The Annals of Probability
1973, Vol. 1, No. 6, 1044–1056

WEAK CONVERGENCE OF SUPERPOSITIONS OF RANDOMLY SELECTED PARTIAL SUMS[1]

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The main results are functional central limit theorems for superpositions of randomly selected partial sums in which the random variables being summed are independent and have distributions in the domain of attraction of stable laws. These results extend those of Tucker and Sreehari concerning when convolutions of distributions are attracted to stable laws. Other functional central limit theorems are presented for more general sums. The results herein extend the central limit theory for additive processes on Markov chains.

1. Introduction. Functional central limit theorems (invariance principles or weak convergence theorems in a function space setting) are presented for superpositions of the form

$$S_n = \sum_{i=1}^{N} \sum_{j=1}^{\nu_i(n)} \xi_{ij} \qquad n \geqq 1, \tag{1.1}$$

where $\xi_{ij}(1 \leqq i \leqq N, j \geqq 1, N \leqq \infty$ being a constant) is a double sequence of random variables (rv's), and $\nu_i(n)$ $(1 \leqq i \leqq N, n \geqq 1)$ are positive integer-valued rv's.

The sums (1.1) appear in many contexts. For example, suppose the ξ_{ij} are independent rv's such that for each i, the $\xi_{i1}, \xi_{i2}, \cdots$ have a common distribution F_i and suppose

$$\nu_i(n) = \sum_{k=1}^{n} I_{\{i\}}(\eta_k), \tag{1.2}$$

where I_A is the indicator function, and $\{\eta_n\}$ is a process taking values in $\{1, \cdots, N\}$ which is independent of the ξ_{ij}. Then (1.1) can be written as $S_n = \sum_{k=1}^{n} X_k$, where all n, $x_1, \cdots, x_n$ and $i_1, \cdots, i_n$

$$P[X_1 \leqq x_1, \cdots, X_n \leqq x_n \mid \eta_1 = i_1, \cdots, \eta_n = i_n] = \prod_{k=1}^{n} F_{i_k}(x_k) .$$

In other words, S_n is a sum of independent rv's whose distributions are randomly selected from the family $\{F_1, \cdots, F_N\}$ by the process η_n. This S_n could be thought of as a random walk in a randomly changing environment, where η_n is the environment process. Recent studies of stochastic systems (viz., branching, queues, Brownian motion, Poisson processes) in random environments appear in [1], [19], [24], [29], [34]. When η_n is a Markov chain, S_n is called an additive

Received October 15, 1971; revised March 14, 1973.
[1] Partially supported by NSF grant GK-27837.
AMS 1970 *subject classifications.* Primary 60F05; Secondary 60G50.
Key words and phrases. Functional central limit theorems, weak convergence, invariance principle, domain of attraction of stable law, randomly selected partial sums, stable processes, functionals of Markov processes, random environments.

process defined on a Markov chain, [7], [9], [13], [20], [25], [26], [30]. Many other examples can be described for an N-dimensional process $(\eta_1(n), \cdots, \eta_N(n))$ $(n \geqq 1)$ where $\nu_i(n) = \sum_{k=1}^{n} f_i(\eta_i(k))$ for appropriate functions f_i.

Our major result is Theorem 3.1, a functional central limit theorem for S_n in (1.1), where ξ_{ij} are independent rv's such that for each i, the $\xi_{i1}, \xi_{i2}, \cdots$ have a common distribution function F_i belonging to the domain of attraction of a stable law. This generalizes Theorem 2 of Tucker (1968) and Theorem 3.1 of Sreehari (1970) concerning the ordinary limit law of S_n for nonrandom $\nu_i(n)$. Theorem 3.2 is a similar result, under weaker hypotheses, for S_n with a random translation term (similar normalizations appear in Sreehari (1968)). Theorems 3.1 and 3.3 are proved by a random time change argument, which was introduced by Billingsley (1968) page 144, and extended by Iglehart and Kennedy (1970) and Whitt (1971a, 1972b) (other applications appear in [16], [18], [29], [30], [40]—[42]). Also used in their proofs are: (i) a weak convergence result involving summations (Theorem 2.1) due to Whitt (1972b), (ii) a weak convergence result involving stable processes (Theorem 2.3), which is a corollary of Theorem 2.7 of Skorohod (1957) as noted in Theorem 1 of Liggett (1968), and (iii) a basic property of stable distributions (implicit in Proposition 2.4), which is also the key to the results of Tucker (1968) and Sreehari (1970).

In Section 4, two functional central limit theorems for S_n, with no independence assumptions on the ξ_{ij} or $\nu_i(n)$, are presented. They are similar to Corollaries 5.1 and 5.2 of Whitt (1972b), which extend Iglehart and Kennedy (1970) and Whitt (1971a). The results of Section 3 and 4 generalize the functional central limit theorems for additive processes on Markov chains [5], [11], [13]—[15], [30], [32], [33], and their classical central limit theorems [7]—[10], [13], [20], [21], [25], [26], [28].

The results of Sections 3 and 4 are proved for $N < \infty$. Modifications required for $N = \infty$ are discussed in Section 5, where we also discuss continuous time versions of our results and other generalizations. Finally, in Section 6 an example due to M. Sreehari is presented which shows that a major assumption in Theorems 3.1 and 3.2 cannot be relaxed.

2. Preliminaries. Let $D = D[0, \infty)$, the set of all real-valued functions on $[0, \infty)$ which are right-continuous and have left limits everywhere. The topology we use on D is the J_1 topology of Stone (1963), which is the extension of Skorohod's J_1 topology on $D[0, 1]$, as discussed in Billingsley (1968). For more details see Lindvall (1972) and Whitt (1972b). Let D^m denote the Cartesian product of m copies of D with the product topology. A major tool that we use is the following, which is directly from Lemma 4.2 and Corollary 4.1 of Whitt (1972b).

THEOREM 2.1. *Suppose $Z_n^1, \cdots, Z_n^m$ $(n \geqq 1)$ and $Z^1, \cdots, Z^m$ are random elements of D such that*

$$(2.1) \qquad (Z_n^1, \cdots, Z_n^m) \to_{\mathscr{D}} (Z^1, \cdots, Z^m) \quad \text{in } D^m \text{ as } n \to \infty ,$$

and

$$P[\bigcap_{i=1}^{m} \text{Disc}(Z^i) = \phi] = 1 , \tag{2.2}$$

where Disc (x) *denotes the discontinuity set of* $x \in D$. *Then*

$$\sum_{i=1}^{m} Z_n^{\ i} \to_{\mathscr{D}} \sum_{i=1}^{m} Z^i \qquad \textit{in} \quad D \quad \textit{as} \quad n \to \infty . \tag{2.3}$$

In particular, (2.3) holds if (2.1) holds and $Z^1, \cdots, Z^m$ are independent and all but one is continuous in probability.

The remainder of this section deals with properties of stable distributions and stable stochastic processes which we use in proving Theorems 3.1 and 3.2. A stable distribution with constants (α, β, a, b), $0 < \alpha \leqq 2$, $-1 \leqq \beta \leqq 1$, $b \geqq 0$ and a any real number, has the characteristic function, Feller (1966) page 542,

$$\begin{aligned} \psi(u) &= \exp\{iau - b|u|^{\alpha}[1 + i\beta(u/|u|)\tan(\pi\alpha/2)]\} && \text{if} \quad \alpha \neq 1 \\ &= \exp\{iau - b|u|[1 + i\beta(u/|u|)(2/\pi)\log|u|]\} && \text{if} \quad \alpha = 1 . \end{aligned} \tag{2.4}$$

We call α and β the characteristic constants of ψ. They determine the distribution type, page 44 of Feller, as seen in the following (apparently unnoticed) result, which is evident using (2.4).

PROPOSITION 2.2. *Two nondegenerate stable distributions with respective constants* (α, β, a, b) *and* $(\alpha^*, \beta^*, a^*, b^*)$ *are of the same type if and only if* $\alpha = \alpha^*$ *and* $\beta = \beta^*$.

Recall, Section IX.8 of Feller, that a distribution F belongs to the domain of attraction of a nondegenerate stable distribution G, if there are location parameters $a_n > 0$ and b_n such that if $\xi_1, \xi_2, \cdots$ are independent rv's with common distribution F, then G is the limiting distribution of

$$b_n^{-1}\{\sum_{k=1}^{n} \xi_k - a_n\} . \tag{2.5}$$

The characteristic constants α and β of G do not depend on the choice of a_n and b_n. The α is such that $\int_{-x}^{x} y\, dF(y) \sim x^{2-\alpha}L(x)$ for some slowly varying function L, page 303 of Feller. And $\beta = 0$ if $\alpha = 2$, and $\beta = 2p - 1$ if $0 < \alpha < 2$, where

$$p = \lim_{x\to\infty} \{1 - F(x)\}/\{1 - F(x) + F(-x)\} ,$$

see Theorem 2 on page 546 of Feller. The other constants a and b of G, which do depend on the choice of a_n and b_n, are also obtainable from page 546 of Feller.

A random element X of D is called a stable process with constants (α, β, a, b) if it has stationary independent increments, is continuous in probability, and $X(1)$ has a stable distribution with constants (α, β, a, b). We call α and β the characteristic constants of X. The existence of processes of this sort is noted in Skorohod (1957), Theorem 14.20 of Breiman (1968), and Liggett (1968); and can also be derived by Theorem 15.7 of Billingsley (1968). For our next result, let $\xi_1, \xi_2, \cdots$ be independent rv's with a common distribution F which is in the domain of attraction of a stable distribution with characteristic constants α and β. Let $a_n > 0$ and b_n denote location parameters as in (2.5), and for each n and

$t \geqq 0$, set

$$X_n(t) = b_n^{-1}\{\sum_{k=1}^{[nt]} \xi_k - ta_n\},$$

where $[s]$ denotes the integer part of s. Let ζ be a random element of D which has stationary independent increments, is continuous in probability, and is such that $\zeta(1)$ has the distribution F. For each $n \geqq 1$ and $t \geqq 0$, set

$$Z_n(t) = b_n^{-1}\{\zeta(nt) - ta_n\}.$$

THEOREM 2.3. *Under the above assumptions, $Y_n \to_{\mathscr{D}} X$ in D, and $Z_n \to_{\mathscr{D}} X$ in D, where X is a stable process with characteristic constants α and β.*

PROOF. By elementary calculations one can show that for each t, $X_n(t) \to_{\mathscr{D}} X(t)$ (this is convergence in distribution of rv's), see Liggett (1968). Then by Theorem 2.7 of Skorohod (1957), $X_n \to_{\mathscr{D}} X$ in $D[0, s]$ for each $s \geqq 0$, and so by Theorem 3 of Lindvall (1968), $X_n \to_{\mathscr{D}} X$ in D. The proof of $Z_n \to_{\mathscr{D}} X$ in D is the same. The only nontrivial step is in showing that $Z_n(t) \to_{\mathscr{D}} X(t)$ as $n \to \infty$ for each t. This follows since

$$Z_n(t) = b_n^{-1}\{\zeta([nt]) - ta_n\} + b_n^{-1}\{\zeta(nt) - \zeta([nt])\}.$$

where

$$b_n^{-1}\{\zeta([nt]) - ta_n\} =_{\mathscr{D}} X_n(t) \to_{\mathscr{D}} X(t),$$

and $b_n^{-1}\{\zeta(nt) - \zeta([nt])\} \to_{\mathscr{D}} 0$. The latter follows as

$$P[b_n^{-1}|\zeta(nt) - \zeta([nt])| > \varepsilon] \leqq P[b_n^{-1}|\sup\{\zeta(s): 0 \leqq s \leqq 1\}| > \varepsilon] \to 0,$$

since ζ has stationary independent increments and $\sup\{\zeta(s): 0 \leqq s \leqq 1\} < \infty$ a.s. by page 307 of Breiman (1968).

The above theorem can be generalized, along the lines of Theorem 2 on page 480 of Gikman and Skorohod (1969), to the case where X has stationary independent increments. It also appears that multiparameter versions of Theorem 2.3, similar to Theorem 5 of Bickel and Wichura (1971), are obtainable. Our last preliminary result is a generalization of the property (7) on page 1387 of Tucker (1968) for stable distributions.

PROPOSITION 2.4. *Let $X^1, \cdots, X^m$ be independent identically distributed stable processes with constants $(\alpha, \beta, 0, b)$. Let $p_1, \cdots, p_m$ be positive real numbers satisfying $\sum_{i=1}^m p_i^\alpha = 1$, and set*

$$\gamma(t) = 2t\beta c\pi^{-1} \sum_{i=1}^m p_i' \log p_i \quad \text{if } \alpha = 1$$
$$= 0 \quad \text{if } \alpha \neq 1. \tag{2.6}$$

Then $\sum_{i=1}^m p_i X^i + \gamma$ is a stable process with constants $(\alpha, \beta, 0, b)$.

PROOF. This follows since the process $\sum_{i=1}^m p_i X^i + \gamma$ has stationary independent increments, is continuous in probability, and by an elementary calculation, the characteristic function of $\sum_{i=1}^m p_i X^i(1) + \gamma(1)$ is stable with constants $(\alpha, \beta, 0, b)$.

3. Weak convergence to stable processes. Theorems 3.1 and 3.2 are based on the following assumptions and notation. Let ξ_{ij} $(1 \leqq i \leqq N, j = 1, 2, \cdots)$, where $N < \infty$, be independent rv's such that for each i, the variables $\xi_{i1}, \xi_{i2}, \cdots$ have the common distribution F_i which is in the domain of attraction of a stable distribution with characteristic constants α_i and β_i. Let $\alpha = \min\{\alpha_1, \cdots, \alpha_N\}$, and take the F_i's to be subscripted such that if $\alpha < 2$, then

$$\alpha = \alpha_1 = \cdots = \alpha_M < \alpha_{M+1} \leqq \cdots \leqq \alpha_N ,$$

and if $\alpha = 2$, then $F_1, \cdots, F_M$ have infinite second moments and $F_{M+1}, \cdots, F_N$ have finite second moments. Assume that at least one of the F_i has an infinite second moment. This insures that $\alpha < 2$, or that $M \geqq 1$ when $\alpha = 2$. The results of this section do not apply to the case where each F_i has a finite second moment. However, this case is covered in Section 4. Assume also that if $\alpha < 2$, then $\beta = \beta_1 = \cdots = \beta_M$. That is, the $F_1, \cdots, F_M$ are in the domain of attraction of stable distributions of the same type, see Proposition 2.2. This assumption cannot be relaxed either in our results, or in Theorem 2 of Tucker (1968), or in Theorem 3.1 of Sreehari (1970). See Section 6. The referee pointed out that Tucker fails to mention this assumption. Because of this assumption we can, and therefore do, take the location parameters $a_i(n)$ and $b_i(n)$ for $1 \leqq i \leqq M$, to be such that each of the sums

$$b_i(n)^{-1}\{\textstyle\sum_{j=1}^{n} \xi_{ij} - a_i(n)\} \tag{3.1}$$

converges to the same stable distribution with constants $(\alpha, \beta, 0, b)$ for some b.

Let $\nu_i(n)$ $(1 \leqq i \leqq N,\ n = 1, 2, \cdots)$ be positive integer-valued rv's. No assumptions are made concerning the dependency between these rv's and the ξ_{ij}. For each $1 \leqq i \leqq N$, $n \geqq 1$ and $t \geqq 0$ let

$$\begin{aligned} \Phi_n^i(t) &= n^{-1}\nu_i([nt]) \\ \Phi^i(t) &= \pi_i t \\ X_n^i(t) &= b_i(n)^{-1}\{\textstyle\sum_{j=1}^{[nt]} \xi_{ij} - ta_i(n)\} \\ X_n(t) &= B_n^{-1}\{\textstyle\sum_{i=1}^{N} \sum_{j=1}^{\nu_i([nt])} \xi_{ij} - tA_n\} \\ \tilde{X}_n(t) &= B_n^{-1}\{\textstyle\sum_{i=1}^{N} \sum_{j=1}^{[\nu_i(n)t]} \xi_{ij} - t\tilde{A}_n\} \\ Y_n^i(t) &= \beta_i(n)^{-1}\{\nu_i([nt]) - nt\pi_i(n)\} , \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \beta_i(n) &= n && \text{if } 0 < \alpha \leqq 1 \\ &= o(n^{\delta}) && \text{if } 1 < \alpha < 2 \quad \text{for some } \delta < 1/\alpha \\ &= O(n^{\frac{1}{2}}) && \text{if } \alpha = 2 \end{aligned}$$

$$\pi_i(n) \to \pi_i > 0 \qquad \text{as } n \to \infty$$

$$\begin{aligned} B_n &= \{\textstyle\sum_{i=1}^{M} \pi_i b_i(n)^{\alpha}\}^{1/\alpha} \\ A_n &= \textstyle\sum_{i=1}^{N} \pi_i(n)a_i(n) - g_n(\alpha) \\ \tilde{A}_n &= \textstyle\sum_{i=1}^{N} \nu_i(n)n^{-1}a_i(n) - g_n(\alpha) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} g_n(\alpha) &= 2\beta c\pi^{-1} \textstyle\sum_{i=1}^{M} \pi_i b_i(n) \log(\pi_i b_i(n)B_n^{-1}) && \text{if } \alpha = 1 \\ &= 0 && \text{if } \alpha \neq 1 . \end{aligned} \tag{3.4}$$

Let $Y^1, \cdots, Y^N$ denote random elements of D, let θ denote the zero function, and let X denote a stable process with constants $(\alpha, \beta, 0, b)$ (recall (3.1)). Under the above assumptions we have the following two results.

THEOREM 3.1. *If* $Y_n^i \to_{\mathscr{D}} Y^i$ *in* D, *for each* $1 \leqq i \leqq N$, *where* $Y^i = \theta$ *if* $0 < \alpha \leqq 1$, *then* $X_n \to X$ *in* D.

THEOREM 3.2. *If* $\nu_i(n)/n \to_P \pi_i$ *for each* $1 \leqq i \leqq N$, *then* $\tilde{X}_n \to_{\mathscr{D}} X$ *in* D.

PROOF OF THEOREM 3.1. We begin by noting some general properties of the norming constants in X_n. It is known (Lemma 5 of Tucker (1968)) that

$$b_i(n) \sim n^{1/\alpha_i} L_i(n) \tag{3.5}$$

for some measurable slowly varying function L_i. We adopt the definition of page 1381 of Tucker (1968) for slowly varying functions. This differs slightly from that on page 269 of Feller. From this and Lemma 1 of Tucker, it follows that

$$B_n \sim n^{1/\alpha} W(n) \tag{3.6}$$

for some measurable slowly varying function W. Any slowly varying function L satisfies

$$x^\varepsilon L(x) \to \infty\ , \qquad \text{and} \qquad x^{-\varepsilon} L(x) \to 0 \qquad \text{as } x \to \infty \tag{3.7}$$

for any $\varepsilon > 0$ (page 302 of Feller). From this we get

$$\lim_{n\to\infty} n^\sigma B_n^{-1} = 0 \qquad \text{for any } \sigma < 1/\alpha\ . \tag{3.8}$$

Our results depend heavily on the property that for $i > M$

$$\lim_{n\to\infty} b_i(n) B_n^{-1} = 0\ . \tag{3.9}$$

This follows if $\alpha < 2$ by (3.5)—(3.8), since

$$b_i(n) B_n^{-1} \sim n^{1/\alpha_i - 1/\alpha} L_i(n)/W(n)\ ,$$

where L_i/W is slowly varying and $\alpha < \alpha_i$. And if $\alpha = 2$ then (3.9) follows since

$$b_i(n) B_n^{-1} = \{b_i(n)^2/n\}^{\frac{1}{2}} \{\textstyle\sum_{j=1}^M b_j(n)^2/n\}^{-\frac{1}{2}}\ ,$$

and from page 304 of Feller, we know that $b_k(n)^2/n \to \infty$ or 0, if $k \leqq M$ or $k > M$ respectively.

We prove $X_n \to_{\mathscr{D}} X$ in D by using a random time transformation argument as on page 144 of Billingsley and in Whitt (1972b). Using the notation (3.2), and letting $\circ$ denote the composition mapping, we can write

$$\begin{aligned} X_n = {} & \textstyle\sum_{i=1}^M p_i(n) X_n^i \circ \Phi_n^i + \gamma_n + \sum_{i=M+1}^N p_i(n) X_n^i \circ \Phi_n^i \\ & + \textstyle\sum_{i=1}^N B_n^{-1} a_i(n) \beta_i(n) n^{-1} Y_n^i\ , \end{aligned} \tag{3.10}$$

where $p_i(n) = b_i(n) B_n^{-1}$ and $\gamma_n(t) = t g_n(\alpha) B_n^{-1}$. Then our result, $X_n \to_{\mathscr{D}} X$ in D, will follow by Theorem 5.1 of Billingsley upon showing that

$$\textstyle\sum_{i=1}^M p_i(n) X_n^i \circ \Phi^i + \gamma_n \to_{\mathscr{D}} X \qquad \text{in} \quad D\ , \tag{3.11}$$

and that the last two summations in (3.10) converge weakly to the zero function in D.

We first consider the limiting behavior of $X_n{}^i \circ \Phi_n{}^i$. Under the hypothesis of Theorem 3.1, $\Phi_n{}^i \to_{\mathscr{D}} \Phi^i$ in D, since $\Phi_n{}^i - \Phi^i = \beta_i(n)n^{-1}Y_n{}^i \to_{\mathscr{D}} \theta$ in D by Theorem 5.1 of Billingsley. By Theorem 2.3 and the fact that $\{X_n{}^i\}$ $(1 \leqq i \leqq N)$ are independent,

$$(X_n{}^1, \cdots, X_n{}^N) \to_{\mathscr{D}} (X^1, \cdots, X^N) \qquad \text{in} \quad D^N ,$$

where the latter are independent stable processes. Moreover, by our assumption that $F_1, \cdots, F_M$ are in the domain of attraction of the same type of stable law and our choice of $a_i(n)$ and $b_i(n)$ $(1 \leqq i \leqq M)$ in (3.1), it follows that $X^1, \cdots, X^M$ are equally distributed stable processes with the same constants $(\alpha, \beta, 0, b)$. Since the Φ^i are constant elements of D, it follows by Theorem 4.4 of Billingsley that

$$(3.12) \qquad (X_n{}^1, \cdots, X_n{}^N, \Phi_n{}^1, \cdots, \Phi_n{}^N) \to_{\mathscr{D}} (X^1, \cdots, X^N, \Phi^1, \cdots, \Phi^N) \qquad \text{in} \quad D^{2N} .$$

Thus by Corollary 3.1 of Whitt (1972b), and the definitions of X^i and Φ^i,

$$(3.13) \qquad (X_n{}^1 \circ \Phi_n{}^1, \cdots, X_n{}^N \circ \Phi_n{}^N) \to_{\mathscr{D}} (X^1 \circ \Phi^1, \cdots, X^N \circ \Phi^N) =_{\mathscr{D}} (\pi_1{}^{1/\alpha_1}X^1, \cdots, \pi_N{}^{1/\alpha_N}X^N) .$$

We now prove (3.11) by an argument similar to that used in the proof Theorem 2 of Tucker (1968) and Theorem 3.1 of Sreehari (1970). From any subsequence of integers, select another subsequence n' such that for each $i \leqq M$

$$(3.14) \qquad \pi_i{}^{1/\alpha}p_i(n') \to p_i \qquad \text{as } n' \to \infty ,$$

for some $0 \leqq p_i \leqq 1$. This can be done as $0 < \pi_i{}^{1/\alpha}p_i(n) \leqq 1$ for each i and n. These p_i satisfy $\sum_{i=1}^M p_i{}^\alpha = 1$ since $\sum_{i=1}^M \pi_i p_i(n)^\alpha = 1$ for each n. By (3.14) we obviously have $\gamma_{n'} \to_{\mathscr{D}} \gamma$ in D where γ is as in (2.6). Then by (3.13) and (3.14),

$$(3.15) \qquad (p_1(n')X_{n'}^1 \circ \Phi_{n'}^1, \cdots, p_M(n')X_{n'}^M \circ \Phi_{n'}^M, \gamma_{n'}) \to_{\mathscr{D}} (p_1X^1, \cdots, p_MX^M, \gamma) \qquad \text{in} \quad D^{M+1} \qquad \text{as } n' \to \infty .$$

Since the processes on the right of (3.15) are independent and continuous in probability it follows by Theorem 2.1 that as $n' \to \infty$,

$$(3.16) \qquad \textstyle\sum_{i=1}^M p_i(n')X_{n'}^i \circ \Phi_{n'}^i + \gamma_{n'} \to_{\mathscr{D}} \sum_{i=1}^M p_iX^i + \gamma \qquad \text{in} \quad D .$$

But the term on the right of (3.16), by Proposition 2.4 is equal in distribution to X, and so (3.11) holds for the subsequence n'. Thus by Theorem 2.3 of Billingsley it follows that (3.11) holds in general.

Now consider the second summation in (3.10). This term converges to the zero function in D by Theorem 5.1 of Billingsley, since for each $M < i \leqq N$ we have $p_i(n) \to 0$ by (3.9), and by (3.13) we have $X_n{}^i \circ \Phi_n{}^i \to_{\mathscr{D}} \pi_i{}^{1/\alpha_i}X^i$ in D. It remains to show that the last summation in (3.10) converges to the zero function in D. To show this it suffices to show for each $1 \leqq i \leqq N$ that

$$(3.17) \qquad B_n{}^{-1}a_i(n)\beta_i(n)n^{-1}Y_n{}^i \to_{\mathscr{D}} \theta \qquad \text{in} \quad D \quad \text{as} \quad n \to \infty .$$

With no loss in generality we may assume (page 305 of Feller) that the location parameters $a_i(n)$ and $b_i(n)$ satisfy

$$nU_i(b_i(n))/b_i(n)^2 \to 1 \qquad \text{as } n \to \infty \tag{3.18}$$

and

$$\begin{aligned} a_i(n) &= 0 && \text{if } 0 < \alpha_i < 1 \\ &= nV_i(b_i(n)) && \text{if } \alpha_i = 1 \\ &= n\mu_i && \text{if } 1 < \alpha_i \leqq 2\,, \end{aligned} \tag{3.19}$$

where μ_i is the mean of F_i,

$$V_i(x) = \textstyle\int_{-x}^{x} t\, dF_i(t) \qquad \text{and} \qquad U_i(x) = \int_{-x}^{x} t^2\, dF_i(t)\,. \tag{3.20}$$

From (3.19) we see that (3.17) is trivially satisfied for those i with $\alpha_i < 1$. For those i with $\alpha_i = 1$ we have for any $\varepsilon > 0$

$$B_n^{-1}a_i(n)\beta_i(n)n^{-1}Y_n^{\,i} = \{b_i(n)^\varepsilon B_n^{-1}\}\{V_i(b_i(n))/b_i(n)^\varepsilon\}Y_n^{\,i}\,, \tag{3.21}$$

(recall $\beta_i(n) = n$ as $\alpha \leqq \alpha_i = 1$). The first term in braces in (3.21), when $\varepsilon < 1/\alpha$, converges to zero as $n \to \infty$ by (3.7), since (3.5) and (3.6) imply

$$b_i(n)^\varepsilon B_n^{-1} \sim n^{\varepsilon - 1/\alpha} L_i(n)^\varepsilon / W(n)\,,$$

where L_i^ε / W is slowly varying. The second term in braces in (3.21) also converges to zero as $n \to \infty$ since $x^{-\varepsilon}V_i(x) \to 0$ as $x \to \infty$. The latter follows since

$$x^{-\varepsilon}V_i(x) \leqq x^{-\varepsilon/2} \textstyle\int_{-x}^{x} |t|^{1-\varepsilon/2}\, dF_i(t) \qquad \text{for } x \geqq 1$$

and the last term converges to zero as $x \to \infty$, since F_i has finite absolute moments of all orders less than $\alpha_i = 1$, Lemma 2 on page 545 of Feller. Then since $Y_n \to_{\mathscr{D}} Y^i$ in D, by assumption, it follows that (3.17) holds for those i with $\alpha_i = 1$.

Lastly, for those i with $1 < \alpha_i \leqq 2$

$$B_n^{-1}a_i(n)\beta_i(n)n^{-1}Y_n^{\,i} = B_n^{-1}\beta_i(n)\mu_i Y_n^{\,i}\,.$$

If $\alpha < 2$ then (3.17) follows, since $B_n\beta_i(n) \to 0$ by (3,3), (3.6) and (3.7). If $\alpha = 2$ then (3.17) follows, since (3.18) implies $b_k(n)^2/n \to \infty$ for each $k \leqq M$, and this in turn implies

$$B_n^{-1}\beta_i(n) = \{\beta_i(n)/n^{\frac{1}{2}}\}\{\textstyle\sum_{k=1}^{M} \pi_k b_k^2(n)/n\}^{-\frac{1}{2}} \to 0\,.$$

We have shown that (3.17) holds for each i, and this completes the proof of Theorem 3.1.

PROOF OF THEOREM 3.2. Similar to (3.10) we can write

$$\tilde{X}_n = \textstyle\sum_{i=1}^{M} p_i(n)X_n^{\,i} \circ \Psi_n^{\,i} + \gamma_n + \sum_{i=M+1}^{N} p_i(n)X_n^{\,i} \circ \Psi_n^{\,i}\,, \tag{3.22}$$

where $\Psi_n^{\,i}(t) = t\nu_i(n)/n$. The hypothesis $\nu_i(n)/n \to \pi_i$ implies, see (17.17) of Billingsley, that $\Psi_n^{\,i} \to_{\mathscr{D}} \Phi^i$ in $D[0, s]$ for any $s > 0$; and so $\Psi_n^{\,i} \to_{\mathscr{D}} \Phi^i$ in D. With this observation in hand, the proof of Theorem 3.2 is the same as that for Theorem 3.1, excluding the arguments involving (3.17).

4. More general results. Consider the summation S_n in (1.1) with no assumptions on the dependency of the ξ_{ij} and $\nu_i(n)$. For each $1 \leqq i \leqq N$, $n \geqq 1$ and $t \geqq 0$ let

$$\begin{aligned}
\Phi_n^i(t) &= n^{-1}\nu_i([nt]) \qquad \Phi^i(t) = \pi_i t \\
X_n^i(t) &= b_i(n)^{-1}\{\textstyle\sum_{j=1}^{[nt]} \xi_{ij} - tna_i(n)\} \\
Y_n^i(t) &= a_i(n)B_n^{-1}\{\nu_i([nt]) - tn\pi_i(n)\} \\
X_n(t) &= B_n^{-1}\{\textstyle\sum_{i=1}^N \sum_{j=1}^{\nu_i([nt])} \xi_{ij} - tna_i(n)\pi_i(n)\} \qquad (4.1)\\
\tilde{X}_n^i(t) &= b_i(n)^{-1}\{\textstyle\sum_{j=1}^{[nt]} \xi_{ij} - tn\mu_i\} \\
\tilde{Y}_n(t) &= B_n^{-1}\{\textstyle\sum_{k=1}^{[nt]} \mu_{\eta_k} - tn \sum_{i=1}^N \mu_i \pi_i(n)\} \\
\tilde{X}_n(t) &= B_n^{-1}\{\textstyle\sum_{i=1}^N (\sum_{j=1}^{\nu_i([nt])} \xi_{ij} - tn\mu_i\pi_i(n))\}
\end{aligned}$$

where $a_i(n)$, $b_i(n)$, B_n, μ_i, $\pi_i(n)$, π_i are constants with $\pi_i(n) \to \pi_i > 0$ and $\{\eta_k\}$ is a stochastic process which takes values in $\{1, \cdots, N\}$. Let X^i, $\tilde{X}^i$, Y^i, Z^i $(1 \leqq i \leqq N)$ and $\tilde{Y}$ denote random elements of D and let θ denote the zero function in D.

THEOREM 4.1. *Suppose the following hold.*

(a) $(X_n^1, \cdots, X_n^N, Y_n^1, \cdots, Y_n^N) \to_{\mathscr{D}} (X^1, \cdots, X^N, Y^1, \cdots, Y^N)$ *in* D^{2N}, *where* $X^1 \circ \Phi^1, \cdots, X^N \circ \Phi^N, Y^1, \cdots, Y^N$ *satisfy condition* (2.2).

(b) *For each* $1 \leqq i \leqq N$, $b_i(n)B_n^{-1} \to r_i$, *and*

$$\begin{aligned} a_i(n)B_n^{-1} &= o(n) \qquad \text{if} \quad Y^i \neq \theta \\ &\sim n \qquad \text{if} \quad Y^i = \theta\,. \end{aligned}$$

Then

$$X_n \to_{\mathscr{D}} \textstyle\sum_{i=1}^N (r_i(X^i \circ \Phi^i) + Y^i) \qquad \text{in} \quad D\,. \tag{4.2}$$

THEOREM 4.2. *Suppose the following hold.*

(a) $\nu_i(n) = \sum_{k=1}^n I_{\{i\}}(\eta_k)$

(b) $\Phi_n^i \to_{\mathscr{D}} \Phi^i$ *in* D *for each* $1 \leqq i \leqq N$

(c) $(\tilde{X}_n^1, \cdots, \tilde{X}_n^N, \tilde{Y}_n) \to_{\mathscr{D}} (\tilde{X}^1, \cdots, \tilde{X}^N, \tilde{Y})$ *in* D^{N+1}, *where* $\tilde{X}^1 \circ \Phi^1, \cdots, \tilde{X}^N \circ \Phi^N$, $\tilde{Y}$ *satisfy condition* (2.2).

(d) *For each* $1 \leqq i \leqq N$, $b_i(n)B_n^{-1} \to r_i$. *Then*

$$\tilde{X}_n \to_{\mathscr{D}} \textstyle\sum_{i=1}^N r_i(\tilde{X}^i \circ \Phi^i) + \tilde{Y} \qquad \text{in} \quad D\,. \tag{4.3}$$

These results follow by applying the random time change argument along with Theorem 2.1, to the respective representations

$$X_n = \textstyle\sum_{i=1}^N \{(b_i(n)B_n^{-1})(X_n^i \circ \Phi_n^i) + Y_n^i\} \tag{4.4}$$

and

$$\tilde{X}_n = \textstyle\sum_{i=1}^N (b_i(n)B_n^{-1})(\tilde{X}_n^i \circ \Phi_n^i) + \tilde{Y}_n\,. \tag{4.5}$$

Theorem 4.2 is easier to apply than Theorem 4.1 when the $\nu_i(n)$ are as in (a) of Theorem 4.2. For example, if $\{\eta_n\}$ is a Markov chain or a strictly stationary process, then there are well-known conditions [4], [11], [30] under which (b) and (c) of Theorem 4.2 hold. But the establishment of the joint convergence of

$(Y_n^1, \cdots, Y_n^N)$, or even of the random variables $(Y_n^1(1), \cdots, Y_n^N(1))$, in these instances, is much harder. (The author is not aware of any widely used references on this.)

Note that Theorem 3.1 is a special case of Theorem 4.1 if each $p_i(n)$ in (3.10) converges as $n \to \infty$. These $p_i(n)$ do not generally converge, see page 1385 of Tucker (1968). They do converge if each $F_i (1 \leqq i \leqq N)$ belongs to the domain of normal attraction of a stable law with exponent α_i, page 547 of Feller, in which case $b_i(n) = n^{1/\alpha_i}$.

5. Comments. Our results can be generalized to the case where $N = \infty$. Theorems 4.1 and 4.2 are valid simply under the condition that the sums in (4.2) and (4.3) exist. Theorems 3.1 and 3.2 would be valid under some additional assumptions which would guarantee that the A_n, B_n and X_n are finite, and that the last two terms in (3.10) and (3.22) converge weakly to the zero function. These more general results would be based on weak convergence in infinite product spaces. However no new difficulties are encountered in going from finite to infinite product spaces. This is due to the fact that a probability measure is tight on an infinite product space if and only if its marginal distributions are tight on each coordinate space, see comments on page 40 and Problem 6 on page 41 of Billingsley.

Our results also hold for the continuous time counterpart of S_n in (1.1), which is $\sum_{i=1}^N \zeta_i(\nu_i(t))$, where $\zeta_1, \cdots, \zeta_N$ are random elements of D and $\nu_1, \cdots, \nu_N$ are positive real-valued processes. Simply replace $\nu_i([nt])$, $\sum_{j=1}^{[nt]} \xi_{ij}$ and $\sum_{j=1}^{\nu_i([nt])} \xi_{ij}$ in (3.2) and (4.1) by $\nu_i(nt)$, $\zeta_i(nt)$ and $\zeta_i(\nu_i(nt))$, respectively. Also in Theorems 3.1 and 3.2 make the assumption that each $\zeta_1, \cdots, \zeta_N$ are independent random elements of D which have stationary independent increments, are continuous in probability and are such that $\zeta_1(1), \cdots, \zeta_N(1)$ have distributions $F_1, \cdots, F_N$ as described in Section 3. The continuous time result $Z_n \to_{\mathscr{D}} X$ in D of Theorem 2.3, is used in the proofs. In the latter setting, if ζ_i is independent of ν_i, then $\zeta_i(\nu_i(t))$ is a process with conditional stationary independent increments, Serfozo (1972a).

Results such as ours can also be obtained for multiparameter stochastic processes of the form

$$X_n(s, t) = B_n^{-1}\{\sum_{i=1}^N \sum_{j=1}^{[\nu_i(n)s]} \zeta_{ij}(t) - sA_n\} \qquad \text{for } s \in [0, 1],\ t \in [0, 1]^q ,$$

where ζ_{ij} are D_q-valued processes, see Theorem 6 of Bickel and Wichura (1971). Notice that random time change arguments can be used here. Other types of results such as functional laws of large numbers and functional laws of the iterated logarithm, can also be obtained by similar arguments under appropriate moment conditions [11], [16], [38], [42].

6. An example of two distributions attracted to stable laws but their convolution is not. Let $F \in \mathscr{D}(\alpha, \beta)$ denote that the distribution F is in the domain of attraction of a stable law with characteristic constants α and β. The following example, related to me by Professor M. Sreehari, shows that one can have an $F_1 \in \mathscr{D}(\gamma, \delta)$ and an $F_2 \in \mathscr{D}(\gamma, -\delta)$ for some γ and δ, whereas their convolution $F_1 * F_2 \notin \mathscr{D}(\alpha, \beta)$ for any α and β. This implies that our assumption $\beta_1 = \cdots = \beta_M$ in Section 3 cannot be relaxed.

Consider the slowly varying functions $\phi_1(x) = \log x = \exp \int_e^x (t \log t)^{-1}\, dt$ and $\phi_2(x) = \exp \int_e^x (\theta_2(t)/t)\, dt$ for $x \geqq e$, where

$$\begin{aligned} \theta_2(t) &= 3/(2 \log t) \qquad \text{if} \quad t_{2n} < t \leqq t_{2n+1} \\ &= 1/(2 \log t) \qquad \text{if} \quad t_{2n-1} < t \leqq t_{2n}\,, \end{aligned}$$

and where $e = t_1 < \cdots < t_n < \cdots$ are chosen so that t_{2n} is the smallest positive integer $x > t_{2n-1}$ for which $\phi_2(x)/\phi_1(x) \leqq \frac{1}{2}$, and t_{2n+1} is the smallest integer $x > t_{2n}$ for which $1 \leqq \phi_2(x)/\phi_1(x) < 2$. Then there are subsequences k_n and m_n so that $\phi_2(k_n)/\phi_1(k_n) \to c$ for some $c \leqq \frac{1}{2}$, and $\phi_2(m_n)/\phi_1(m_n) \to d$ for some $1 \leqq d \leqq 2$.

Choose $F_1 \in \mathscr{D}(\gamma, \delta)$ and $F_2 \in \mathscr{D}(\gamma, -\delta)$ as on page 1383 of Tucker (1968), where for simplicity we take $\gamma = \frac{1}{2}$ and $\delta = 1$, such that $F_1^{n*}(n^2\phi_1(n)x)$ and $F_2^{n*}(n^2\phi_2(n)x)$ converge as $n \to \infty$ to stable distributions with respective constants $(\frac{1}{2}, 1, 0, 1)$ and $(\frac{1}{2}, -1, 0, 1)$. Let $G_n(x) = F_1^{n*} * F_2^{n*}(n^2\{\phi_1(n)^{\frac{1}{2}} + \phi_2(n)^{\frac{1}{2}}\}^2 x)$. Then G_n does not converge as $n \to \infty$, since the subsequences G_{k_n} and G_{m_n} converge to the two different stable distributions with respective constants $(\frac{1}{2}, \beta_1, 0, 1)$ and $(\frac{1}{2}, \beta_2, 0, 1)$, where $\beta_1 = (1 - c^{\frac{1}{2}})/(1 + c^{\frac{1}{2}})$ and $\beta_2 = (1 - d^{\frac{1}{2}})/(1 + d^{\frac{1}{2}})$, and obviously $\beta_1 \neq \beta_2$. Furthermore, by the convergence of types lemma (page 246 of Feller) and Proposition 2.2, it follows that $F_1 * F_2 \notin \mathscr{D}(\alpha, \beta)$ for any α and β.

Acknowledgments. I would like to thank J. David Mason and M. Sreehari for their comments on this paper. I also want to thank Ward Whitt and the referee for pointing out some of my errors and for their suggestions which lead to significant improvements.

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