [2], conditions on the covariance of a zero-mean stationary Gaussian process are given, under which the distribution of $\alpha T[y, \alpha]$, conditioned on the event $T > 0$, has a limiting distribution as $\alpha \to \infty$. This is then generalized in [3] to include a wider class of covariances with the normalization $\alpha^c T[y, \alpha]$, and the conditioning events $T > 0$ and $y(0) = \alpha$. The principal differences of the present paper are that the normalization $\alpha^c T[y, \alpha]$ is not used, and that sample continuity is the only condition imposed.

To formulate the result, let $L_2[0, 1]$ denote the Hilbert space of real-valued, square-integrable functions on $[0, 1]$ with inner product $(x, u) = \int_0^1 x(\tau)u(\tau)\, d\tau$. Denote the covariance function of the process by ρ (necessarily continuous), and let A denote the operator defined by $Ax \doteq \int_0^1 \rho(\cdot, s)x(s)\, ds$, $x \in L_2[0, 1]$. The space of real-valued, continuous functions on $[0, 1]$ is denoted by $C[0, 1]$, and the supremum norm on $C[0, 1]$ by $\|\cdot\|$. We define $\inf B = +\infty$ whenever B is the empty set.

THEOREM 1.1. *As* $\alpha \to \infty$

$$P\{T[y, \alpha] > \beta\} = \exp\{-(\alpha^2/2)k_\beta(1 + o(1))\}$$

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where

(1) $$k_\beta = \inf\{(Ax, x): x \in C[0, 1],\ T[Ax, 1] > \beta\}\,.$$

Theorem 1.1 is first proved in the finite-dimensional case; the function space version is obtained by passage to the limit using an estimate of the supremum distribution due to Marcus and Shepp [5].

When $\beta = 0$, $P\{T[y, \alpha] > \beta\}$ is equal to $P\{\sup_{0 \leq \tau \leq 1} y(\tau) > \alpha\}$. Using results of [5] (see Lemma 3.1), it follows that

$$P\{T[y, \alpha] > 0\} = \exp\{-\alpha^2(2v^2)^{-1}(1 + o(1))\}$$

where $v^2 = \sup_{0 \leq \tau \leq 1} \operatorname{Var} y(\tau)$. At the other extreme,

$$P\{T[y, \alpha] = 1\} = P\{\inf_{0 \leq \tau \leq 1} y(\tau) > \alpha\}\,.$$

In this case, it is known [6] that

$$P\{T[y, \alpha] = 1\} = \exp\{-\alpha^2(4\sigma)^{-1}(1 + o(1))\}$$

where

$$\sigma = \sup_{x \in C[0,1]} \{\inf_{0 \leq \tau \leq 1} (Ax)(\tau) - \tfrac{1}{2}(Ax, x)\}\,.$$

The extremal problem (1) is solved for the following class of covariance functions in Section 5: *If, for each* s, $\rho(\tau, s)$ *is non-decreasing (non-increasing) in* τ, *then* $k_\beta = 1/\rho(1 - \beta, 1 - \beta)$ *(respectively,* $1/\rho(\beta, \beta)$*) for all* $0 \leq \beta < 1$. Note that neither condition is satisfied if ρ is the covariance function of a stationary process, except in the trivial case when $P\{y(\tau) = y(0), 0 \leq \tau \leq 1\} = 1$. Also, if $1 \leq \gamma \leq 2$, the covariance function

$$\rho(s, \tau) = \tfrac{1}{2}[s^\gamma + \tau^\gamma - |s - \tau|^\gamma]$$

satisfies the first condition, while for $0 < \gamma < 1$, it satisfies neither.

2. Finite-dimensional results. If $x \in \mathscr{R}^{n+1}$ is a column vector with components $x_0, \cdots, x_n$, we denote the transpose of x by x^T and define $|x| = \max_{0 \leq k \leq n} |x_k|$. For each $x \in \mathscr{R}^{n+1}$ with components $x_0, \cdots, x_n$ we define the polygonal function $x^{n+1} \in C[0, 1]$ where

$$x^{n+1}(\tau) = (j - n\tau)x_{j-1} + (n\tau - j + 1)x_j\,,$$
$$j - 1 \leq n\tau \leq j;\ j = 1, 2, \cdots, n;\ 0 \leq \tau \leq 1\,.$$

If $x \in \mathscr{R}^{n+1}$, we define $t_n(x, \alpha) = T[x^{n+1}, \alpha]$ and $t_n(|x|, \alpha) = T[||x^{n+1}||, \alpha]$.

LEMMA 2.1. *Let* $y_0, \cdots, y_n$ *be zero-mean, jointly Gaussian random variables with nonsingular covariance matrix* H. *Write* $y = (y_0, \cdots, y_n)$. *Then*

$$\lim_{\alpha \to \infty} (1/\alpha^2) \log P\{t_n(y, \alpha) > \beta\} = -\tfrac{1}{2} \inf\{x^T Hx: x \in \mathscr{R}^{n+1},\ t_n(Hx, 1) > \beta\}\,.$$

PROOF. Let $P(\alpha) = P\{t_n(y, \alpha) > \beta\}$. Since H is nonsingular,

$$P(\alpha) = C_n \int_{t_n(u,\alpha) > \beta} \exp\{-\tfrac{1}{2}u^T H^{-1} u\}\, du$$

where $C_n = (\det H)^{-\frac{1}{2}}(2\pi)^{-(n+1)/2}$. Making the linear transformation $u = \alpha x$, and using $t_n(\alpha x, \alpha) = t_n(x, 1)$, $\alpha > 0$, we obtain

$$P(\alpha) = \alpha^{n+1} C_n \int_\Lambda \exp\{-(\alpha^2/2) x^T H^{-1} x\}\, dx ,$$

where $\Lambda = \{x : t_n(x, 1) > \beta\}$. We shall assume that Λ is not empty since the conclusion of the lemma is clearly true when Λ is empty. Since the L_p norm converges to the L_∞ norm as $p \to \infty$, we have (with $p = \alpha^2$)

$$\lim_{\alpha\to\infty}(1/\alpha^2) \log P(\alpha) = \operatorname{ess\,sup}_{x\in\Lambda} \{-\tfrac{1}{2} x^T H^{-1} x\} .$$

Because Λ is open, and $x^T H^{-1} x$ is continuous on Λ, it follows that the essential supremum equals the supremum. Nonsingularity of H then implies that

$$\inf \{x^T H^{-1} x : t_n(x, 1) > \beta\} = \inf \{x^T H x : t_n(Hx, 1) > \beta\} ,$$

and the proof is complete.

The next two lemmas will be used to remove the nonsingularity assumption of Lemma 2.1.

LEMMA 2.2. *Let $y_0, \cdots, y_n$ be zero-mean, jointly Gaussian random variables, and let $z_0, \cdots, z_n$ be independent, standard normal random variables with $(y_0, \cdots, y_n)$ and $(z_0, \cdots, z_n)$ independent. Then*

$$\lim_{\varepsilon\downarrow 0} \lim_{\alpha\to\infty} (1/\alpha^2) \log P\{t_n(y + \varepsilon z, \alpha) > \beta\} = \lim_{\alpha\to\infty} (1/\alpha^2) \log P\{t_n(y, \alpha) > \beta\} .$$

PROOF. Note first that

(2) $$\lim_{\alpha\to\infty} (1/\alpha^2) \log P\{|z| > \alpha\} = -\tfrac{1}{2} .$$

Let $p > 0$, $q > 0$, $p + q = 1$. If $\varepsilon > 0$,

(3) $$\begin{aligned} t_n(y + \varepsilon z, \alpha) &\leqq t_n(y, \alpha p) + t_n(\varepsilon z, \alpha q) \\ &\leqq t_n(y, \alpha p) + t_n(\varepsilon|z|, \alpha q) \\ &= t_n(y, \alpha p) + V(|z| - (\alpha q/\varepsilon)) . \end{aligned}$$

Similarly,

(4) $$t_n(y, \alpha) \leqq t_n(y + \varepsilon z, \alpha p) + V(|z| - (\alpha q/\varepsilon)) .$$

If ξ and η are real numbers, then $\xi + \eta > \beta$ implies that $\xi > \beta$ or $\eta > 0$. Thus, from (3) and (4)

(5) $$P\{t_n(y + \varepsilon z, \alpha) > \beta\} \leqq P\{t_n(y, \alpha p) > \beta\} + P\{|z| > (\alpha q/\varepsilon)\}$$

(6) $$P\{t_n(y, \alpha) > \beta\} \leqq P\{t_n(y + \varepsilon z, \alpha p) > \beta\} + P\{|z| > (\alpha q/\varepsilon)\} .$$

Using the inequality $\log(|x| + |y|) \leqq \log 2 + \max(\log|x|, \log|y|)$, it follows from (2), (5), and (6) that

(7) $$\begin{aligned} &\lim_{\alpha\to\infty} (1/\alpha^2) \log P\{t_n(y + \varepsilon z, \alpha) > \beta\} \\ &\qquad \leqq \max [p^2 \liminf_{\alpha\to\infty} (1/\alpha^2) \log P\{t_n(y, \alpha) > \beta\}, -q^2/(2\varepsilon^2)] \end{aligned}$$

(8) $$\begin{aligned} &\limsup_{\alpha\to\infty} (1/\alpha^2) \log P\{t_n(y, \alpha) > \beta\} \\ &\qquad \leqq \max [p^2 \lim_{\alpha\to\infty} (1/\alpha^2) \log P\{t_n(y + \varepsilon z, \alpha) > \beta\}, -q^2/(2\varepsilon^2)] . \end{aligned}$$

The limits appearing in (7) and (8) exist by virtue of Lemma 2.1 and the fact that the covariance matrix of $y_0 + \varepsilon z_0, \cdots, y_n + \varepsilon z_n$ is nonsingular. The proof is completed by letting $\varepsilon \downarrow 0$ and then $p \uparrow 1$ in (7) and (8).

LEMMA 2.3. *Let H be an $(n + 1) \times (n + 1)$ covariance matrix, and let I denote the $(n + 1) \times (n + 1)$ identity matrix. Then*

$$\lim_{\varepsilon \downarrow 0} \inf \{x^T(H + \varepsilon I)x : t_n(Hx + \varepsilon x, 1) > \beta\} = \inf \{x^T Hx : t_n(Hx, 1) > \beta\} .$$

PROOF. Let

$$u(\varepsilon) = \inf \{x^T(H + \varepsilon I)x : t_n(Hx + \varepsilon x, 1) > \beta\} , \qquad \varepsilon > 0 .$$

If $p > 0,\ q > 0,\ p + q = 1$, then

$$t_n(Hx + \varepsilon x, 1) \leqq t_n(Hx, p) + V(|x| - (q/\varepsilon)) .$$

It follows that

$$\{x^T(H + \varepsilon I)x : t_n(Hx + \varepsilon x, 1) > \beta\} \\ \subset \{x^T(H + \varepsilon I)x : t_n(Hx, p) > \beta\} \cup \{x^T(H + \varepsilon I)x : |x| > (q/\varepsilon)\} ;$$

hence

$$\begin{aligned} u(\varepsilon) &\geqq \min [\inf \{x^T(H + \varepsilon I)x : t_n(Hx, p) > \beta\}, \\ &\qquad \inf \{x^T(H + \varepsilon I)x : |x| > (q/\varepsilon)\}] \\ &\geqq \min [\inf \{x^T Hx : t_n(Hx, p) > \beta\}, \inf \{\varepsilon x^T x : |x| > (q/\varepsilon)\}] . \end{aligned} \tag{9}$$

Using $t_n(Hx, p) = t_n((1/p)Hx, 1)$ and $x^T x \geqq |x|^2$, it follows from (9) that

$$u(\varepsilon) \geqq \min [p^2 \inf \{x^T Hx : t_n(Hx, 1) > \beta\}, q^2/\varepsilon] . \tag{10}$$

Letting $\varepsilon \downarrow 0$ and then $p \uparrow 1$ in (10) we obtain

$$\liminf_{\varepsilon \downarrow 0} u(\varepsilon) \geqq \inf \{x^T Hx : t_n(Hx, 1) > \beta\} . \tag{11}$$

Next, let $p > 0,\ q > 0,\ p + q = 1$. Then

$$t_n(Hx, 1) \leqq t_n(Hx + \varepsilon x, p) + V(|x| - (q/\varepsilon)) .$$

Hence

$$\inf \{x^T(H + \varepsilon I)x : t_n(Hx, 1) > \beta\} \geqq \min [p^2 u(\varepsilon), \inf \{\varepsilon x^T x : |x| > (q/\varepsilon)\}] .$$

Writing $m(\varepsilon) = \inf \{x^T(H + \varepsilon I)x : t_n(Hx, 1) > \beta\}$, and using $x^T x \geqq |x|^2$ we obtain

$$m(\varepsilon) \geqq \min [p^2 u(\varepsilon), q^2/\varepsilon] . \tag{12}$$

Letting $\varepsilon \downarrow 0$ and then $p \uparrow 1$ in (12), we have

$$\limsup_{\varepsilon \downarrow 0} m(\varepsilon) \geqq \limsup_{\varepsilon \downarrow 0} u(\varepsilon) . \tag{13}$$

Next, if $t_n(Hx, 1) > \beta$ then $x^T Hx + \varepsilon x^T x \geqq m(\varepsilon)$; hence $\limsup_{\varepsilon \downarrow 0} m(\varepsilon) \leqq x^T Hx$. It follows that

$$\limsup_{\varepsilon \downarrow 0} m(\varepsilon) \leqq \inf \{x^T Hx : t_n(Hx, 1) > \beta\} , \tag{14}$$

and the proof is completed by combining (11), (13), and (14).

The finite-dimensional version of Theorem 1.1 in the zero-mean case now follows directly from Lemmas 2.1, 2.2, and 2.3:

THEOREM 2.1. *Let $y_0, \cdots, y_n$ be zero-mean, jointly Gaussian random variables with covariance matrix H (possibly singular). Then, as $\alpha \to \infty$*

$$P\{t_n(y, \alpha) > \beta\} = \exp\{-(\alpha^2/2)k_\beta{}^n(1 + o(1))\}$$

where $k_\beta{}^n = \inf\{x^T Hx : t_n(Hx, 1) > \beta\}$.

3. Lemmas. Several lemmas required in the proof of Theorem 1.1 are given in this section. The following notation will be used: for $0 < h \leqq 1$,

$$\psi^2(h) = \sup\{E[y(s) - y(\tau)]^2 : |s - \tau| \leqq h, 0 \leqq s \leqq 1, 0 \leqq \tau \leqq 1\}.$$

If $x \in C[0, 1]$, then $\pi_n x$ denotes the polygonal function

$$(\pi_n x)(\tau) = (j - n\tau)x[(j - 1)/n] + (n\tau - j + 1)x[j/n]$$
$$j - 1 \leqq n\tau < j;\ j = 1, 2, \cdots, n;\ 0 \leqq \tau \leqq 1.$$

The first lemma is due to Marcus and Shepp [5], and provides the link between the finite-dimensional and function space versions of the results of this paper.

LEMMA 3.1. (Marcus–Shepp) *Let $X_1, X_2, \cdots$ be a Gaussian sequence with arbitrary covariance and means. If* $P\{\sup_{n\geqq 1} |X_n| < \infty\} > 0$, *then*

$$\lim_{\alpha\to\infty} (1/\alpha^2) \log P\{\sup_{n\geqq 1} |X_n| > \alpha\} = -(2v^2)^{-1}$$

where $v^2 = \sup_{n\geqq 1} \operatorname{Var} X_n$.

Note that since $\sup_{n\geqq 1} P\{X_n > \alpha\} \leqq P\{\sup_{n\geqq 1} X_n > \alpha\}$, the above lemma holds for $\sup_{n\geqq 1} X_n$ as well.

Applying the above to a sample continuous Gaussian process $\{y(\tau), 0 \leqq \tau \leqq 1\}$, it follows from separability that

$$\lim_{\alpha\to\infty} (1/\alpha^2) \log P\{||y|| > \alpha\} = -(2v^2)^{-1}$$

where $v^2 = \sup_{0\leqq\tau\leqq 1} \operatorname{Var} y(\tau)$.

LEMMA 3.2. *For all* $0 \leqq \tau \leqq 1$, $E[y(\tau) - (\pi_n y)(\tau)]^2 \leqq \psi^2(1/n)$.

The proof of Lemma 3.2 is straightforward.

LEMMA 3.3. *Let $x \in C[0, 1]$ and define*

$$(B_n x)(\tau) = (1/n) \sum_{j=0}^n \rho(\tau, j/n)x(j/n), \qquad 0 \leqq \tau \leqq 1.$$

Then $||Ax - \pi_n B_n x|| \to 0$ *as* $n \to \infty$.

PROOF. If $x \in C[0, 1]$, then $||Ax - \pi_n Ax|| \to 0$ and $||Ax - B_n x|| \to 0$ as $n \to \infty$. The result now follows from the inequalities

$$||Ax - \pi_n B_n x|| \leqq ||Ax - \pi_n Ax|| + ||\pi_n Ax - \pi_n B_n x||$$
$$\leqq ||Ax - \pi_n Ax|| + ||Ax - B_n x||.$$

LEMMA 3.4. *Let*

$$k_\beta = \inf\{(Ax, x) : T(Ax, 1) > \beta, x \in C[0, 1]\} ,$$
$$k_\beta{}^* = \inf\{(Ax, x) : T(Ax, 1) > \beta, x \in L_2[0, 1]\} .$$

Then $k_\beta = k_\beta{}^*$.

PROOF. Since $k_\beta \geqq k_\beta{}^*$, it suffices to prove that $k_\beta \leqq k_\beta{}^*$. Assume $k_\beta{}^* < \infty$. Let $u \in L_2[0, 1]$, and suppose that $T(Au, 1) > \beta$. By the projection theorem [7], page 71, $u = v + w$ where $v \in \overline{AL_2}$, $w \perp AL_2$, and $\overline{AL_2}$ denotes the L_2-closure of the range of A. Symmetry of A implies that $Aw = 0$, hence $(Au, u) = (Av, v)$ and $T(Au, 1) = T(Av, 1)$. Choose $v_n \in AL_2$ such that $(v_n - v, v_n - v) \to 0$. Then $||Av_n - Av|| \to 0$ [7], page 244. If $p > 0$, $q > 0$, $p + q = 1$, then

$$\beta < T(Av, 1) \leqq T(Av_n, p) + V(||Av - Av_n|| - q) .$$

Thus, for all $n \geqq N(v, q)$ we have $T(Av_n, p) > \beta$ hence $k_\beta \leqq (1/p^2)(Av_n, v_n)$. Letting $n \to \infty$ and then $p \uparrow 1$ we obtain $k_\beta \leqq (Av, v) = (Au, u)$. Since the last holds for all $u \in L_2[0, 1]$ for which $T(Au, 1) > \beta$, it follows that $k_\beta \leqq k_\beta{}^*$.

LEMMA 3.5. *Let* $x = (x_0, \cdots, x_n) \in \mathscr{R}^{n+1}$ *and define*

$$(U_n x)(\tau) = \sum_{j=0}^n \rho(\tau, j/n)x_j \qquad 0 \leqq \tau \leqq 1 .$$

Then, there is a sequence $x_m \in L_2[0, 1]$ *such that*:

(i) $\lim_{m\to\infty} ||U_n x - Ax_m|| = 0$;
(ii) $\lim_{m\to\infty} (Ax_m, x_m) = \sum_{i=0}^n \sum_{j=0}^n \rho(i/n, j/n)x_i x_j$.

PROOF. The proof is based on choosing a sequence $x_m \in L_2[0, 1]$ which converges to

$$\sum_{j=0}^n x_j \delta(\tau - (j/n)) ,$$

where δ denotes the δ-function. Such a sequence is given by

$$\begin{aligned} x_m(\tau) = {} & m x_0 V((1/m) - \tau) \\ & + \sum_{j=1}^{n-1} (m x_j/2)[V((j/n) + (1/m) - \tau) - V((j/n) - (1/m) - \tau)] \\ & + (m x_n/(m - 1))V(\tau - 1 + (1/m)) \end{aligned}$$

where $m > n$ and $0 \leqq \tau \leqq 1$.

LEMMA 3.6. *Define* $U_n : \mathscr{R}^{n+1} \to C[0, 1]$ *as in Lemma* 3.5, *and define* $k_\beta{}^*$ *as in Lemma* 3.4. *Let* A_n *denote the* $(n + 1) \times (n + 1)$ *matrix* $(\rho(i/n, j/n))$. *Then, if* $0 < p < 1$,

$$\inf\{x^T A_n x : T((1/p)U_n x, 1) > \beta, x \in \mathscr{R}^{n+1}\} \geqq p^4 k_\beta{}^* .$$

PROOF. Let $x \in \mathscr{R}^{n+1}$. By Lemma 3.5 there is a sequence $x_m \in L_2[0, 1]$ such that $||U_n x - Ax_m|| \to 0$ as $m \to \infty$. Suppose that $T((1/p)U_n x, 1) > \beta$. Then, if $q = 1 - p$,

$$\begin{aligned} \beta < T((1/p)U_n x, 1) & \leqq T((1/p)Ax_m, p) + V((1/p)||U_n x - Ax_m|| - q) \\ & = T((1/p^2)Ax_m, 1) + V(||U_n x - Ax_m|| - pq) . \end{aligned}$$

Thus, if $m \geqq M(x, p)$, we have $T((1/p^2)Ax_m, 1) > \beta$ hence, by definition of $k_\beta{}^*$, $(Ax_m, x_m) \geqq p^4 k_\beta{}^*$. Letting $m \to \infty$, and using part (ii) of Lemma 3.5, we obtain $x^T A_n x \geqq p^4 k_\beta{}^*$. Since the last holds for all $x \in \mathscr{R}^{n+1}$ such that $T((1/p)U_n x, 1) > \beta$, the assertion follows.

LEMMA 3.7. *Define* $U_n: \mathscr{R}^{n+1} \to C[0, 1]$ *as in Lemma* 3.5, *and let* A_n *denote the* $(n + 1) \times (n + 1)$ *matrix* $(\rho(i/n, j/n))$. *Then, if* $Ey(\tau) = 0,\ 0 \leqq \tau \leqq 1$, $||\pi_n U_n x - U_n x||^2 \leqq \psi^2(1/n)(x^T A_n x)$ *for all* $x \in \mathscr{R}^{n+1}$.

PROOF. Let $x = (x_0, \cdots, x_n) \in \mathscr{R}^{n+1}$ and $\tau \in [0, 1]$. For some j we have $j - 1 \leqq n\tau \leqq j$ and

$$\begin{aligned}(\pi_n U_n x)(\tau) - (U_n x)(\tau) = {} & [(U_n x)((j/n) - (1/n)) - (U_n x)(\tau)](j - n\tau) \\ & + [(U_n x)(j/n) - (U_n x)(\tau)](n\tau - j + 1)\,,\end{aligned}$$

hence

$$||\pi_n U_n x - U_n x||^2 \leqq \sup_{|s-\tau| \leqq (1/n)} [(U_n x)(s) - (U_n x)(\tau)]^2\,.$$

If $|s - \tau| \leqq (1/n)$, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}[(U_n x)(s) - (U_n x)(\tau)]^2 &= [E\{[y(s) - y(\tau)] \textstyle\sum_{j=0}^n y(j/n)x_j\}]^2 \\ &\leqq \psi^2(1/n)(x^T A_n x)\,.\end{aligned}$$

4. Proof of Theorem 1.1. Let $m(\tau) = Ey(\tau),\ 0 \leqq \tau \leqq 1$. We show first that, without loss of generality, we may assume that $m(\tau) = 0,\ 0 \leqq \tau \leqq 1$. Let $p > 0$, $q > 0$, $p + q = 1$. Then

$$\begin{aligned}T(y, \alpha) &\leqq T(y - m, \alpha p) + V(||m|| - \alpha q) \\ T(y - m, \alpha) &\leqq T(y, \alpha p) + V(||m|| - \alpha q)\,.\end{aligned}$$

Thus, if the theorem is true for zero-mean processes,

$$\begin{aligned}&\limsup_{\alpha\to\infty} (1/\alpha^2) \log P\{T(y, \alpha) > \beta\} \\ &\qquad\leqq p^2 \lim_{\alpha\to\infty} (1/\alpha^2) \log [P\{T(y - m, \alpha) > \beta\}] \\ &p^2 \liminf_{\alpha\to\infty} (1/\alpha^2) \log P\{T(y, \alpha) > \beta\} \\ &\qquad\geqq \lim_{\alpha\to\infty} (1/\alpha^2) \log [P\{T(y - m, \alpha) > \beta\}]\,.\end{aligned}$$

Letting $p \uparrow 1$, it follows that we may assume $m(\tau) = 0,\ 0 \leqq \tau \leqq 1$. Next, note that, by Lemmas 3.1 and 3.2,

$$\lim_{\alpha\to\infty} (1/\alpha^2) \log P\{||y - \pi_n y|| \geqq \alpha\} \leqq -(2\psi^2(1/n))^{-1}\,. \tag{15}$$

Also, by Theorem 2.1,

$$\lim_{\alpha\to\infty} (1/\alpha^2) \log P\{T(\pi_n y, \alpha) > \beta\} = -(k_\beta{}^n/2)\,, \tag{16}$$

where $k_\beta{}^n = \inf\{x^T A_n x : t_n(A_n x, 1) > \beta, x \in \mathscr{R}^{n+1}\}$, and A_n denotes the $(n + 1) \times (n + 1)$ matrix $(\rho(i/n, j/n))$. If $p > 0,\ q > 0,\ p + q = 1$ then

$$T(y, \alpha) \leqq T(\pi_n y, \alpha p) + V(||y - \pi_n y|| - \alpha q)$$

hence

$$P\{T(y, \alpha) > \beta\} \leqq P\{T(\pi_n y, \alpha p) > \beta\} + P\{||y - \pi_n y|| > \alpha q\}\,. \tag{17}$$

Similarly,

$$(18) \qquad P\{T(\pi_n y, \alpha) > \beta\} \leqq P\{T(y, \alpha p) > \beta\} + P\{\|y - \pi_n y\| > \alpha q\} .$$

Taking logarithms of both sides of (17) and (18), using the inequality $\log(|x| + |y|) \leqq \log 2 + \max(\log|x|, \log|y|)$, dividing by α^2 and letting $\alpha \to \infty$ we obtain

$$(19) \qquad \limsup_{\alpha\to\infty} (1/\alpha^2) \log P\{T(y, \alpha) > \beta\} \leqq \max[-(p^2/2)k_\beta{}^n, -q^2(2\psi^2(1/n))^{-1}]$$

$$(20) \qquad -(k_\beta{}^n/2) \leqq \max[p^2 \liminf_{\alpha\to\infty} (1/\alpha^2) \log P\{T(y, \alpha) > \beta\}, -q^2(2\psi^2(1/n))^{-1}]$$

where (15) and (16) have been used. Letting $n \to \infty$ and then $p \uparrow 1$ in (19) and (20) we obtain

$$\lim_{\alpha\to\infty} (1/\alpha^2) \log P\{T(y, \alpha) > \beta\} = -\tfrac{1}{2} \lim_{n\to\infty} k_\beta{}^n .$$

We complete the proof by showing that

$$\lim_{n\to\infty} k_\beta{}^n = \inf\{(Ax, x) : T(Ax, 1) > \beta, x \in C[0, 1]\} .$$

Let $x \in C[0, 1]$ and define

$$(B_n x)(\tau) = (1/n) \textstyle\sum_{j=0}^n \rho(\tau, j/n)x(j/n) , \qquad 0 \leqq \tau \leqq 1 .$$

By Lemma 3.3, $\|Ax - \pi_n B_n x\| \to 0$. Suppose now that $x \in C[0, 1]$ and $T(Ax, 1) > \beta$. If $p > 0$, $q > 0$, $p + q = 1$, then

$$\beta < T(Ax, 1) \leqq T(\pi_n B_n x, p) + V(\|Ax - \pi_n B_n x\| - q) .$$

Thus, for all $n \geqq N(x, q)$, $T(\pi_n B_n x, p) > \beta$ or, equivalently,

$$t_n((1/p)A_n x^*, 1) > \beta$$

where $x^* = (1/n)(x(0), x(1/n), \cdots, x(1))$. Thus,

$$k_\beta{}^n \leqq (1/p^2)(A_n x^*, x^*) = (n^2p^2)^{-1} \textstyle\sum_{i=0}^n \sum_{j=0}^n \rho(i/n, j/n)x(i/n)x(j/n) .$$

Letting $n \to \infty$ and then $p \uparrow 1$, we obtain

$$\lim_{n\to\infty} k_\beta{}^n \leqq (Ax, x) .$$

Since the last holds for all $x \in C[0, 1]$ such that $T(Ax, 1) > \beta$ it follows that

$$(21) \qquad \lim_{n\to\infty} k_\beta{}^n \leqq k_\beta .$$

Next, define for $x \in \mathscr{R}^{n+1}$ with components $x_0, \cdots, x_n$,

$$(U_n x)(\tau) = \textstyle\sum_{j=0}^n \rho(\tau, j/n)x_j , \qquad 0 \leqq \tau \leqq 1 .$$

Then $t_n(A_n x, 1) = T(\pi_n U_n x, 1)$ hence, if $p > 0$, $q > 0$, $p + q = 1$,

$$t_n(A_n x, 1) \leqq T(U_n x, p) + V(\|\pi_n U_n x - U_n x\| - q) .$$

It follows that

$$\{x : t_n(A_n x, 1) > \beta\} \subset \{x : T(U_n x, p) > \beta\} \cup \{x : \|\pi_n V_n x - U_n x\| > q\}$$

hence, by Lemma 3.6,

$$(22) \qquad k_\beta{}^n \geqq \min[p^4 k_\beta{}^*, \inf\{x^T A_n x : \|\pi_n U_n x - U_n x\| > q\}] ,$$

where $k_\beta^* = \inf\{(Ax, x): T(Ax, 1) > \beta, x \in L_2[0, 1]\}$. Applying Lemmas 3.4 and 3.7 it follows from (22) that

$$k_\beta^n \geqq \min[p^4 k_\beta, q^2/\psi^2(1/n)] .$$

Letting $n \to \infty$ and then $p \uparrow 1$ we obtain

$$\lim_{n\to\infty} k_\beta^n \geqq k_\beta ,$$

and this together with (21) completes the proof.

5. Solution of the extremal problem for a class of covariances. Suppose that $\rho(\tau, s)$ is nondecreasing in τ for each s. We shall prove that for all $0 < \beta < 1$,

$$k_\beta = 1/\rho(1-\beta, 1-\beta) , \tag{23}$$

where, formally, the infimum k_β is attained by the δ-function $[\rho(1-\beta, 1-\beta)]^{-1}\delta_{1-\beta}(\cdot)$. To prove (23), note first that if $\sup_{0\leqq\tau\leqq 1-\beta} y(\tau) \leqq \alpha$ then $T[y, \alpha] \leqq \beta$. Thus,

$$P\{T[y, \alpha] > \beta\} \leqq P\{\sup_{0\leqq\tau<1-\beta} y(\tau) > \alpha\} .$$

Applying Theorem 1.1 and Lemma 3.1 we obtain $k_\beta \geqq 1/\rho(1-\beta, 1-\beta)$. If $\rho(1-\beta, 1-\beta) = 0$, the proof is complete. Assume now that $\rho(1-\beta, 1-\beta) > 0$. By Lemma 3.4,

$$k_\beta = \inf\{(Ax, x): T(Ax, 1) > \beta, x \in L_2[0, 1]\} . \tag{24}$$

Let $0 < \varepsilon < 1$, $(1/m) < \min(\beta, 1-\beta)$, and define

$$\begin{aligned} x_m(\tau) &= 1/C_m \qquad 1-\beta-(1/m) \leqq \tau \leqq 1-\beta+(1/m) \\ &= 0 \qquad\quad \text{otherwise,} \end{aligned}$$

where

$$C_m = (1-\varepsilon)\int_{1-\beta-(1/m)}^{1-\beta+(1/m)} \rho(1-\beta, s)\,ds .$$

For m sufficiently large, $C_m > 0$. Next,

$$(Ax_m)(\tau) = (1/C_m)\int_{1-\beta-(1/m)}^{1-\beta+(1/m)} \rho(\tau, s)\,ds .$$

If $\tau \geqq 1-\beta$, then $\rho(\tau, s) \geqq \rho(1-\beta, s)$, hence $(Ax_m)(\tau) \geqq (1-\varepsilon)^{-1} > 1$. It follows that $T[Ax_m, 1] > \beta$, and from (24),

$$\begin{aligned} k_\beta &\leqq (Ax_m, x_m) \\ &= [1/C_m^2]\int_{1-\beta-(1/m)}^{1-\beta+(1/m)}\int_{1-\beta-(1/m)}^{1-\beta+(1/m)} \rho(\tau, s)\,d\tau\,ds . \end{aligned}$$

Letting $m \to \infty$ we obtain $(1-\varepsilon)^2 k_\beta \leqq 1/[\rho(1-\beta, 1-\beta)]$. Since $0 < \varepsilon < 1$ was arbitrary, the proof is complete. Note that if $\rho(\tau, s)$ is non-increasing in τ for each s, application of the above to the process $y(1-\tau)$ gives $k_\beta = 1/\rho(\beta, \beta)$. A similar argument shows that (23) holds when $\beta = 0$.

EXAMPLE. If $\{y(\tau), 0 \leqq \tau \leqq 1\}$ is the Wiener process, then $\rho(s, \tau) = \min(s, \tau)$, and application of the above gives

$$P\{T[y, \alpha] > \beta\} = \exp\{-(\alpha^2/2)(1-\beta)^{-1}(1 + o(1))\}$$

as $\alpha \to \infty$. This can be verified directly using results of Kac [4] to obtain the distribution of T:

$$P\{T[y, \alpha] > \beta\} = \alpha(2/\pi)^{\frac{1}{2}} \int_{(1-\beta)^{-\frac{1}{2}}}^{\infty} K(\beta, x) \exp\{-(\alpha^2 x^2/2)\}\, dx$$

where

$$K(\beta, x) = 1 - (2/\pi) \sin^{-1}\{[(\beta x^2)/(x^2 - 1)]^{\frac{1}{2}}\}.$$

REFERENCES

[1] Berman, S. M. (1971). Excursions above high levels for stationary Gaussian processes. *Pacific J. Math.* **36** 63–79.

[2] Berman, S. M. (1971). Maxima and high level excursions of stationary Gaussian processes. *Trans. Amer. Math. Soc.* **160** 65–85.

[3] Berman, S. M. (1972). A class of limiting distributions of high level excursions of Gaussian processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **21** 121–134.

[4] Kac, M. (1949). On distributions of certain Wiener functionals. *Trans. Amer. Math. Soc.* **65** 1–13.

[5] Marcus, M. B. and Shepp, L. A. (1972). Sample behavior of Gaussian processes. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* Univ. of California Press.

[6] Marlow, N. A. (1971). Asymptotic tail probabilities for a class of functionals of a continuous Gaussian process. To appear.

[7] Riesz, F. and Sz-Nagy, B. (1955). *Functional Analysis.* Unger, New York.

[8] Volkonskii, V. A. and Rozanov, Ju. A. (1961). Some limit theorems for random functions II. *Theor. Probability Appl.* **6** 186–198.

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