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ON THE LOCAL BEHAVIOR OF CHARACTERISTIC FUNCTIONS[1]

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Some theorems are obtained relating the asymptotic behavior of a distribution function with the local behavior of its characteristic function.

1. Introduction and Summary. We first introduce some notation. Let $F(x)$ be a distribution function with characteristic function $f(t)$. Let $\rho_k(t)$ be the kth symmetric difference of f at 0, i.e. $\rho_k(t) = \sum_{k=0}^{n} (-1)^n \binom{k}{n} f[(k-2n)t]$, and let $D_k f(0)$ denote the kth symmetric derivative of f at 0, i.e. $\lim_{t\to+0} \rho_k(t)/(2t)^k$ if this limit exists. If $f(t) = 1 + \sum_{j=1}^{k} c_j t^j/j! + o(t^k)$ as $t \to 0$ then the number c_k is called the kth generalized derivative of f at 0, denoted by $f_{(k)}(0)$. The characteristic function $f(t)$ is said to satisfy the smoothness condition S_k at 0 if $\rho_{k+1}(t) = o(t^k)$ as $t \to +0$.

The primary purpose of this paper is to give proofs of the following three theorems:

THEOREM 1. *Let k be a positive even integer and let $0 < \lambda < k$. Then $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to \infty$ if and only if $\rho_k(t) = o(t^\lambda)$ as $t \to +0$. This statement remains true if o is replaced by O.*

THEOREM 2. *Let k be a positive even integer and let $0<\lambda<k$. Then $\int_{-\infty}^{\infty} |x|^\lambda \, dF(x) < \infty$ if and only if $\int_0^\varepsilon t^{-\lambda-1}|\rho_k(t)| \, dt < \infty$ for some $\varepsilon > 0$ in which case*

(1) $$\int_{-\infty}^{\infty} |x|^\lambda \, dF(x) = [2^k \int_0^\infty v^{-\lambda-1}(\sin v)^k \, dv]^{-1} \int_0^\infty t^{-\lambda-1}|\rho_k(t)| \, dt .$$

THEOREM 3. *Let k be a positive integer and let $k < \lambda < k+1$. The characteristic function $f(t)$ admits the expansion*

(2) $$f(t) = 1 + \sum_{j=1}^{k} c_j t^j/j! + o(|t|^\lambda) \qquad \text{as } t \to 0$$

if and only if $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to \infty$. This statement remains true if o is replaced by O.

Several well-known results can be combined to yield the following theorem:

THEOREM A. *If k is a positive even integer, the following four statements are equivalent:*

(3) $$f^{(k)}(0) \quad \text{exists},$$

(4) $$f_{(k)}(0) \quad \text{exists},$$

(5) $$D_k f(0) \quad \text{exists},$$

(6) $$\int_{-\infty}^{\infty} x^k \, dF(x) < \infty .$$

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Proofs of the equivalence of (3), (4), and (6) can be found in [6]. The equivalence of (5) and (6) can be shown quite easily using Fatou's Lemma and the Lebesgue Dominated Convergence Theorem. If one of the conditions (3)—(6) holds then

$$f^{(k)}(0) = f_{(k)}(0) = D_k f(0) = i^k \int_{-\infty}^{\infty} x^k \, dF(x) \, .$$

Theorem 1 can be combined with theorems of Zygmund and Pitman to yield an analogue to Theorem A.

THEOREM B. *If k is a positive odd integer, the following four statements are equivalent*:

(7) $$f^{(k)}(0) \quad \textit{exists} \, ,$$

(8) $$f_{(k)}(0) \quad \textit{exists} \, ,$$

(9) $$D_k f(0) \quad \textit{exists and} \quad f(t) \quad \textit{satisfies condition} \quad S_k \quad \textit{at} \quad 0 \, ,$$

(10) $$\lim_{T\to\infty} \int_{-T}^{T} x^k \, dF(x) \quad \textit{exists and} \quad 1 - F(x) + F(-x) = o(x^{-k}) \quad \textit{as} \quad x \to \infty \, .$$

It is easily seen that (7) implies (8). If (8) holds, then $\rho_k(t) = c_k(2t)^k + o(t^k)$ as $t \to +0$ and $\rho_{k+1}(t) = o(t^k)$ as $t \to +0$ and so (9) holds. If (9) holds then it follows from Theorem 2 of Zygmund [15] that $\lim_{T\to\infty} \int_{-T}^{T} x^k \, dF(x)$ exists and from Theorem 1 that $1 - F(x) + F(-x) = o(x^{-k})$ as $x \to \infty$. If (10) holds then (7) follows from a theorem of Pitman [7]. If one of the conditions (7)—(10) holds then $f^{(k)}(0) = f_{(k)}(0) = D_k f(0) = i^k \lim_{T\to\infty} \int_{-T}^{T} x^k \, dF(x)$.

Let $F(x)$, $F_1(x)$, and $F_2(x)$ be distribution functions with characteristic functions $f(t)$, $f_1(t)$ and $f_2(t)$ respectively. Let $F = F_1 * F_2$ and let $\lambda > 0$. It has been shown in ([13] Theorem 1) that $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to \infty$ if and only if $1 - F_j(x) + F_j(-x) = o(x^{-\lambda})$ as $x \to \infty$ for $j = 1$ and $j = 2$. Wintner proved ([12] page 47) that $\int_{-\infty}^{\infty} |x|^\lambda \, dF(x) < \infty$ if and only if $\int_{-\infty}^{\infty} |x|^\lambda \, dF_j(x) < \infty$ for $j = 1$ and $j = 2$. Boas proved ([2] Theorem 1) that $f(t) \in \mathrm{Lip}\, \alpha$ for $0 < \alpha < 1$ if and only if $1 - F(x) + F(-x) = o(x^{-\alpha})$ as $x \to \infty$. The following corollaries follow from Theorems 1 and 2 and the above remarks:

COROLLARY 1. *Let $f(t)$, $f_1(t)$, and $f_2(t)$ be characteristic functions such that $f(t) = f_1(t)f_2(t)$. Let k be a positive even integer and let $0 < \lambda < k$. Then $\Delta_k{}^t f(0) = o(t^\lambda)$ as $t \to +0$ if and only if $\Delta_k{}^t f_j(0) = o(t^\lambda)$ as $t \to +0$ for $j = 1$ and $j = 2$. Also $\int_0^\varepsilon t^{-\lambda-1} |\Delta_k{}^t f(0)| \, dt < \infty$ if and only if $\int_0^\varepsilon t^{-\lambda-1} |\Delta_k{}^t f_j(0)| \, dt < \infty$ for $j = 1$ and $j = 2$.*

COROLLARY 2. *Let $f(t)$, $f_1(t)$, and $f_2(t)$ be characteristic functions such that $f(t) = f_1(t)f_2(t)$. Let $0 < \alpha < 1$. Then $f(t) \in \mathrm{Lip}\, \alpha$ if and only if $f_j(t) \in \mathrm{Lip}\, \alpha$ for $j = 1$ and $j = 2$.*

The proof of Theorem 1 for $k = 2$ and $0 < \lambda < 1$ is contained in a Theorem of Boas ([2] Theorem 1). Binmore and Stratton [1] obtained Theorem 1 for $k = 2$ and $0 < \lambda < 2$ by an entirely different method of proof.

Theorem 2 is a generalization of a Theorem ([10] Theorem 5) of Ramachandran. Ramachandran's method of proof, however, is somewhat different and does not yield a formula for the absolute moments of a distribution function. Formula (1) should be compared with other formulae for the absolute moments of a distribution function that were obtained by Brown ([3] page 658) and by Zolotarev ([14] page 440).

Theorem 3 sharpens some results that were obtained by other authors. Loève ([5] page 199), showed that if $F(x)$ has an absolute moment of the λth order where $k < \lambda < k+1$ then $f(t)$ admits an expansion of the form (2) where o is replaced by O. Ramachandran ([10] Theorem 3) showed that if $f(t)$ admits an expansion of the form (2) where o is replaced by O then $F(x)$ has absolute moments of all orders less than λ. Both of these results are contained in Theorem 3. Theorem 3 is closely related to a result of Brown ([3] page 659) that states that if k is an even positive integer and if $k < \lambda < k+2$, then $\int_{-\infty}^{\infty} |x|^{\lambda}\, dF(x) < \infty$ if and only if $f(t) = 1 + \sum_{j=1}^{k} c_j t^j/j! + R(t)$ where $\int_0^{\varepsilon} \operatorname{Re}\{R(t)\} t^{-\lambda-1}\, dt < \infty$ for $\varepsilon > 0$. Brown's result makes use of a Lemma of Von Bahr ([11] Lemma 4).

Pitman ([8] and [9]) has obtained some more precise theorems about the relationship between the asymptotic behavior of distribution functions and the local behavior of their characteristic functions when the tails of the distribution functions are functions of regular growth or when the distribution functions satisfy some similar condition.

2. Proof of Theorem 1. Let $G(x) = F(x) - F(-x)$. It is easily seen that

$$\rho_k(t) = 2^k(-1)^{k/2} \int_0^{\infty} (\sin yt)^k\, dG(y)\,.$$

Assume that $\rho_k(t) = o(t^{\lambda})$ as $t \to +0$. Then if $t > 0$,

$$\int_0^{1/t} (\sin yt)^k\, dG(y) = o(t^{\lambda}) \qquad \text{as } t \to +0\,.$$

If $0 < \theta < 1$, then $\sin\theta/\theta \geqq 1 - \theta^2/6 \geqq \frac{5}{6}$. Thus $(\sin yt)^k \geqq (\frac{5}{6})^k y^k t^k$ if $0 < y < 1/t$ and

$$t^k \int_0^{1/t} y^k\, dG(y) = o(t^{\lambda}) \qquad \text{as } t \to +0\,,$$

or equivalently

(11) $$A(x) = \int_0^x y^k\, dG(y) = o(x^{k-\lambda}) \qquad \text{as } x \to \infty\,.$$

If $x > 0$,

$$\begin{aligned} 1 - F(x) + F(-x) &= 1 - G(x) = \int_x^{\infty} dG(y) \\ &= \int_x^{\infty} y^{-k}\, dA(y) = -x^{-k}A(x) + k\int_x^{\infty} y^{-k-1} A(y)\, dy \\ &= o(x^{-\lambda}) \qquad \text{as } x \to \infty\,. \end{aligned}$$

Assume conversely that $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to \infty$. Then if $t > 0$,

$$\begin{aligned} |\rho_k(t)| &\leqq 2^k \int_0^{\infty} (\sin yt)^k\, dG(y) \\ &\leqq 2^k t^k \int_0^{1/t} y^k\, dG(y) + 2^k \int_{1/t}^{\infty} dG(y)\,. \end{aligned}$$

The assumption that $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to \infty$ implies that

$$\int_{1/t}^{\infty} dG(y) = o(t^{\lambda}) \qquad \text{as } t \to +0$$

and that

$$\begin{aligned} t^k \int_0^{1/t} y^k\, dG(y) &= t^k \int_0^{1/t} y^k\, d[G(y) - 1] \\ &= G(1/t) - 1 + kt^k \int_0^{1/t} y^{k-1}[1 - G(y)]\, dy \\ &= o(t^\lambda) \qquad \text{as } t \to +0 . \end{aligned}$$

Thus $\rho_k(t) = o(t^\lambda)$ as $t \to +0$.

3. Proof of Theorem 2. Assume that $\int_{-\infty}^{\infty} |x|^\lambda\, dF(x) < \infty$. Then $\int_0^\infty x^\lambda\, dG(x) < \infty$ and if $\varepsilon > 0$ then

$$\begin{aligned} \int_0^\varepsilon t^{-\lambda-1}|\rho_k(t)|\, dt &= 2^k \int_0^\varepsilon t^{-\lambda-1}[\int_0^\infty (\sin xt)^k\, dG(x)]\, dt \\ &= 2^k \int_0^\infty [\int_0^\varepsilon t^{-\lambda-1}(\sin xt)^k\, dt]\, dG(x) \qquad (12) \\ &= 2^k \int_0^\infty x^\lambda[\int_0^{\varepsilon x} v^{-\lambda-1}(\sin v)^k\, dv]\, dG(x) . \end{aligned}$$

Since $\int_0^\infty v^{-\lambda-1}(\sin v)^k\, dv < \infty$, it follows that $\int_0^\varepsilon t^{-\lambda-1}|\rho_k(t)|\, dt < \infty$.

Assume conversely that $\int_0^\varepsilon t^{-\lambda-1}|\rho_k(t)|\, dt < \infty$ where $\varepsilon > 0$. Then it follows from the above argument that

$$\int_0^\infty x^\lambda[\int_0^{\varepsilon x} v^{-\lambda-1}(\sin v)^k\, dv\}\, dG(x) < \infty ,$$

and so

$$\int_1^\infty x^\lambda[\int_0^{\varepsilon x} v^{-\lambda-1}(\sin v)^k\, dv]\, dG(x) < \infty .$$

But if $x > 1$, then

$$\int_0^{\varepsilon x} v^{-\lambda-1}(\sin v)^k\, dv \geqq \int_0^\varepsilon v^{-\lambda-1} (\sin v)^k\, dv > 0 .$$

Thus $\int_1^\infty x^\lambda\, dG(x) < \infty$ and so $\int_{-\infty}^\infty |x|^\lambda\, dF(x) < \infty$.

Formula (1) follows from (12) when ε goes to infinity.

4. Proof of Theorem 3. Let l be an even integer that is greater than λ. Assume that (2) holds. Then $\rho_l(t) = o(t^\lambda)$ as $t \to +0$ and it follows from Theorem 1 that $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to \infty$.

Assume conversely that $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to \infty$. Let $G(x) = F(x) - F(-x)$. By a Lemma of Feller ([4] page 512), if $\theta > 0$ then

$$|e^{i\theta} - 1 - \sum_{j=1}^n (i\theta)^j/j!| \leqq \theta^{n+1}/(n + 1)!$$

Thus $f(t)$ admits an expansion of the form

$$f(t) = 1 + \sum_{j=1}^k c_j t^j/j! + R(t)$$

where

$$|R(t)| \leqq (1/(k + 1)!) \int_0^{1/|t|} |tx|^{k+1}\, dG(x) + (2/k!) \int_{1/|t|}^\infty |tx|^k\, dG(x) .$$

If $1 - G(x) = o(x^{-\lambda})$ as $x \to \infty$ then

$$\begin{aligned} \int_x^\infty y^k\, dG(y) &= -x^k(1 - G(x)) + k \int_x^\infty y^{k-1}(1 - G(y))\, dy \\ &= o(x^{k-\lambda}) \qquad \text{as } x \to \infty \text{ and so} \end{aligned}$$

$$\int_{1/|t|}^\infty |tx|^k\, dG(x) = o(|t|^\lambda) \qquad \text{as } t \to 0 .$$

By an argument similar to that used in the proof of Theorem 1, it can be shown that if $1 - G(x) = o(x^{-\lambda})$ as $x \to \infty$ then

$$\int_0^{1/|t|} |tx|^{k+1} \, dG(x) = o(|t|^\lambda) \qquad \text{as } t \to 0 .$$

It follows that $R(t) = o(|t|^\lambda)$ as $t \to 0$.

5. A counter-example. Statement (11) remains true if $\lambda = k$. It follows that if k is a positive even integer and $\rho_k(t) = O(t^k)$ as $t \to +0$, then $\int_{-\infty}^{\infty} |x|^k \, dF(x) < \infty$ and thus $1 - F(x) + F(-x) = O(x^{-k})$ as $x \to \infty$. Also, if $\rho_k(t) = o(t^k)$ as $t \to +0$, then $F(x)$ is degenerate at 0 and thus $1 - F(x) + F(-x) = o(x^{-k})$ as $x \to \infty$. It is easy to show that the converse statements are not true. Let $c = [\int_2^\infty (1/y^{k+1} \ln y) \, dy]^{-1}$, let $p(x) = 0$ if $x < 2$, let $p(x) = c/x^{k+1} \ln x$ if $x \geqq 2$, and let $F(x) = \int_{-\infty}^x p(y) \, dy$. Then $1 - F(x) + F(-x) = o(x^{-k})$ as $x \to \infty$. However $\int_{-\infty}^{\infty} |x|^k p(x) \, dx = \infty$, so it is not true that $\rho_k(t) = O(t^k)$ as $t \to +0$.

REFERENCES

[1] Binmore, K. G. and Stratton, H. H. (1969). A note on characteristic functions. *Ann. Math. Statist.* **40** 303–307.

[2] Boas, R. P. (1967). Lipschitz behavior and integrability of characteristic functions. *Ann. Math. Statist.* **38** 32–36.

[3] Brown, B. M. (1970). Characteristic functions, moments, and the central limit theorem. *Ann. Math. Statist.* **41** 658–664.

[4] Feller, W. (1971). *An Introduction to Probability Theory and its Applications* **2**. Wiley, New York.

[5] Loève, M. (1963). *Probability Theory*. VanNostrand, Princeton.

[6] Lukacs, E. (1970). *Characteristic Functions*. Hafner, New York.

[7] Pitman, E. J. G. (1956). On the derivatives of a characteristic function at the origin. *Ann. Math. Statist.* **27** 1156–1160.

[8] Pitman, E. J. G. (1962). Some theorems on characteristic functions of probability distributions. *Fourth Berkeley Symp. Math. Statist. Prob.* **2**. Univ. of California Press, Berkeley.

[9] Pitman, E. J. G. (1968). On the behavior of the characteristic function of a probability distribution in the neighborhood of the origin. *J. Austral. Math. Soc.* **8** 423–443.

[10] Ramachandran, B. (1969). On characteristic functions and moments. *Sankhyā Ser. A.* **31** 1–12.

[11] Von Bahr, B. (1965). On the convergence of moments in the central limit theorem. *Ann. Math. Statist.* **36** 808–818.

[12] Wintner, A. (1947). *The Fourier Transforms of Probability Distributions.* Privately published, Baltimore.

[13] Wolfe, S. J. (1971). On moments of infinitely divisible distribution functions. *Ann. Math. Statist.* **42** 2036–2043.

[14] Zolotarev, V. M. (1957). Mellin-Stieltjes transforms in probability theory. *Theor. Probability App.* **2** 433–460.

[15] Zygmund, A. (1947). A remark on characteristic functions. *Ann. Math. Statist.* **18** 272–276.

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