

## THE GROWTH OF SUPERCRITICAL BRANCHING PROCESSES WITH RANDOM ENVIRONMENTS

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For the supercritical branching process with random environments, the rate of growth of the generation size  $Z_n$  is studied in the marginal distribution. It is shown that unless the environmental process yields a constant conditional expectation  $E(Z_1 | \zeta)$ , the asymptotic distribution of

$$(Z_n \exp(-nE_\zeta(\log E(Z_1 | \zeta))))^{n^{-1/2}}$$

is that of  $Ue^V$  where  $U$  and  $V$  are independent,  $P(U = 0) = 1 - P(U = 1) = P(Z_n \rightarrow 0)$  and  $V$  is normal  $(0, V_\zeta(\log E(Z_1 | \zeta)))$ .

**1. Introduction.** In a supercritical Galton-Watson branching process  $Z_0, Z_1, \dots$  an important result states that if  $E(Z_1 \log Z_1) < \infty$ , then  $Z_n/EZ_n \rightarrow W$  a.s. where  $P\{W = 0\} = q = P\{Z_n \rightarrow 0\}$  and where otherwise the distribution of  $W$  is absolutely continuous (Kesten and Stigum (1966)). In their recent treatment of branching processes with random environments (BPRE), Athreya and Karlin ((1971 b) page 1845) for the supercritical process "evaluate the rate of growth of  $Z_n$  on the set of nonextinction." They propose to study the limiting distribution of the random variable  $W_n = Z_n/E(Z_n | \zeta_0, \dots, \zeta_{n-1})$ , the denominator denoting the conditional expectation of  $Z_n$  given the environmental process up to time  $n - 1$ . Athreya and Karlin obtain results similar to those by Kesten and Stigum.

It seems, however, in some situations more appropriate to consider a "marginal" norming of  $Z_n$  which does not depend on the realized environmental process and it is therefore proposed in this note to study the asymptotic distribution of the variables  $X_n = Z_n e^{-n\mu}$ ,  $\mu = E_\zeta(\log E(Z_1 | \zeta))$  for the case of independent environmental variables with  $\sigma^2 = V_\zeta(\log E(Z_1 | \zeta)) < \infty$ . If  $\sigma^2 = 0$ ,  $E(Z_1 | \zeta)$  is a.s. constant and the results of the deterministic environment will essentially apply. If  $\sigma^2 > 0$ , however, there appear rather different results which can be informally stated as follows.

Apart from the event of extinction the fluctuations in  $X_n$  are determined by the fluctuations in  $E(Z_n | \zeta_0, \dots, \zeta_{n-1})$  which in turn can be described by the central limit theorem. Furthermore, the event of extinction is asymptotically independent of the latter fluctuations.

An application of the BPRE that has received some attention is demographic forecasting (Pollard (1968), Sykes (1969), Schweder and Hoem (1972)). It appears that the variation due to the random environment is much greater than

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the intrinsic random variation given the environment. The result in this note illustrates these empirical findings.

**2. Results.** Let  $Z_0, Z_1, Z_2, \dots$  be a BPRE. The environmental process is denoted by  $\check{\zeta} = (\zeta_0, \zeta_1, \dots)$  where it is assumed that  $\zeta_0, \zeta_1, \dots$  are independent and identically distributed. Assume  $Z_0 = 1$  and define  $m(\zeta) = m_1(\zeta) = E(Z_1 | \zeta)$  and

$$m_n(\zeta_0, \zeta_1, \dots, \zeta_{n-1}) = E(Z_n | \check{\zeta}) = \prod_{i=0}^{n-1} m(\zeta_i).$$

For details, see Athreya and Karlin (1971 a).

**THEOREM** Let  $Z_n, n = 0, 1, 2, \dots$  be a supercritical BPRE, i.e.,

$$E_\zeta(-\log P\{Z_1 > 0 | \zeta\}) < \infty \quad \text{and} \quad E_\zeta(\log m(\zeta))^- < E_\zeta(\log m(\zeta))^+ \leq \infty$$

and assume that  $E_\zeta(E(Z_1 \log Z_1 | \zeta)/m(\zeta)) < \infty$ .

Let  $\mu = E_\zeta(\log m(\zeta))$  and  $\sigma^2 = V_\zeta(\log m(\zeta)) < \infty$ .

- (a) If  $\sigma^2 = 0$ ,  $Z_n e^{-n\mu} \rightarrow W$  a.s. as  $n \rightarrow \infty$  where  $P\{W = 0\} = q = P\{Z_n \rightarrow 0\}$ ,
- (b) If  $\sigma^2 > 0$ , the asymptotic distribution of  $(Z_n e^{-n\mu})^{n^{-\frac{1}{2}}}$  as  $n \rightarrow \infty$  is that of  $Ue^V$  where  $U$  and  $V$  are independent,  $P(U = 0) = 1 - P(U = 1) = q$  and  $V$  is normal  $(0, \sigma^2)$ .

**PROOF.** Recall from Athreya and Karlin ((1971 b) Theorem 1) the following properties under the stated assumptions:  $W_n = Z_n/m_n(\zeta_0, \zeta_1, \dots, \zeta_{n-1}) \rightarrow W$  a.s. as  $n \rightarrow \infty$ .  $E(W | \check{\zeta}) = 1$  and  $P\{W = 0 | \check{\zeta}\} = q(\check{\zeta}) = P\{Z_n \rightarrow 0 | \check{\zeta}\} < 1$  a.s. It follows that  $\{W = 0\} = \{Z_n \rightarrow 0\}$  a.s. and that these events have probability  $q = E_\zeta q(\check{\zeta})$ .

**PROOF OF (a).** When  $\sigma^2 = 0$ ,  $m(\zeta) = m$  a.s.,  $\mu = \log m$  and  $m_n(\zeta_0, \dots, \zeta_{n-1}) = e^{n\mu}$  a.s., so that the result is a trivial corollary of the facts just stated.

**PROOF OF (b).** Write  $Z_n e^{-n\mu} = W_n Y_n$  so that  $Y_n = m_n(\zeta_0, \dots, \zeta_{n-1})e^{-n\mu}$ .

Let  $k_n$  be a sequence of integers such that  $k_n \rightarrow \infty$  and  $k_n n^{-\frac{1}{2}} \rightarrow 0$ . Define  $U_n = I\{Z_{k_n} > 0\}$  and  $X_n = m(\zeta_{k_n}) \dots m(\zeta_{n-1})e^{-n\mu}$ . Since  $U_n$  depends on  $(Z_0, \zeta_0, \dots, Z_{k_n-1}, \zeta_{k_n-1}, Z_{k_n})$  only it is seen that  $U_n$  and  $X_n$  are independent. The proof will now be carried out by proving that

- (i)  $U_n - W_n^{n^{-\frac{1}{2}}} \rightarrow 0$  a.s. and hence in probability,
- (ii)  $X_n^{n^{-\frac{1}{2}}} - Y_n^{n^{-\frac{1}{2}}} \rightarrow_p 0$ , and
- (iii)  $n^{-\frac{1}{2}} \log Y_n$  is asymptotically normal  $(0, \sigma^2)$ .

The proof will then be complete by the remark that by (i) and (ii), the limiting distribution of the pair  $(W_n^{n^{-\frac{1}{2}}}, Y_n^{n^{-\frac{1}{2}}})$  is the same as that of the pair of independent variables  $(U_n, X_n^{n^{-\frac{1}{2}}})$ . But  $U_n \rightarrow I\{Z_n \rightarrow \infty\}$  a.s. with  $P\{Z_n \rightarrow \infty\} = 1 - P\{Z_n \rightarrow 0\} = 1 - q$  and the limiting distribution of  $X_n^{n^{-\frac{1}{2}}}$  is, by (ii), the same as that of  $Y_n^{n^{-\frac{1}{2}}}$  given by (iii).

**PROOF OF (i).** It follows from the results quoted above that  $W_n \rightarrow W$  a.s. and  $P\{W = 0\} = q$ .

On the set  $\{W = 0\}$ ,  $W_n = 0$  from some  $n$  on a.s., and it follows that  $W_n^{n^{-\frac{1}{2}}} \rightarrow 0$  a.s. On the set  $\{W > 0\}$ , since  $E(W | \zeta) = 1$ , it is true a.s. that  $0 < W < \infty$  and hence that  $W_n^{n^{-\frac{1}{2}}} \rightarrow 1$ . The proof is complete since  $\{W = 0\} = \{Z_n \rightarrow 0\}$  a.s.

PROOF OF (iii).

$$n^{-\frac{1}{2}} \log Y_n = \frac{\sum_{i=0}^{n-1} \log m(\zeta_i) - n\mu}{n^{\frac{1}{2}}}$$

and the result is immediate from the central limit theorem.

PROOF OF (ii).

$$Y_n^{n^{-\frac{1}{2}}} - X_n^{n^{-\frac{1}{2}}} = (m_n(\zeta_0, \zeta_1, \dots, \zeta_{n-1})e^{-n\mu})^{n^{-\frac{1}{2}}}(1 - m_{k_n}(\zeta_0, \dots, \zeta_{k_n-1})^{-(n^{-\frac{1}{2}})}).$$

Since  $[\log m_{k_n}(\zeta_0, \dots, \zeta_{k_n-1}) - k_n \mu] / k_n^{\frac{1}{2}}$  is asymptotically normal  $(0, \sigma^2)$  by the central limit theorem cf. the proof above, and

$$k_n n^{-\frac{1}{2}} \rightarrow 0, \quad m_{k_n}(\zeta_0, \dots, \zeta_{k_n-1})^{-(n^{-\frac{1}{2}})} \rightarrow_p 1.$$

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