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A WEAK CONVERGENCE THEOREM FOR GAUSSIAN SEQUENCES

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In this note a weak convergence result in the Skorohod space $D^2[a, b]$ for a sequence of stochastic processes generated by the sample extrema of a stationary Gaussian sequence is obtained.

1. Introduction and main theorem. Let $\{X_n : 1 \leqq n < \infty\}$ be a stationary, Gaussian sequence of random variables with $E(X_1) = 0$, $E(X_1^2) = 1$ and $E(X_1 X_{n+1}) = r_n$. In this note we obtain a weak convergence result for a sequence of stochastic processes related to the extreme order-statistics of $X_1, X_2, \cdots, X_n$. Specifically, we consider the joint behavior of the maximum and the minimum of $X_1, X_2, \cdots, X_n$. The results parallel those of R. E. Welsch in [5] and [6] wherein he investigates the joint behavior of the maximum and the second maximum.

Let $0 < a < b < \; < \infty$. Define the sequences a_n, b_n by $a_n = (2 \log n)^{-\frac{1}{2}}$ and $b_n = (2 \log n)^{\frac{1}{2}} - \frac{1}{2}(2 \log n)^{-\frac{1}{2}}(\log \log n + \log 4\pi)$. We consider the stochastic process $\{(m_n(t), M_n(t)) : a \leqq t \leqq b\}$ where

$$m_n(t) = a_n^{-1}\{\min(X_1, X_2, \cdots, X_{[nt]}) + b_n\}$$

and

$$M_n(t) = a_n^{-1}\{\max(X_1, X_2, \cdots, X_{[nt]}) - b_n\},$$

$[\cdot]$ being the greatest integer function. If $[nt] < 1$ then write $m_n(t) = a_n^{-1}(X_1 + b_n)$ and $M_n(t) = a_n^{-1}(X_1 - b_n)$. Let $D[a, b]$ be the Skorohod space of right-continuous functions on $[a, b]$ having left-limits and let $D^2[a, b] = D[a, b] \times D[a, b]$. Clearly the stochastic process $\{(m_n(t), M_n(t)) : a \leqq t \leqq b\}$ has sample paths in $D^2[a, b]$.

Let $\Lambda(x)$ be the type III extreme-value distribution function $\Lambda(x) = \exp(-e^{-x})$ and let $\{M(t) : a \leqq t \leqq b\}$ denote the "extremal process" corresponding to Λ. Such processes are described in Dwass (1964) and in Lamperti (1964). $\{M(t)\}$ is a Markov process whose sample paths are right-continuous, non-decreasing, step-functions (and hence in $D[a, b]$) and for which

(i) $P\{M(t) \leqq x\} = \Lambda^t(x)$, $-\infty < x < \infty$, $a \leqq t \leqq b$.

(ii) $P\{M(s + t) \leqq y \mid M(s) = x\} = \Lambda^t(y)$ if $y > x$; $= 0$ if $y \leqq x$ with $a \leqq s \leqq s + t \leqq b$.

Let $\{m(t) : a \leqq t \leqq b\}$ denote a stochastic process which is independent of $\{M(t)\}$ and has the distribution of $\{-M(t) : a \leqq t \leqq b\}$. Then the two-dimensional stochastic process $\{(m(t), M(t)) : a \leqq t \leqq b\}$ can be regarded as a random element in $D^2[a, b]$.

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The object of this note is to show that if $\{X_n\}$ satisfies one of the mixing conditions introduced by Berman (1964) viz. either

(i) $r_n \log n \to 0$ or
(ii) $\sum r_n^2 < \infty$

then the sequence of stochastic processes $\{(m_n(t), M_n(t)) : a \leqq t \leqq b\}$ converges weakly, in $D^2[a, b]$, to the stochastic process $\{(m(t), M(t)) : a \leqq t \leqq b\}$.

THEOREM. *Let either $r_n \log n \to 0$ or $\sum r_n^2 < \infty$. Then the sequence of stochastic processes $\{(m_n(t), M_n(t)) : a \leqq t \leqq b\}$ converges weakly in the Skorohod space $D^2[a, b]$ to $\{(m(t), M(t)) : a \leqq t \leqq b\}$.*

2. Proof. The proof is broken up into the following lemmas. Lemma 1 is probably well known.

LEMMA 1. *If $\{X_n)$ are* i.i.d. *standard normal variables then the random variables $M_n(1)$ and $m_n(1)$ are asymptotically independent with limiting distribution functions $\Lambda(x)$ and $1 - \Lambda(-x)$ respectively.*

PROOF. Let x, y be fixed real numbers. Note that $a_n x - b_n$ is eventually less than $a_n y + b_n$. For such n, then, $P\{m_n(1) > x, M_n(1) < y\}$ has logarithm equal to $n \log P(a_n y + b_n > X_1 > a_n x - b_n\}$. Now $n \log P(a_n y + b_n > X_1 > a_n x - b_n\}$ is asymptotically equal to $-n[1 - P\{a_n y + b_n > X_1 > a_n x - b_n\}]$. But $-n[1 - P\{a_n y + b_n > X_1 > a_n x - b_n\}] = -nP\{X_1 > a_n y + b_n\} - nP\{X_1 < a_n x - b_n\}$. And $-nP\{X_1 > a_n y + b_n\} \to \log \Lambda(y)$ and $-nP\{X_1 < a_n x - b_n\} \to \log \Lambda(-x)$. Thus the probability $P\{m_n(1) > x, M_n(1) < y\} \to \Lambda(-x)\Lambda(y)$ and the proof of the lemma is complete.

LEMMA 2. *Under conditions of Lemma* 1 *the finite dimensional distributions of the process $\{(m_n(t), M_n(t)) : a \leqq t \leqq b\}$ converge to those of $\{(m(t), M(t)) : a \leqq t \leqq b\}$.*

PROOF. Let $a \leqq t_1 \leqq t_2 \leqq b$. Fix real numbers $x_2 < x_1$ and $y_1 < y_2$. We have,

$$\begin{aligned} & P\{m_n(t_1) > x_1, m_n(t_2) > x_2; M_n(t_1) < y_1, M_n(t_2) < y_2\} \\ & \quad = P\{a_n x_1 - b_n < X_j < a_n y_1 + b_n \text{ for } 1 \leqq j \leqq [nt_1] \text{ and} \\ & \qquad a_n x_2 - b_n < X_j < a_n y_2 + b_n \text{ for } [nt_1] + 1 \leqq j \leqq [nt_2]\} \end{aligned} \tag{2.1}$$

$$\begin{aligned} & \quad = P\{a_n x_1 - b_n < X_j < a_n y_1 + b_n; 1 \leqq j \leqq [nt_1]\} \\ & \qquad \times P\{a_n x_2 - b_n < X_j < a_n y_2 + b_n; [nt_1] + 1 \leqq j \leqq [nt_2]\}. \end{aligned} \tag{2.2}$$

Now proceeding as in Lemma 1 it is easy to see that the first factor in (2.2) converges to $\Lambda^{t_1}(y_1)\Lambda^{t_1}(-x_1)$ and the second factor converges to $\Lambda^{t_2-t_1}(y_2)\Lambda^{t_2-t_1}(-x_2)$. Thus the probability in (2.2) converges to $\Lambda^{t_1}(y_1)\Lambda^{t_2-t_1}(y_2)\Lambda^{t_1}(-x_1)\Lambda^{t_2-t_1}(-x_2)$. This verifies the assertion of the lemma for two-dimensional distributions. The convergence of the higher-dimensional distributions can be similarly verified. The proof of the lemma is complete.

LEMMA 3. *Suppose now that $\{X_n\}$ is a stationary, Gaussian sequence with $E(X_1) = 0$, $E(X_1^2) = 1$ and $E(X_1 X_{n+1}) = r_n$ and suppose further that either $r_n \log n \to 0$ or*

$\sum r_n^2 < \infty$. *Then the finite dimensional distributions of* $\{(m_n(t), M_n(t)) : a \leqq t \leqq b\}$ *converge to those of* $\{(m(t), M(t)) : a \leqq t \leqq b\}$.

PROOF. The proof is based on inequality (4.5) of Berman (1971). Again we restrict ourselves to two-dimensional distributions. The proof for higher-dimensional distributions is exactly the same. Let $x_2 < x_1$ and $y_1 < y_2$ be real numbers and let $a \leqq t_1 < t_2 \leqq b$. Denote by Δ_n the difference in the probabilities of the event $\{m_n(t_1) > x_1, m_n(t_2) > x_2; M_n(t_1) < y_1, M_n(t_2) < y_2\}$ computed under the hypotheses of Lemma 2 and Lemma 3 respectively. We want to estimate Δ_n using the inequality (4.5) in [2]. Toward this end let $\varphi(u, v; \lambda)$ be the standard bivariate normal density with marginal means zero, variance one, and correlation coefficient λ i.e.

$$\varphi(u, v; \lambda) = [2\pi(1 - \lambda^2)^{\frac{1}{2}}]^{-1} \exp\left\{-\frac{u^2 - 2\lambda uv + v^2}{2(1 - \lambda^2)}\right\}.$$

Let us write $u_i = a_n x_i - b_n$, $i = 1, 2$; and $v_i = a_n y_i + b_n$, $i = 1, 2$. Let α be any one of the four numbers u_1, u_2, v_1, v_2 and let β also be any one of these four numbers. It is easy to see that

$$\begin{aligned} \alpha^2 &= 2 \log n - \log\log n + O_n(1)\,, \\ \beta^2 &= 2 \log n - \log\log n + O_n(1)\,, \qquad \text{and} \\ |\lambda\alpha\beta| &\leqq |\lambda|(2 \log n - \log\log n) + O_n(1)\,, \qquad -1 < \lambda < 1\,. \end{aligned} \tag{2.3}$$

Under either of the two conditions $r_n \log n \to 0$ or $\sum r_n^2 < \infty$ it follows that $r_n \to 0$. By stationary, then, $\sup_n |r_n| = \delta < 1$. Using (2.3) it is easy to see that

$$\varphi(\alpha, \beta; \lambda) \leqq Kn^{-2/(1+|\lambda|)} \log n \qquad \text{for all} \quad |\lambda| \leqq \delta\,, \tag{2.4}$$

where K is a constant independent of n and λ.

Now, by the inequality (4.5) of [2],

$$\begin{aligned} \Delta_n \leqq\ & \textstyle\sum_{1\leqq j \leqq [nt_1]-1} ([nt_2] - j) \int_0^{|r_j|} \{\varphi(v_1, v_1; \lambda) \\ & + 2\varphi(v_1, u_1; \lambda) + \varphi(u_1, u_1; \lambda)\}\, d\lambda \\ & + \textstyle\sum_{[nt_1]\leqq j \leqq [nt_2]-1} ([nt_2] - j) \int_0^{|r_j|} \{\varphi(v_2, v_2; \lambda) + 2\varphi(v_2, u_2; \lambda) \\ & + \varphi(u_2, u_2; \lambda)\}\, d\lambda\,. \end{aligned} \tag{2.5}$$

Using (2.4) and (2.5) we get

$$\Delta_n \leqq 4K \textstyle\sum_{1\leqq j\leqq [nb]-1} ([nb] - j) n^{-2/(1+|r_j|)} (\log n)\,. \tag{2.6}$$

One can now imitate the proof of Theorem 3.1 of Berman (1964) to show that the right side of (2.6) goes to zero if, either $r_n \log n \to 0$ or $\sum r_n^2 < \infty$. This completes the proof of Lemma 3.

Now to complete the proof of the main theorem we need only show that both sequences of stochastic processes $\{M_n(t) : a \leqq t \leqq b\}$ and $\{m_n(t) : a \leqq t \leqq b\}$ are tight in the Skorohod space $D[a, b]$ under either of the two conditions $r_n \log n \to 0$ or $\sum r_n^2 < \infty$. The tightness of $\{M_n(t)\}$ is shown by Welsch in [6]. The tightness of $\{m_n(t)\}$ follows from that of $\{M_n(t)\}$ since in distribution, $\{m_n(t)\}$ is equivalent to $\{-M_n(t)\}$. This completes the proof of the main theorem.

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