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RADIAL PROCESSES[1]

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Let $X = \{X_t = (X_t^1, \cdots, X_t^d),\ t \geqq 0\}$ be an isotropic stochastic process with stationary independent increments having its values in d-dimensional Euclidean space, $d \geqq 2$. Let $R_t = |X_t|$ be the radial process. It is proved (except for a rather trivial exception) that the Markov process $\{R_t\}$ hits points if and only if the real process $\{X_t^1\}$ hits points; a simple analytic criterion for the latter possibility has been known now for some time. If $x > 0$, the sets $\{t: R_t = x\}$ and $\{t: X_t^1 = 0\}$ are then shown to have the same size in the sense that there is an exact Hausdorff measure function that works for both. Finally, if X^1 hits points, it is shown that then X will hit any reasonable smooth surface.

1. Introduction. Let $X = \{X_t, t \geqq 0\}$ be a stochastic process with stationary independent increments having values in d-dimensional Euclidean space E^d, $d \geqq 2$. Then $E\, e^{i(u, X_t)} = \exp\{-t\psi(u)\}$ where

$$\psi(u) = i(a, u) + \tfrac{1}{2}(\sigma u, u) + \int_{E^d} [1 - e^{i(u,y)} + i(u, y)/(1 + |y|^2)]\nu(dy)\,. \tag{1.1}$$

The measure ν is called the Lévy measure, and ψ is called the exponent. The vector X_t may be written

$$X_t = (X_t^1, \cdots, X_t^d)\,. \tag{1.2}$$

The process $X^k = \{X_t^k, t \geqq 0\}$ will be called the kth coordinate process of X. Of course, X^k is a real process with stationary independent increments whose exponent ψ_k is given by

$$\psi_k(u) = \psi(0, \cdots, 0, u, 0, \cdots, 0)\,, \tag{1.3}$$

the real variable u appearing in the kth place.

Assume that X is isotropic: if $\tau: E^d \to E^d$ is any rotation about 0, then τX and X have the same distribution under P^0. Define the radial process $R = \{R_t, t \geqq 0\}$ by

$$R_t = |X_t| \tag{1.4}$$

where $|\cdot|$ denotes ordinary Euclidean distance in E^d. It follows from isotropy that R is a Markov process. Finally, assume further that either

$$\nu(E^d) = \infty \quad \text{or} \quad (\sigma u, u) \not\equiv 0 \tag{1.5}$$

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so that the processes moving only by a finite number of jumps in a finite time interval are excluded from consideration. Let us say that R hits points if, for every $x > 0$ and every $y \geqq 0$,

$$P^y\{R_t = x \text{ for some } t > 0\} > 0 ;$$

and that X^1 hits points if, for every real x and y,

$$P^y\{X_t^1 = x \text{ for some } t > 0\} > 0 .$$

Here P^y is the measure for the process starting at y.

One of the main results of this paper is that, under the assumptions of the preceding paragraph, R hits points if and only if X^1 hits points (see Theorem 3.1). Since a simple necessary and sufficient criterion is now known for the latter possibility, one obtains:

$$R \text{ hits points if and only if } \int_0^\infty (\lambda + \psi_1(x))^{-1}\, dx < \infty \quad \text{for some} \quad \lambda > 0 .$$

One may then deduce that, if R hits points, then each $x > 0$ is regular for itself; i.e., if $T_x = \inf\{t > 0: R_t = x\}$, then $P^x\{T_x > 0\} = 0$. If $x > 0$, define $Z_x = \{t > 0: R_t = x\}$, and $Z = \{t > 0: X_t^1 = 0\}$. Then according to Theorem 3.2 of this paper, the random sets Z_x and Z have the same size in the sense that there is a continuous increasing function f that serves as an exact measure function for each set (see Section 3 for further explanation). In particular, these two sets have the same Hausdorff dimension. Since the set Z is relatively well understood, this result is of considerable help in understanding the nature of the sets Z_x. (See, for example, [1], [6] for various estimates of the dimension of Z; and see [9] for a summary of what is known about exact measure functions for Z.)

Blumenthal and Getoor [1] have obtained these results in case X is an isotropic stable process; their method (involving subordination) is not applicable in the more general case considered here.

Of course, the process R hits points if and only if the process X hits the spherical shells $S = S(x) = \{z \in E^d: |z| = x\}$; and X^1 hits points if and only if X hits hyperplanes $M = M(x) = \{z \in E^d: z = (x_1, \cdots, x_n), x_1 = x\}$. A small patch of S can be obtained by bending a patch of M "just a little," and so it seems likely that if X hits one patch with positive probability then it "should" hit the other also. Thus the basic problem of this paper is part of a much more general problem that may be formulated as follows. Let A be a Borel set in E^d, let X be an E^d-valued process with stationary independent increments, and let f be a mapping of E^d to E^d. Let $C^\lambda(A)$ be the λ-capacity of A (for X); see [2], [8], and Section 3 for definitions. Under mild regularity conditions, $C^\lambda(A) > 0$ if and only if X will hit A with positive probability, starting from any point in E^d. The basic problem is then: under what conditions on f, X, A does it follow that

$$C^\lambda(A) > 0 \quad \text{if and only if} \quad C^\lambda(f(A)) > 0 . \tag{1.6}$$

The methods of this paper do yield (1.6) for very smooth f, A if X is isotropic—

see the remarks in Section 3, and the main result of Section 4. Hopefully, further results in the direction of (1.6) would provide a deeper understanding of capacity theory as well as some insight into the (relatively poorly understood) sample function behavior of processes with values in E^d, $d \geqq 2$.

The organization of the paper is as follows. Section 2 contains a number of rather simple facts necessary for proving the main results, while Section 3 contains the proofs of the main results for the radial process. Section 4 contains a result bearing on the problem (1.6). Terminology and notation referring to the theory of Markov processes will be that of [2]. The process X is, as usual, assumed to be a Hunt process.

Finally, I thank Professor Harry Kesten for a number of helpful conversations in connection with this paper.

2. Preliminary facts. This section gathers together a number of facts needed for proving the main results in Sections 3 and 4. Throughout this section X will be an isotropic process with stationary independent increments having values in E^d, $d \geqq 2$, and satisfying (1.5).

If f, g, h are nonnegative universally measurable functions on E^d, $[0, \infty)$, and E^1 respectively, define for $\lambda > 0$:

$$\begin{aligned} U^\lambda f(x) &= E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt\,, && x \in E^d \\ V^\lambda g(x) &= E^x \int_0^\infty e^{-\lambda t} g(R_t)\, dt\,, && x \in [0, \infty) \\ W^\lambda h(x) &= E^x \int_0^\infty e^{-\lambda t} h(X_t^1)\, dt\,, && x \in E^1\,. \end{aligned} \tag{2.1}$$

If A is a Borel set in E^d, define $U^\lambda(x, A) = U^\lambda f(x)$ with f as the indicator function of A, and define the measures $V^\lambda(x, \cdot)$, $W^\lambda(x, \cdot)$ similarly. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product (with respect to Lebesgue measure) for real-valued functions on E^d. It follows from symmetry that for any two nonnegative universally measurable functions f, g on E^d (respectively E^1):

$$\langle U^\lambda f, g \rangle = \langle g, U^\lambda f \rangle \tag{2.2a}$$

$$\langle W^\lambda f, g \rangle = \langle g, W^\lambda f \rangle\,. \tag{2.2b}$$

Moreover, according to Zabczyk [10], the random variable X_t $(t > 0)$ has a density relative to Lebesgue measure on E^d, so that X_t^1 and R_t likewise have densities relative to Lebesgue measure on E^1. It follows from this that there exist potential kernels $u^\lambda(x, y)$ $w^\lambda(x, y)$ such that for bounded universally measurable functions f (with the appropriate domain):

$$\begin{aligned} U^\lambda f(x) &= \int_{E^d} u^\lambda(x, y) f(y)\, dy \\ u^\lambda(x, y) &= u^\lambda(|x - y|)\,, && x, y \in E^d \end{aligned} \tag{2.3a}$$

$$\begin{aligned} W^\lambda f(x) &= \int_{E^1} w^\lambda(x, y) f(y)\, dy \\ w^\lambda(x, y) &= w^\lambda(|x - y|)\,, && x, y \in E^1 \end{aligned} \tag{2.3b}$$

(2.3c) $u^\lambda(\cdot)$ and $w^\lambda(\cdot)$ are λ-excessive for X, X^1 respectively.

See [2] chapter VI for details and further properties.

The first proposition of this section establishes the same sort of thing for V^λ.

PROPOSITION 2.1. *For $x \geqq 0$, let $\xi(dx) = x^{d-1}\,dx$, and define the inner product $(f, g) = \int_0^\infty f(x)g(x)\xi(dx)$. Let $\lambda > 0$. Then there exists a kernel $v^\lambda(x, y) = v^\lambda(y, x)$, $x, y \geqq 0$, such that*

(2.4i) $$x \to v^\lambda(x, y) \quad \text{is } \lambda\text{-excessive for } R$$

(2.4ii) $$V^\lambda f(x) = \int_{[0,\infty)} v^\lambda(x, y)f(y)\xi(dy),\ f \geqq 0 \quad \text{universally measurable.}$$

PROOF. (a) Let us first verify that the measure $V^\lambda(x, \cdot)$ is absolutely continuous with respect to Lebesgue measure on $[0, \infty)$. Indeed, if $A \subset [0, \infty)$ is a set of Lebesgue measure 0, then let $A_s \subset E^d$ be defined by $A_s = \{z \in E^d : |z| \in A\}$. Since A has (linear) Lebesgue measure 0, A_s has d-dimensional Lebesgue measure 0 and $V^\lambda(x, A) = U^\lambda(x', A_s)$, where $x \in [0, \infty)$ and $x' = (x, 0, \cdots, 0)$. Here we have used the fact that X is isotropic so that $U^\lambda(y, B)$ depends only on $|y|$ whenever B is a spherically symmetric set about 0. By (2.3a), $U^\lambda(x', A_s) = 0$ if A_s has zero Lebesgue measure, and so $V^\lambda(x, y)$ is absolutely continuous relative to Lebesgue measure.

(b) Next let us verify that if f, g are nonnegative universally measurable functions on $[0, \infty)$, then

(2.5) $$(V^\lambda f, g) = (g, V^\lambda f)\,.$$

It will be enough to verify this for $f = I_A$, $g = I_B$ indicators of sets A, B. Define A_s, B_s as in part (a). Then for known constants c_d (depending only on the dimension d):

$$\begin{aligned}(V^\lambda f, g) &= \int_B V^\lambda(x, A)x^{d-1}\,dx \\ &= c_d \int_{B_s} U^\lambda(y, A_s)\,dy \\ &= c_d\langle U^\lambda I_{A_s}, I_{B_s}\rangle \\ &= c_d\langle I_{A_s}, U^\lambda I_{B_s}\rangle \qquad \text{by (2.2a)} \\ &= (f, V^\lambda g)\,.\end{aligned}$$

The proposition now follows from (a), (b) by [2], VI. 1, Theorem 1.4, page 254.

PROPOSITION 2.2. *If $\lambda > 0$ and f is a real, bounded, universally measurable function on $[0, \infty)$ having compact support, then $V^\lambda f(x)$ is a continuous function of x. In particular, every λ-excessive function is lower semicontinuous.*

PROOF. The second assertion is a well-known consequence of the first. To verify the first assertion, define for $x \in E^d$, $f^1(x) = f(|x|)$. It follows from (2.3a) that $U^\lambda f^1(x)$ is a continuous function of $x \in E^d$. Since

$$\begin{aligned}U^\lambda f^1(x) &= \int_0^\infty e^{-\lambda t}E^x f^1(X_t)\,dt \\ &= \int_0^\infty e^{-\lambda t}E^{|x|} f^1(X_t)\,dt \\ &= \int_0^\infty e^{-\lambda t}E^{|x|} f(|X_t|)\,dt \\ &= V^\lambda f(|x|)\,,\end{aligned}$$

the result follows.

This completes the collection of facts needed for carrying through the arguments of Section 3. For the amusement of the reader we conclude this section with a simple proposition that complements Corollary 3.2.

PROPOSITION 2.3. *Let X be an isotropic process in E^d, $d \geqq 2$, with $\nu(E^d) = \infty$. Let $R = \{R_t\}$ be the radial process. If $z > 0$, then (relative to R) z is regular for both (z, ∞) and $[0, z)$.*

Recall that if $T_A = \inf\{t > 0 : R_t \in A\}$, then z regular for A means $P^z\{T_A > 0\} = 0$. The proposition therefore states that, starting from $z > 0$, R will hit $[0, z)$ and (z, ∞) "immediately."

PROOF. From Proposition 2.1 it follows that R cannot remain at z for a positive initial time interval, so that, starting from z, it must with positive probability hit one of the sets (z, ∞), $[0, z)$ immediately. By the Blumenthal zero-one law, R will then hit one of these sets immediately with probability 1.

Suppose that R hits $(0, z)$ immediately with probability 1. Then by isotropy, X starting from $z' = (z, 0, \cdots, 0)$ must hit the sphere $B_z = \{x \in E^d : |x| < z\}$ immediately with probability 1. Again by isotropy X will then hit B_z immediately starting from $-z'$. But by translation invariance, X starting from z' will have to hit $B_z + 2z'$ immediately. *A fortiori*, X starting from z' hits $E^d - B_z$ immediately; i.e., R starting from z hits (z, ∞) immediately with probability 1.

On the other hand, suppose that R starting from z hits (z, ∞) immediately with probability 1. For ease of exposition, suppose $d = 2$. Then since X is right continuous, X starting from $(z, 0) \in E^2$ must hit $S_1 \cup S_2 \cup S_3$ immediately, where S_i, $i = 1, 2, 3$ are open spheres of radius z centered at (z, z), $(2z, 0)$, $(z, -z)$ respectively. Hence by the Blumenthal zero-one law, X must hit at least one of the S_i immediately with probability one. Suppose S_1 is hit immediately from the point $(z, 0)$ on its boundary. By isotropy, X then enters B_z immediately from $(z, 0)$; i.e., R starting from z hits $[0, z)$ immediately with probability 1.

3. **Main results.** In this section we obtain a useful criterion for deciding precisely when the radial process $R_t = |X_t|$ hits points (equivalently, the isotropic process X hits spherical shells.) The ideas leading to the solution of this problem are then applied to show that the sets $\{t : R_t = z\}$ and $\{t : X_t^1 = 0\}$ have essentially the "same size." Unless explicitly stated to the contrary, X is assumed to be an isotropic E^d-valued ($d \geqq 2$) process satisfying (1.5).

We begin by establishing a preliminary criterion for deciding when X hits spherical shells. A main tool for this first lemma will be capacity theory as developed by Port and Stone [8]. If B is a Borel set in E^d, let $T_B = \inf\{t > 0 : X_t \in B\}$. If $\lambda > 0$, let $\mu_B{}^\lambda$ be the unique measure supported on $\bar{B}$ (= closure of B) whose λ-potential has the density (relative to Lebesgue measure on E^d) $E^{-x}e^{-\lambda T - B}$. This measure always exists and

$$C^\lambda(B) = \mu_B{}^\lambda(\bar{B}) \tag{3.1}$$

is the λ-capacity of B ([8] Theorem 6.1). Since X_t has a density for each t,

(3.2) $$C^\lambda(B) > 0 \quad \text{if and only if} \quad P^x\{T_B < \infty\} > 0 \quad \text{for every} \quad x \in E^d .$$

Finally, as an elementary consequence of the definition of $\mu_B{}^\lambda$ and the first passage relation we have the following formula ([8] Equation 6.1):

(3.3) $$\int_{\bar{B}} \mu_B{}^\lambda(dz) U^\lambda(z, A) = \int_A E^{-x} e^{-\lambda T_{-B}}\, dx = \int_A E^x e^{-\lambda T_B}\, dx$$

where A, B, are Borel sets in E^d, and the second equality comes from isotropy.

Let $a > 0$ be fixed. Define

(3.4) $$S = S(a) = \{z \in E^d : |z| = a\}$$
$$S_h = S_h(a) = \{z \in E^d : |z - S| < h\} .$$

Then we have the following criteria for deciding when X hits the spherical shell S.

LEMMA 3.1. *Let $a > 0$, and $a' = (a, 0, \cdots, 0) \in E^d$.*

(a) *If $\liminf_{h \downarrow 0} h^{-1} U^\lambda(a', S_h) < \infty$, then $C^\lambda(S) > 0$.*

(b) *If $\limsup_{h \downarrow 0} h^{-1} U^\lambda(a', S_h) = \infty$, then $C^\lambda(S) = 0$.*

PROOF. In (3.3) choose $A = S_h$, $B = S_{2h}$. Obviously $A \subset B$ and if $x \in A$, $P^x\{T_B = 0\} = 1$. Thus the right side of (3.3) yields

(3.5) $$\int_{S_h} E^x e^{-\lambda T_{S_{2h}}}\, dx = \int_{S_h} dx \sim \text{const.} \times h \qquad \text{as } h \downarrow 0$$

(where the positive constant depends on a and d). From the left side of (3.3)

(3.6) $$\int_{S_h} E^x e^{-\lambda T_{S_{2h}}}\, dx = \int_{\bar{S}_{2h}} \mu_{S_{2h}}^\lambda(dz) U^\lambda(z, S_h)$$
$$\leqq \sup_{z \in \bar{S}_{2h}} U^\lambda(z, S_h) C^\lambda(S_{2h}) .$$

Hence, combining (3.5), (3.6),

(3.7) $$C^\lambda(S_{2h}) \geqq \text{const.}\, h[\sup_{z \in \bar{S}_{2h}} U^\lambda(z, S_{2h})]^{-1} .$$

But using the isotropy,

$$\sup_{z \in S_{2h}} U^\lambda(z, S_{2h}) = \sup_{a-2h \leqq z \leqq a+2h} U^\lambda(z', S_{2h}) , \qquad z' = (z, 0, \cdots, 0) \in E^d$$
$$= \sup_{a-2h \leqq z \leqq a+2h} U^\lambda(0, S_{2h} - z') .$$

However, if $a - 2h \leqq z \leqq a + 2h$, then

$$S_{2h} - z' \subset S_{4h} - a' ,$$

whence

(3.8) $$\sup_{z \in \bar{S}_{2h}} U^\lambda(z, S_{2h}) \leqq U^\lambda(0, S_{4h} - a') = U^\lambda(a', S_{4h}) .$$

Since $C^\lambda(\cdot)$ is a Choquet capacity ([8] Section 6)

(3.9) $$C^\lambda(S_{2h}) \downarrow C^\lambda(S) \qquad \text{as } h \downarrow 0 .$$

From (3.7), (3.8), (3.9) it then follows that $C^\lambda(S) > 0$ provided that

$$\limsup_{h \downarrow 0} h[U^\lambda(a', S_{4h})]^{-1} > 0$$

and this implies the conclusion (a).

In an entirely analogous way one obtains conclusion (b). Merely choose $B = S_h$, $A = S_{2h}$ in (3.3) to obtain

$$\int_{S_{2h}} E^x e^{-\lambda T_{S_h}} dx \leqq \int_{S_{2h}} dx \sim \text{const. } h \quad \text{as } h \downarrow 0 \tag{3.10}$$

and

$$\int_{\bar{S}_h} \mu^\lambda_{S_h}(dz) U^\lambda(z, S_{2h}) \geqq \inf_{z \in \bar{S}_h} U^\lambda(z, S_{2h}) C^\lambda(S_h) , \tag{3.11}$$

implying that

$$C^\lambda(S_h) \leqq \text{const. } h[\inf_{z \in \bar{S}_h} U^\lambda(z, S_{2h})]^{-1} . \tag{3.12}$$

If z is real and $|z - a| \leqq h$, then $S_{2h} - z' \supset S_h - a'$, so that as before

$$\begin{aligned}\inf_{z \in \bar{S}_h} U^\lambda(z, S_{2h}) &= \inf_{a-h \leqq z \leqq a+h} U^\lambda(0, S_{2h} - z') \\ &\geqq U^\lambda(0, S_h - a')\end{aligned}$$

completing the proof.

The reader no doubt will have noticed that the proof above can be modified slightly to give criteria for other nice surfaces in E^d. We will need the following additional result. The proof is omitted; using the proof of Lemma 3.1 as a model, the reader will have no difficulty in supplying the proof himself. For further extensions, see the remarks after the proof of Theorem 3.1.

Let $a > 0$, and let $M^a = \{x = (x_1, \cdots, x_d) \in E^d : x_1 = 0, x_2^2 + \cdots + x_d^2 < a^2\}$, and $M_h^a = \{x \in E^d : |x - M^a| < h\}$.

LEMMA 3.2. (a) *If* $\liminf_{h \downarrow 0} h^{-1} U^\lambda(0, M_h^a) < \infty$, *then* $C^\lambda(M^a) > 0$.
(b) *If* $\limsup_{h \downarrow 0} h^{-1} U^\lambda(0, M_h^a) = \infty$, *then* $C^\lambda(M^a) = 0$.

It is easily checked that if $C^\lambda(M^a) > 0$ for one point a, then $C^\lambda(M^a) > 0$ for all $a > 0$. Let $M = \{x \in E^d : x_1 = 0\}$. One also easily checks that $C^\lambda(M) > 0$ if and only if $C^\lambda(M^a) > 0$ for some a. Finally, writing the process X_t as $X_t = (X_t^1, \cdots, X_t^d)$ one sees that $C^\lambda(M) > 0$ if and only if the one-dimensional process $X^1 = \{X_t^1, t \geqq 0\}$ hits points. In particular,

COROLLARY 3.1. *X^1 hits points if and only if* $\lim_{h \downarrow 0} h^{-1} U^\lambda(0, M_h^a) < \infty$ *for some (hence all)* $a > 0$.

The idea of proving the next lemma by comparing arcs was suggested by Professor Harry Kesten.

LEMMA 3.3. *Fix $q \in E^d$, $|q| = a > 0$. Then there are positive constants c, C such that for all sufficiently small h*

$$cU^\lambda(0, M_h^a) \leqq U^\lambda(q, S_h(a)) \leqq CU^\lambda(0, M_{4h}^{2a}) .$$

PROOF. First consider the two-dimensional case, and let us begin here with the left-hand inequality. By isotropy, we may assume $q = (a, 0)$; it will then be enough to show

$$cU^\lambda(0, M_h^a) \leqq U^\lambda(0, S_h(a) - q) .$$

Elementary plane geometry shows (by comparing arc lengths) that if (r, θ) are the polar co-ordinates of a point in E^2, then for fixed r

$$(3.13) \qquad (2/\pi)\mu\{\theta : (r, \theta) \in M_h^a\} \leqq \mu\{\theta : (r, \theta) \in S_h(a) - q\}$$

where μ is Lebesgue measure on the line. Since X_t is isotropic, it has a density f_t on E^2 depending only on r (see Section 2), so that in terms of polar co-ordinates, for any Borel set $A \subset E^2$:

$$(3.14) \qquad P^\circ\{X_t \in A\} = \int_{\{(r,\theta)\in A\}} d\theta f_t(r) r\, dr .$$

Using (3.13) and (3.14), we see by integrating on θ first, then r, that

$$(3.15) \qquad (2/\pi)P^\circ\{X_t \in M_h^a\} \leqq P^\circ\{X_t \in S_h(a) - q\} .$$

The left-hand inequality of the lemma now follows (for E^2) from (3.16) since

$$U^\lambda(0, A) = \int_0^\infty e^{-\lambda t} P^\circ\{X_t \in A\}\, dt .$$

The proof of the right-hand inequality is slightly more involved. We will show, assuming $q = (a, 0)$

$$CU^\lambda(0, M_{4h}^{2a}) \geqq U^\lambda(0, S_h(a) - q) .$$

Let $S_h^1 = \{S_h(a) - q\} \cap \{(x, y) \in E^2 : x \geqq -a\}$ be the right half of the annulus $S_h(a) - q$, and $S_h^2 = \{S_h(a) - q\} \cap \{(x, y) \in E^2 : x \leqq -a, y \geqq 0\}$, $S_h^3 = \{S_h(a) - q\} \cap \{(x, y) \in E^2 : x \leqq -a, y < 0\}$ be the northwest and southwest sectors respectively. If $0 \leqq r \leqq |a + h|2^{\frac{1}{2}}$ (so as θ ranges over $[0, 2\pi)$ and r over the range indicated, (r, θ) will just barely cover S_h^1), then some analytic geometry comparing arc lengths reveals that there is a constant C_1 such that for sufficiently small h, and r in the above range,

$$\mu\{\theta : (r, \theta) \in S_h^1\} \leqq C_1 \mu\{\theta : (r, \theta) \in M_h^{2a}\} .$$

It follows as in the first part of the proof that

$$U^\lambda(0, S_h^1) \leqq C_1 U^\lambda(0, M_h^{2a}) .$$

Next, let $T = \inf\{t > 0 : X_t \in S_h^2\}$. Then

$$U^\lambda(0, S_h^2) = E^\circ\{e^{-T\lambda} U^\lambda(X_T, S_h^2)\} .$$

But using isotropy and translation, if $x \in S_h^2$ then (rotating S_h^2 around to the eastern side of the annulus and then sliding it over so that the point x is placed at 0):

$$U^\lambda(x, S_h^2) \leqq U^\lambda(0, S_{4h}^1)$$

so that

$$U^\lambda(0, S_h^2) \leqq C_1 U^\lambda(0, M_{4h}^{2a}) .$$

Hence,

$$\begin{aligned} U^\lambda(0, S_h(a) - q) &= U^\lambda(0, S_h^1) + U^\lambda(0, S_h^2) + U^\lambda(0, S_h^3) \\ &\leqq 3C_1 U^\lambda(0, M_{4h}^{2a}) , \end{aligned}$$

as desired.

Finally, the d-dimensional case can be treated as follows. As in the two dimensional case, we may assume $q = (a, 0, \cdots, 0)$ and must compare $U^\lambda(0, M_h^a)$ to $U^\lambda(0, S_h(a) - q)$. Introduce cylindrical co-ordinates as follows: the point $(x_1, \cdots, x_d)$ has the representation $(x_1, \rho\varphi_1, \cdots, \rho\varphi_{d-1})$ with $\sum \varphi_i^2 = 1$, $\rho \geqq 0$. The density of X_t then depends only on $x_1^2 + \rho^2$, and because the sets M_h^a, $S_h(a) - q$ are symmetric about the x_1 axis, the calculation of $P^\circ\{X_t \in M_h^a\}$, $P^\circ\{X_t \in S_h(a) - q\}$ will involve only x_1 and ρ (the φ_i's will be integrated out.) The calculation of these probabilities is thereby reduced to essentially the problem treated in the two dimensional case. Since the reader should now have no difficulty carrying out the details here, this completes the proof.

The next theorem is one of the main results of this section.

THEOREM 3.1. *Let $X_t = (X_t^1, \cdots, X_t^d)$ be an isotropic process in E^d, $d \geqq 2$, with stationary independent increments satisfying* (1.5). *Let $R_t = |X_t|$ be the radial process. Define ψ_1 as in* (1.3). *Then the following statements are equivalent*:

(a) *For every $x \geqq 0$ and every $y > 0$, $P^x\{R_t = y \text{ for some } t\} > 0$.*
(b) *For every real x, y, $P^x\{X_t^1 = y \text{ for some } t\} > 0$.*
(c) *$\int_0^\infty [\lambda + \psi_1(r)]^{-1}\, dr < \infty$ for some $\lambda > 0$.*

Briefly, the radial process R hits points if and only if the co-ordinate process X^1 hits points.

PROOF. Since $\psi_1(r)$ is the exponent of X^1, the equivalence of (b) and (c) is due to Kesten [5]. The radial process will hit points if and only if spherical shells have positive capacity for X. The theorem now follows from Lemmas 3.1, 3.3 and Corollary 3.1.

REMARK. The method of proof in the preceding theorem can be used to establish somewhat more general results. Let X be isotropic and let B be a nice smooth surface in E^d, $d \geqq 2$. Then with a few alterations, the preceding method yields: X hits B if and only if X^1 hits points. Here "nice smooth" means that if $B_h = \{x: |x - B| < h\}$ then the volume of B_h is $\sim$ const. h as $h \downarrow 0$ (so that an analogue of Lemma 3.1 will hold) and that B not be so convoluted that the arcwise comparison of Lemma 3.3 cannot be carried out. Of course, in order to prove B is hit, one need consider only some small patch of B where these smoothness "conditions" are satisfied. See Section 4 for a more precise treatment of this general problem by different methods. Turning to more general processes in E^d, one finds the following analogue of the criteria of Lemma 3.1 for hitting the spherical shell $S(a)$:

(a) If $\liminf_{h \downarrow 0} h^{-1}\{\sup_{\{z:|z|=a\}} U^\lambda(z, S_h(a))\} < \infty$, then $C^\lambda(S(a)) > 0$.
(b) If $\limsup_{h \downarrow 0} h^{-1}\{\inf_{\{z:|z|=a\}} U^\lambda(z, S_h(a))\} = \infty$, then $C^\lambda(S(a)) = 0$.

COROLLARY 3.2. *If* (a), (b), *or* (c) *of Theorem* 3.1. *holds, then every $z > 0$ is regular for itself* (*for R*).

PROOF. If z were not regular for itself, then z would be a thin, hence semi-polar, set. Since $v^\lambda(x, y) = v^\lambda(y, x)$ (Proposition 2.1), all semi-polar sets are polar ([2] VI, Proposition 4.10, page 289: this result is applicable here by virtue of the work in Section 2). This contradicts the hypothesis that singletons in $(0, \infty)$ are not polar.

COROLLARY 3.3. *If* (a), (b) *or* (c) *of Theorem* 3.1 *holds and if* $z > 0$, *then* $v(z, \cdot)$ *is continuous at* z.

PROOF. Let $T_y = \inf\{t > 0 : R_t = y\}$. According to [2], II. 2, $E^z e^{-\lambda T_y}$ is λ-excessive as a function of z, so $E^{\cdot} e^{-\lambda T_y}$ is lower semi-continuous (Proposition 2.2). Hence $E^{\cdot} e^{-\lambda T_y}$ is continuous at $y > 0$; for if not,

$$1 \geqq \limsup_{z \to y} E^z e^{-\lambda T_y} > \liminf_{z \to y} E^z e^{-\lambda T_y} \geqq E^y e^{-\lambda T_y} ,$$

implying that y is not regular for itself, a contradiction. From capacity theory (see [2] VI. 4)

$$E^z e^{-\lambda T_y} = C^\lambda(\{y\}) v^\lambda(z, y)$$

where $C^\lambda(A)$ is the λ-capacity of the set A. Hence from the preceding argument $v^\lambda(\cdot, y)$ is continuous at $y > 0$. Since $v^\lambda(x, y) = v^\lambda(y, x)$, $v^\lambda(z, \cdot)$ is continuous at $z > 0$.

Our final project in this section is to give a useful estimate of the size of the set $\{t : R_t = z\}$. In order to do this, we first sharpen Lemma 3.3.

LEMMA 3.4. *Fix* $z > 0$ *and suppose* R *hits points. Then there are positive constants* c, C *such that for all sufficiently large* λ,

$$cw^\lambda(0, 0) \leqq v^\lambda(z, z) \leqq Cw^\lambda(0, 0) \tag{3.16}$$

where $w^\lambda(x, y) = w^\lambda(|y - x|)$ *is the potential kernel for the co-ordinate process* X^1 *of* $(X^1, \cdots, X^d)$.

PROOF. According to Corollary 3.3, $v^\lambda(z, \cdot)$ is a continuous function, and since the symmetric process X^1 hits points, $w^\lambda(\cdot)$ is also continuous (see Bretagnolle [3]). Let $M_h = \{x \in E^d : x = (x_1, \cdots, x_d),\ -h < x_1 < h\}$. Then from Lemma 3.3

$$U^\lambda(z', S_h(z)) \leqq C_1 U^\lambda(0, M_{4h}) , \qquad z' = (z, 0, \cdots, 0)$$

so that

$$\int_{z-h}^{z+h} v^\lambda(z, y) y^{d-1}\, dy \leqq C_2 \int_{-4h}^{4h} w^\lambda(y)\, dy .$$

The continuity of the integrands at z then implies that

$$v^\lambda(z, z) z^{d-1} \leqq 4C_2 w^\lambda(0) ,$$

establishing the right-hand side of (3.16) (with C depending on z).

To obtain the left-hand side, let $M_h^z = \{x \in E^d : -h < x_1 < h,\ x_2^2 + \cdots + x_d^2 < z^2\}$ and $N_h^z = M_h - M_h^z$. Then

$$U^\lambda(0, M_h) = U^\lambda(0, M_h^z) + U^\lambda(0, N_h^z) . \tag{3.17}$$

Let T_λ be a stopping time independent of X, having an exponential distribution with density $\lambda e^{-\lambda t}$, $t \geqq 0$. Then since $U^\lambda(0, N_h^z)$ is the expected time in N_h^z

before T_λ, we have, if $T^h = \inf\{t > 0: X_t \in N_h^z\}$, $T^\circ = \inf\{t > 0: X_t \in N_0^z\}$ $N_0^z = \bigcap_h N_h^z$:

$$U^\lambda(0, N_h^z) \leqq P^\circ\{T^h < T_\lambda\} U^\lambda(0, M_{2h}) . \tag{3.18}$$

Also, by Lemma 3.3,

$$U^\lambda(0, M_h^z) \leqq c_1 U^\lambda(z', S_h(z)) , \qquad z' = (z, 0, \cdots, 0) . \tag{3.19}$$

Combining (3.17)–(3.19), we have

$$U^\lambda(0, M_h) \leqq c_1 U^\lambda(z', S_h(z)) + P^\circ\{T^h < T_\lambda\} U^\lambda(0, M_{2h}) ; \tag{3.20}$$

that is,

$$\int_{-h}^{h} w^\lambda(y)\, dy \leqq c_2 \int_{-h}^{h} v^\lambda(z, y) y^{d-1}\, dy + P^\circ\{T^h < T_\lambda\} \int_{-2h}^{2h} w^\lambda(y)\, dy . \tag{3.21}$$

Since $w^\lambda(\cdot)$ is continuous, and $v^\lambda(z, \cdot)$ is continuous at z, this implies

$$w^\lambda(0) \leqq c_2 v^\lambda(z, z) + 2P^\circ\{T^\circ \leqq T_\lambda\} w^\lambda(0) . \tag{3.22}$$

Lemma 3.3 now follows from (3.22), since $P^\circ\{T^\circ > 0) = 1$ and $T_\lambda \to 0$ in probability as $\lambda \to \infty$.

Before stating the final theorem of this section, it will be convenient to present further background. Let $T = \{T_t, t \geqq 0\}$ be a subordinator; i.e., a process with stationary independent increments and increasing paths. Then

$$Ee^{-\lambda T_t} = \exp\{-tg(\lambda)\}$$

where g is called the exponent of T. Let η be the inverse of g (near infinity), let

$$h_\gamma(t) = \log|\log t| / \eta(\gamma t^{-1} \log|\log t|) , \qquad \gamma > 0$$

and let f_γ be the inverse of h_γ. Then Fristedt and Pruitt [4] have shown that for any $\gamma > 0$, f is an exact Hausdorff measure function for the range of T. (This means that the Hausdorff f-measure of Range T is positive and finite a.s.) Let $x > 0$, and

$$Z_x = \{t > 0: R_t = x\} , \qquad Z = \{t > 0: X_t^1 = 0\} . \tag{3.23}$$

If x is regular for $\{x\}$ (for R), then it is a known consequence of the theory of local time (see [1], [2]) that Z_x may be regarded as the range of a subordinator $\{T_R(t), t \geqq 0\}$ (possibly truncated at a positive stopping time), where the exponent g_R of T_R is

$$g_R(\lambda) = [v^\lambda(x, x)]^{-1} ;$$

and similarly Z is the range of a subordinator T_Z whose exponent is

$$g_Z(\lambda) = [w^\lambda(0)]^{-1} .$$

Finally, note that the measure function f_γ is obtained from $g(\lambda)$ by applying monotone functions to g for large values of λ. From the foregoing discussion and Lemma 3.4, an easy argument then gives the following theorem which implies in particular that dim $Z =$ dim Z_x.

THEOREM 3.2. *Let $X_t = (X_t^1, \cdots, X_t^d)$ be an isotropic process in E^d, $d \geqq 2$, with stationary independent increments. Let $R = \{R_t, t \geqq 0\}$ be the radial process, and assume $x > 0$ is regular (for R). Define Z_x, Z by (3.23). Then there exists a monotone continuous function f which is an exact measure function for Z_x and for Z.*

REMARK. Theorem 3.2 of course gives information on the dimension (and on the exact measure function) for the set $\{t > 0 : X_t \in S(x)\}$. It seems likely that the same measure function would work for the set $\{t > 0 : X_t \in C\}$ where C is any nice smooth surface in E^d, but I have no proof.

4. A generalization of Theorem 3.1. According to the results of Section 3, if X_1 hits points then X hits spherical shells. The purpose of this section is to show that indeed under this hypothesis X will hit any reasonably smooth surface. To make this precise, let $f: [0, 1]^{d-1} \to E^d$ be a surface in E^d. Assume f is smooth in the sense that if

$$f(t_1, \cdots, t_{d-1}) = (f_1(t_1, \cdots, t_{d-1}), \cdots, f_d(t_1, \cdots, t_{d-1}))$$

then in a neighborhood of the origin, $f_1, \cdots, f_d$ all have continuous second partial derivatives and

$$\sum_{i=1}^d [(\partial/\partial t_j) f_i(t_1, \cdots, t_{d-1})|_{(t_1,\cdots,t_{d-1})=0}]^2 > 0$$

for each $j = 1, 2, \cdots, d-1$. Then the following theorem is the main result of this section.

THEOREM 4.1. *Let X be an isotropic process in E^d satisfying (1.5). Let f be a smooth surface in E^d as described above, and let $A = \{f(t_1, \cdots, t_{d-1}) : 0 \leqq t_i \leqq 1, 1 \leqq i < d\}$. If X^1 hits points, then $C^\lambda(A) > 0$.*

REMARKS. If A is part of a spherical shell, then one can use the method of proof below to establish the converse, so by this method one of the results of Section 3 is recovered. While the proof below does not yield the useful double inequality of Lemma 3.3, it nevertheless yields directly a result valid for a much greater variety of surfaces.

PROOF. For ease of exposition we will assume $d = 2$; the case $d > 2$ is proved in essentially the same way. If X^1 hits points, then with positive probability X hits the line $L = \{(t, s) \in E^2 : 0 \leqq t \leqq 1, s = 0\}$. As in Section 2, $u^\lambda(x, y) = u^\lambda(x - y) = u^\lambda(|x - y|)$ is the λ-potential kernel for X; if $x = (t, 0) \in E^2$ and $y = (s, 0) \in E^2$, then we will abuse notation further and write $u^\lambda(x, y) = u^\lambda(t - s) = u^\lambda(|t - s|)$.

Let us verify that

$$\int_{s=0}^1 \int_{t=0}^1 u^\lambda(|t - s|)\, dt\, ds < \infty . \tag{4.1}$$

Since X hits L, there is a finite measure μ carried by L such that

$$\int_L u^\lambda(x, y)\mu(dy) \leqq 1 \qquad \text{for all } x \in E^2$$

(see [2] VI. 4.3, page 285). Hence, letting $x = (t, 0) \in E^2$,

$$\int_{t=0}^{2} \int_{L} u^{\lambda}((t, 0), y)\mu(dy)\, dt = \int_{t=0}^{2} \int_{s=0}^{1} u^{\lambda}((t, 0), (s, 0))\mu(ds)\, dt \leqq 2 .$$

Hence there exists $0 \leqq s_0 \leqq 1$ such that

$$\infty > \int_{t=0}^{2} u^{\lambda}(t - s_0)\, dt \geqq \int_{-s_0}^{2-s_0} u^{\lambda}(t)\, dt \geqq \int_0^1 u^{\lambda}(t)\, dt .$$

Therefore,

$$\int_0^1 \int_0^1 u^{\lambda}(t - s)\, dt\, ds = \int_0^1 \int_{-s}^{1-s} u^{\lambda}(t)\, dt\, ds$$
$$\leqq 2 \int_0^1 \int_0^1 u^{\lambda}(t)\, dt\, ds < \infty$$

proving (4.1).

Since X is isotropic, for every finite measure μ:

$$\sup_x \int u^{\lambda}(x, y)\mu(dy) = \sup_{x \in \text{support}\,\mu} \int u^{\lambda}(x, y)\mu(dy)$$

(see [2] VI, 1.26). By a lemma of Orey ([7], Lemma 1.1), in order to show that a set $A \subset E^d$ has positive λ-capacity it is sufficient to produce a finite measure μ with support in A such that

$$\int_A \int_A u^{\lambda}(x, y)\mu(dx)\mu(dy) < \infty .$$

We will do this for the set A of the hypothesis by showing that for ε small

$$\int_0^{\varepsilon} \int_0^{\varepsilon} u^{\lambda}[(f_1(t), f_2(t)), (f_1(s), f_2(s))]\, dt\, ds < \infty .$$

Since for $x, y \in E^2$, $u^{\lambda}(x, y) = u^{\lambda}(|x - y|)$, the integrand may be written $u^{\lambda}[|t - s| H(t, s)]$ where

$$H(t, s)^2 = F(t, s)^2 + G(t, s)^2$$
$$F(t, s) = f_1(t) - f_1(s)/(t - s)$$
$$G(t, s) = f_2(t) - f_2(s)/(t - s) .$$

Make the change of variable $y = tH(t, s)$, $z = sH(t, s)$. The regularity of (f_1, f_2) yields $dy\, dz = B(s, t)\, dt\, ds$ where for s and t near zero, $B(s, t)$ is bounded away from 0 (and bounded above). Hence after this change of variable, if ε is small

$$\int_0^{\varepsilon} \int_0^{\varepsilon} u^{\lambda}[(f_1(t), f_2(t)), (f_1(s), f_2(s))]\, dt\, ds$$
$$\leqq \text{const.} \int_0^1 \int_0^1 u^{\lambda}(|y - z|)\, dy\, dz < \infty ,$$

finishing the proof.

REFERENCES

[1] Blumenthal, R. M. and Getoor, R. K. (1964). Local times for Markov processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.* **3** 50–74.

[2] Blumenthal, R. M. and Getoor, R. K. (1968). *Markov Processes and Potential Theory.* Academic Press, New York.

[3] Bretagnolle, J. (1969). Résultats de Kesten sur les processus à accroisements indépendantes. Séminaire de Probabilités V, 21–36. Springer Lecture Notes in Mathematics **191**.

[4] Fristedt, B. and Pruitt, W. (1971). Lower functions for increasing random walks and subordinators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.* **18** 167–82.

[5] Kesten, H. (1969). Hitting probabilities of single points for processes with stationary independent increments. *Memoir* 93, American Mathematical Society.

[6] MILLAR, P. W. (1972). Some remarks on asymmetric processes. *Ann. Math. Statist.* **43** 597-601.

[7] OREY, S. (1967). Polar sets for processes with stationary independent increments, in *Markov Processes and Potential Theory*. Wiley, New York, 117-126.

[8] PORT, S. C. and Stone, C. J. (1971). Infinitely divisible processes and their potential theory, I. *Ann. Inst. Fourier* **21** 167-275.

[9] TAYLOR, S. J. (1972). Sample path properties of processes with stationary independent increments. To appear in *Stochastic Geometry and Analysis*. Wiley, London.

[10] ZABCZYK, J. (1970). Sur la theorie semiclassique du potential pour les processus a accroisements independantes. *Studia Math.* **35** 227-247.

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