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A METHOD FOR COMPUTING THE ASYMPTOTIC LIMIT OF A CLASS OF EXPECTED FIRST PASSAGE TIMES[1]

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The precise asymptotic behavior of certain expected first passage times plays an important role in C. Stone's theory of weak convergence of Markov processes. For a special class of random walks studied by Harris (1952), Lamperti (1962) and Karlin-McGregor (1959) we present a new method, using a maximum principle for a linear second order difference operator, that yields these asymptotic estimates. As a corollary we obtain an alternative proof of Lamperti's (1962) invariance principle.

1. Introduction. In his thesis "Limit theorems for birth and death processes and diffusion processes" C. Stone (1961) derived a set of necessary and sufficient conditions for the weak convergence of a sequence of one dimensional Markov processes $X_n(t)$ to a limiting diffusion process $X(t)$. The results to be presented in this paper arose out of the author's attempt to apply the general theory of Stone to a class of random walks that has been extensively studied by T. E. Harris (1952) and J. Lamperti (1962). As we shall see in a moment the method of Stone requires that one be able to compute the *precise* asymptotic behavior of a sequence of expected first passage times based on the processes $X_n(t)$. A consequence of our work would be an alternate derivation of Lamperti's invariance principle based on the general theory of Stone.

2. The method of C. Stone—a brief sketch. We shall suppose that the state space of the process $X(t)$ is an interval I with left and right hand end points denoted by r_1 and r_2 respectively. We define the first passage times $\tau(x_1, x_2)$ and $\tau(x_1)$ via the formulas

$$\begin{aligned} \tau(x_1, x_2) &= \inf\{t : X(t) \geqq x_2 \text{ or } X(t) \leqq x_1\} && \text{if } x_1 \leqq X(0) \leqq x_2 \\ \tau(x_1) &= \inf\{t : X(t) \geqq x_1\} && \text{if } X(0) \leqq x_1 \\ \tau(x_1) &= \inf\{t : X(t) \leqq x_1\} && \text{if } x_1 \leqq X(0)\,. \end{aligned} \tag{2.1}$$

Replacing the Markov process $X(t)$ in the above definition by the processes $X_n(t)$ produces a sequence of first passage times $\tau_n(x_1, x_2)$, $\tau_n(x_1)$ whose asymptotic behavior plays a crucial role in the general theory of Stone. As is well known the functions $M(x, x_1, x_2) = E_x(\tau(x_1, x_2))$ and $p(x, x_2, x_1) = P_x(\tau(x_1, x_2) < +\infty$ and $X(\tau(x_1, x_2)) = x_2)$ are of major importance in the modern theory of

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diffusion processes. The functions $M_n(x, x_1, x_2)$ and $p_n(x, x_2, x_1)$ are defined in exactly the same way replacing the $X(t)$ process in the definition by $X_n(t)$.

With these preliminary definitions out of the way we can now give a brief sketch of part of the main theorem of Stone (Theorem 3.1 on page 19 of [11]).

THEOREM. *The sequence of Markov processes $X_n(t)$ is weakly convergent to $X(t)$ if*

(2.2) *for every compact subinterval $I' \subset I$*

(a) $\lim_{n\to\infty} p_n(x, x_2, x_1) = p(x, x_2, x_1)$

(b) $\lim_{n\to\infty} M_n(x, x_1, x_2) = M(x, x_2, x_1)$, *uniformly for $x \in I'$,*

with other conditions to be imposed depending on the nature of the boundaries r_1 and r_2.

In the special case to be studied in this paper r_2 will be a natural boundary (so no boundary condition can be imposed) and r_1 will either be an entrance or a regular boundary in which case Stone's boundary conditions for convergence are

$$\lim_{n\to\infty} E_{r_1}(\tau_n(x)) = E_{r_1}(\tau(x)) \qquad \text{for every } x > r_1 . \tag{2.3}$$

The main contribution of this paper is to give a new method for evaluating the limit (2.3). The method will be illustrated by the following specific example, other applications will undoubtedly occur to the interested reader.

Let $X(0), X(1), \cdots, X(n)$, denote the successive positions of a particle performing a random walk on the nonnegative integers $I^+ = \{0, 1, 2, \cdots, n, \cdots\}$ with transition probabilities given by

$$p(i) = P(X(n+1) = i+1 \mid X(n) = i) = \tfrac{1}{2}(1 + (\gamma/i))$$
$$q(i) = P(X(n+1) = i-1 \mid X(n) = i) = \tfrac{1}{2}(1 - (\gamma/i)) = 1 - p(i)$$

where $0 \leqq \gamma < 1$ and $0 \neq i = 1, 2, \cdots$. At 0 we have the "reflecting barrier condition" $p(0) = +1$. These random walks are a special case of a more general class of random walks studied by Harris (1952), Lamperti (1962) and Brezis, Rosenkrantz and Singer (1971 b). The Markov processes $X_n(t)$ to which we shall apply the general theory of Stone are defined by the equation $X_n(t) = X([nt])/n^{\frac{1}{2}}$ where $[t]$ denoted the integer part of t. Using another method, Lamperti has established an invariance priniple for the sequence of Markov processes $X_n(t)$; that is to say he has demonstrated the weak convergence of the sequence $X_n(t)$ to $X(t)$ where $X(t)$ is a time homogeneous Markov process with state space $\bar{R}^+ = [0, \infty)$ and with infinitesimal generator G given by $Gf(x) = \frac{1}{2}f''(x) + (\gamma/x)f'(x)$, $0 \leqq x \leqq +\infty$. In the language of the analytic theory of semi-groups we have $U(x, t) = E_x f(X(t)) = T(t)f(x)$ satisfies the singular parabolic partial differentiable equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \frac{\gamma}{x}\frac{\partial U}{\partial x} = Gu\,, \qquad x \geqq 0 \tag{2.4}$$

with initial condition $U(x, 0) = f(x)$ and boundary condition $\partial U(0, t)/\partial x = 0$ for all $t \geqq 0$.

The precise conditions on f for which there exists a unique solution to (2.4) is that $f \in$ domain of the infinitesimal generator G, denoted by $D(G)$—a characterization of which has been given in Brezis, Rosenkrantz and Singer (1971 a).

It is to be observed that 0 is a regular or entrance boundary according as $0 \leqq \gamma < \frac{1}{2}$ or $\gamma \geqq \frac{1}{2}$ and that ∞ is a natural boundary. Here $r_1 = 0$ and $r_2 = \infty$.

In Part 4 it will be shown using a suitable martingale that $E_0(\tau(x)) = x^2/(2\gamma + 1)$ and therefore we must prove that $\lim_{n\to\infty} E_0(\tau_n(x)) = x^2/(2\gamma + 1)$.

Let $\tau'(a)$ denote the first passage time of the Markov chain $\{X(n), X(0) = 0\}$ to the integer a. Clearly $\tau_n(x) = \tau'([xn^{\frac{1}{2}}])/n$ and it therefore is sufficient to establish the following exact asymptotic estimate

$$E_0(\tau'(a)) \sim \frac{a^2}{2\gamma + 1} \qquad \text{as } a \uparrow +\infty . \tag{2.5}$$

Exact formulas for these expected first passage terms have been given by T. E. Harris, *op. cit.*, but it is very doubtful that they can be exploited to obtain (2.5). We shall proceed by another road.

Let a denote a fixed positive integer and set $D_a(i) = E_i(\tau'(a))$ where $0 \leqq i \leqq a$. In Feller's terminology $D_a(i)$ is called the expected duration of the game with a reflecting boundary at the origin and an absorbing boundary at a (cf. Feller (1968) **1**). It is easily checked that $D_a(i)$ satisfies the following inhomogeneous difference equation

$$p(i)D_a(i + 1) + q(i)D_a(i - 1) - D_a(i) = -1 , \qquad 1 \leqq i \leqq a - 1 \tag{2.6}$$

with the boundary conditions $D_a(0) = D_a(1) + 1$, $D_a(a) = 0$.

We denote this difference operator by $\hat{G}$ i.e.

$$\hat{G}f(i) = p(i)f(i + 1) + q(i)f(i - 1) - f(i) , \qquad 1 \leqq i \leqq a - 1 .$$

It is to be observed that if the function f has an interior maximum at j i.e. $f(i) \leqq f(j)$ for all i, $0 \leqq i \leqq a$, $1 \leqq j \leqq a - 1$, then $\hat{G}f(j) \leqq 0$, hence if f satisfies the difference inequality $Gf(i) > 0$, $1 \leqq i \leqq a - 1$, then f *cannot have an interior maximum* or in other words $\max\{f(i), 0 \leqq i \leqq a\} = \max\{f(0), f(a)\}$. Similarly if $\hat{G}f(i) < 0$, then f cannot have an interior minimum.

Consider the functions $W(i)$ and $Z(i)$ defined by the formulas

$$W(i) = \frac{a^2 - i^2}{2\gamma + 1} \qquad 0 \leqq i \leqq a$$

$$Z(i) = \left(\frac{2\gamma}{2\gamma + 1}\right)(a - i)$$

An elementary calculation yields

$$\hat{G}W(i) = -1 , \qquad 1 \leqq i \leqq a - 1$$

$$W(a) = 0 , \qquad W(0) = W(1) + 1/(2\gamma + 1), \text{ thus}$$

$W(i)$ satisfies the same inhomogeneous difference equation but with a slightly different boundary condition at 0. Another calculation shows that

$$\hat{G}Z(i) < 0\,, \qquad 1 \leqq i \leqq a-1 \text{ and that}$$
$$Z(a) = 0\,, \qquad Z(0) = Z(1) + 2\gamma/(2\gamma+1)\,.$$

Let $V(i) = D_a(i) - (W(i) + Z(i))$. Since $\hat{G}$ is a linear operator we get that $\hat{G}V(i) > 0$ for $1 \leqq i \leqq a-1$ and hence the maximum of $V(i)$ is taken on at 0 or at a. Now at 0 we have that $D_a(0) = D_a(1) + 1$, $W(0) = W(1) + 1/2\gamma + 1$ and $Z(0) = Z(1) + 2\gamma/2\gamma + 1$ and this implies that $V(0) = V(1)$. If $\max_{0 \leqq i \leqq a} V(i) = V(0) = V(1)$ then it would follow that V would have an interior maximum at $i = 1$ and so $\hat{G}V(1) \leqq 0$, this contradicts the fact that $\hat{G}V(1) > 0$. Therefore $\max_{0 \leqq i \leqq a} V(i) = V(a) = 0$, so $V(i) \leqq 0$ and therefore $D_a(i) \leqq W(i) + Z(i)$. In particular $D_a(0) \leqq a^2/(2\gamma+1) + 2\gamma/(2\gamma+1)a$. Therefore $\limsup_{a\to\infty} D_a(0)/a^2 \leqq 1/(2\gamma+1)$.

To obtain the inequality in the opposite direction we proceed in exactly the same way except that we use a minimum principle.

Let $U(i) = D_a(i) - (W(i) - Z(i))$. Then $\hat{G}U(i) = GZ(i) < 0$, $1 \leqq i \leqq a-1$, hence $U(i) \geqq \min(U(0), U(a))$. But $U(0) = U(1) + 4\gamma/(2\gamma+1)$ i.e. $U(0) \geqq U(1)$ and this implies that $\min_{0 \leqq i \leqq a} U(i) = U(a) = 0$. Thus $D_a(i) \geqq W(i) - Z(i)$ for $0 \leqq i \leqq a$, and in particular $\liminf_{a\to\infty} D_a(0)/a^2 \geqq 1/(2\gamma+1)$. The proof is now complete.

Actually the asymptotic estimate (2.5) holds for a much wider class of random walks including the "ultra spherical" random walks of Karlin and McGregor, see [8]. For example suppose $p(i)$ satisfies the less restrictive condition $p(i) = 1 - q(i) = 1/2(1 + \gamma/i + O(i^{-2}))$ as $i \to +\infty$ and $p(i) - q(i) > 0$ for all $i \neq 0$.

As before we assume $p(0) = +1$. Then the asymptotic estimate $D_a(0) \sim a^2/(2\gamma+1)$ as $a \to +\infty$ still holds.

PROOF. Let $\hat{G}$ denote the difference operator corresponding to the random walk whose transition probabilities satisfies the less restrictive growth condition $p(i) = 1/2(1 + \gamma/i + O(i^{-2}))$ as $i \to +\infty$. Let $Z(i) = C(a-i)$, where $C > 0$ is to be determined later. The calculations are in the same spirit as before and so we merely sketch the details. One checks easily enough that

$$GW(i) = -\frac{1}{2\gamma+1} - \frac{2i}{2\gamma+1}(p(i) - q(i))\,.$$

From our hypotheses on the asymptotic behavior of $p(i)$ we get $p(i) - q(i) = (\gamma/i)(1 + O(i^{-1}))$ and therefore $\hat{G}W(i) = -1 + O(i^{-1})$. Moreover $\hat{G}Z(i) = C(p(i) - q(i)) = -C(\gamma/i)(1 + O(i^{-1})) < 0$. Let $V(i) = D_a(i) - (W(i) + Z(i))$. $\hat{G}V(i) = -\hat{G}Z(i) + O(i^{-1}) = +C(\gamma/i)(1 + O(i^{-1})) + O(i^{-1}) = +C\gamma/i + O(i^{-1}) > 0$ provided C is large enough and $i \geqq k$, where k is some sufficiently large positive integer depending only on the $O(i^{-1})$ term. As for $1 \leqq i \leqq k-1$ we have

$$GV(i) = -\frac{2\gamma}{2\gamma+1} + \left(C - \frac{2i}{2\gamma+1}\right)(p(i) - q(i)) > 0$$

for $1 \leqq i \leqq k-1$ provided C is sufficiently large and positive. Hence $GV(i) > 0$ for $1 \leqq i \leqq a-1$, where C was chosen independently of a. Our maximum principle now yields $\max_{0\leqq i\leqq a} V(i) = \max(V(0), V(a)$. But $V(0) = V(1) + 2\gamma/(2\gamma+1) - C$ and therefore when $C > 2\gamma/(2\gamma+1)$ we have $V(1) > V(0)$, so $\max_{0\leqq i\leqq a} V(i) = V(a) = 0$. Thus $D_a(i) \leqq a^2/(2\gamma+1) + Ca$, from which we conclude as before $\limsup_{a\to\infty}(2\gamma+1)a^{-2}D_a(0) \leqq 1$. The inequality in the opposite direction is proved in exactly the same way using a minimum principle applied to the function $U(i) = D_a(i) - (W(i) - Z(i))$ and noting that $U(0) = U(1) + 2\gamma/(2\gamma+1) + C$ or $U(1) < U(0)$ and hence $\min_{0\leqq i\leqq a} U(i) = U(a) = 0$.

3. Uniform convergence of p_n, M_n to p and M on compact subintervals of $(0, \infty)$. We denote the state space of the process $X_n(t)$ by $I_n = \{j\cdot n^{-\frac{1}{2}} : j = 0, 1, 2, \cdots\}$ and define the function $e_n(x) : [0, \infty) \to I_n$ via the formula $e_n(x) = [xn^{\frac{1}{2}}]/n^{\frac{1}{2}} = \sup\{\alpha \in I_n : \alpha \leqq x\}$. If $x_1 \leqq x \leqq x_2$ are all points in I_n then $p_n(x, x_2, x_1)$, $M_n(x, x_1, x_2)$ are defined as in Section 2. If not then we extend the domains of these functions to all of $(0, \infty)$ by setting

$$p_n(x, x_2, x_1) = p_n(e_n(x), e_n(x_2), e_n(x_1)) \tag{3.1}$$
$$M_n(x, x_1, x_2) = M_n(e_n(x), e_n(x_1), e_n(x_2)) .$$

From the continuity properties of M and p one easily concludes the uniform convergence for $x \in [x_1, x_2]$, as n tends to infinity, of $p(e_n(x), x_2, x_1)$ to $p(x, x_2, x_1)$ and $M(e_n(x), x_1, x_2)$ to $M(x, x_1, x_2)$, since $\lim_{n\to\infty} e_n(x) = x$. In order, therefore, to conclude the uniform convergence of p_n, M_n to p and M respectively it suffices to prove that

$$\lim_{n\to\infty} |p_n(x, x_2, x_1) - p(e_n(x), x_2, x_1)| = 0 ,$$
$$\lim_{n\to\infty} |M_n(x, x_1, x_2) - M(e_n(x), x_1, x_2)| = 0$$

uniformly for $x \in I_n \cap [x_1, x_2]$.

To simplify the notation set $g(x) = p(x, x_2, x_1)$, $h(x) = M(x, x_1, x_2)$, $g_n(x) = p_n(x, x_2, x_1)$ and $h_n(x) = M_n(x, x_1, x_2)$. Our proof of the uniform convergence of g_n and h_n to g and h respectively uses the fact that g_n and h_n satisfy difference equations which are approximations to the differential equations satisfied by g and h. Using standard techniques of numerical analysis, in particular we establish a suitable "a priori estimate" for the difference operator $\hat{G}_n$ defined at (3.6), the uniform convergence is readily established.

The differential equations satisfied by g and h are

$$Gg(x) = 0 , \qquad x_1 < x < x_2 , \tag{3.2}$$
$$g(x_1) = 0 , \qquad g(x_2) = +1 ,$$

and

$$Gh(x) = -1 , \qquad x_1 < x < x_2$$
$$h(x_1) = h(x_2) = 0 ,$$

where G is the differential operator defined at (2.4).

Similarly the difference equations satisfied by g_n and h_n are

$$(3.4)\qquad \hat{G}_n g_n(x) = 0 \qquad x \in I_n \cap (x_1, x_2)\,,$$
$$g_n(x_1) = 0\,, \qquad g_n(x_2) = +1\,,$$

$$(3.5)\qquad \hat{G}_n h_n(x) = -1\,, \qquad x \in I_n \cap (x_1, x_2)$$
$$h_n(x_1) = h_n(x_2) = 0$$

where

$$(3.6)\qquad \hat{G}_n f(x) = n\left[E\left\{f\left(\frac{x_1}{n^{\frac{1}{2}}}\right)\middle|\frac{x_0}{n^{\frac{1}{2}}} = x\right\} - f(x)\right]$$
$$= n[f(x + n^{-\frac{1}{2}})p([xn^{\frac{1}{2}}]) + f(x - n^{-\frac{1}{2}})q([xn^{\frac{1}{2}}]) - f(x)]\,,$$

and $p(i) = \frac{1}{2}(1 + (\gamma/i) + o(i^{-1}))$ as $i \to +\infty$.

We are now ready to state the main result of this section.

THEOREM. $\lim_{n\to\infty} g_n(x) = g(x)$, $\lim_{n\to\infty} h_n(x) = h(x)$ *uniformly for* $x \in I_n \cap [x_1, x_2]$ *and hence conditions* a *and* b *of* (2.2) *are satisfied.*

The proof depends on two additional facts, one of which we present in the form of a

LEMMA. *Let* $\hat{G}_n\psi(x) = \phi(x)$, $x \in I_n \cap (x_1, x_2)$ *and set* $A = \max(|\psi(x_1)|, |\psi(x_2)|)$. *Then*

(3.7) $\|\psi\| \leqq A + K\|\phi\|$ *where* $\|\phi\| = \max_{x\in I_n\cap[x_1,x_2]}|\phi(x)|$ *and similarly* $\|\psi\| = \max_{x\in I_n\cap[x_1,x_2]}|\psi(x)|$. *K is a constant depending only on* x_1 *and* x_2 *and is independent of* ϕ *and* ψ.

REMARK. Estimates of the type just given are very important in the modern theory of partial differential equations and are referred to as "a priori estimates." A useful reference here is Hellwig (1964) especially pages 95–96.

We postpone the proof of the lemma to the end of this section in order to avoid interrupting our immediate task which is to establish the theorem. To this end we observe that the solutions to the differential equation (3.2) and (3.3) are smooth in the sense that $g \in C^{(3)}[x_1, x_2]$ and $h \in C^{(3)}[x_1, x_2]$ where $C^{(n)}[a, b]$ denotes the class of continuous functions with continuous derivatives up to and including order n on the closed interval $[a, b]$. This can be established directly or as a consequence of the results in Brezis, *et al.*, (1971 a). In addition it was shown, see Brezis *et al.*, (1971 b), using a three term Taylor expansion that for $f \in C^{(3)}[x_1, x_2]$.

$$(3.8)\qquad Gf(x) - \hat{G}_n f(x) = 0(\|f^{(3)}\| \cdot n^{-\frac{1}{2}}) \qquad \text{uniformly for } x \in I_n \cap (x_1, x_2)$$

where $f^{(i)}(x)$ denotes the ith derivative of f and $\|f^{(i)}\| = \sup_{x_1\leqq x\leqq x_2}|f^{(i)}(x)|$.

A more precise version of (3.8) is

$$(3.9)\qquad \lim_{n\to\infty}|\Pi_n Gf(x) - \hat{G}_n\Pi_n f(x)| = 0 \qquad \text{uniformly for } x \in (x_1, x_2)$$

where $\Pi_n f(x) = f(e_n(x))$.

In particular then we have

(3.10) $\lim_{n\to\infty} |\Pi_n Gg(x) - \hat{G}_n \Pi_n g(x)| = 0$ uniformly for $x \in (x_1, x_2)$ and

(3.11) $\lim_{n\to\infty} |\Pi_n Gh(x) - \hat{G}_n \Pi_n h(x)| = 0$ uniformly for $x \in (x_1, x_2)$.

Given $\varepsilon > 0$ pick $N(\varepsilon)$ so that for $n \geqq N(\varepsilon)$, $|\Pi_n Gg(x) - \hat{G}_n \Pi_n g(x)| < \varepsilon \cdot K^{-1}$, $x \in (x_1, x_2)$. Now $\Pi_n Gg(x) = 0 = \hat{G}_n g_n(x)$, $x \in I_n \cap (x_1, x_2)$ from which it is easily concluded that $|\hat{G}_n(\Pi_n g(x) - g_n(x))| < \varepsilon \cdot K^{-1}$, noting that $\hat{G}_n$ is a linear operator. We now apply (3.7) of our lemma with $\psi(x) = \Pi_n g(x) - g_n(x)$ and $\phi(x) = G_n(\Pi_n g(x) - g_n(x))$ and obtain the estimate $|\Pi_n g(x) - g_n(x)| \leqq \varepsilon$ because $||\phi|| < \varepsilon \cdot K^{-1}$ and $A = 0$ since $\psi(x_1) = \psi(x_2) = 0$. But this is equivalent to the assertion of the uniform convergence of g_n to g on $I_n \cap [x_1, x_2]$. The uniform convergence of h_n to h is established the same way. One first observes that $\Pi_n Gh(x) = -1 = \hat{G}_n h_n(x)$ and thus $\lim_{n\to\infty} |\hat{G}_n h_n(x) - \hat{G}_n \Pi_n h(x)| = 0$ uniformly $x \in I_n \cap (x_1, x_2)$. Set $\psi(x) = \Pi_n h(x) - h_n(x)$ and $\phi = \hat{G}_n(\Pi_n h - h_n)$, apply estimate (3.7) once again, noting that $\psi(x_1) = \psi(x_2) = 0$, and get as before $|\Pi_n h(x) - h_n(x)| \leqq \varepsilon$, $x \in I_n \cap [x_1, x_2]$. This establishes the theorem except for the lemma to the proof of which we now turn.

LEMMA. *Assume $0 < a < x < b$ and a, x, b are all points of the grid I_n. Let $f(x)$ denote a function satisfying the difference inequality $\hat{G}_n f(x) \geqq 0$, $x \in (a, b) \cap I_n$ and $f(a) \leqq 0$, $f(b) \leqq 0$. Then $f(x) \leqq 0$, provided n is sufficiently large.*

PROOF. Let $W(x) = x^2 - c^2$ where c is a constant satisfying the inequality $c > b$. We observe that $W(a) < 0$, $W(b) < 0$ and from the definition of $\hat{G}_n$ given at (3.6) we conclude $\lim_{n\to\infty} \hat{G}_n W(x) = (2\gamma + 1)$ uniformly on compact subsets of the interior of $(0, \infty)$. In the calculation it is assumed that $p(i) = \frac{1}{2}(1 + (\gamma/i) + o(i^{-1}))$ as $i \to \infty$ and $q(i) = \frac{1}{2}(1 - (\gamma/i) + o(i^{-1}))$ as $i \to +\infty$.

We now choose $\varepsilon > 0$ but otherwise arbitrary and note that $\hat{G}_n(f(x) + \varepsilon W(x)) \geqq \varepsilon > 0$, provided n is large enough, and that $f(a) + \varepsilon W(a) < 0$, $f(b) + \varepsilon W(b) < 0$. We claim that $f(x) + \varepsilon W(x) < 0$ otherwise it would have a nonnegative interior maximum—but this is impossible. Hence $f(x) \leqq -\varepsilon W(x)$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$ yields the result $f(x) \leqq 0$ and completes the proof of the lemma. A similar argument yields the

COROLLARY. *With a, x, b as in the previous lemma let $f(x)$ denote a function satisfying the difference inequality $\hat{G}_n f(x) \leqq 0$, $f(a) \geqq 0$, $f(b) \geqq 0$. Then, provided n is sufficiently large, $f(x) \geqq 0$, $x \in I_n \cap [a, b]$.*

We now apply this result to the proof (3.7).

Let $u(x) = \alpha(b^2 - x^2)$, where α satisfies the condition $\alpha(\gamma + \frac{1}{2}) > 1$. Then $Gu(x) = -\alpha(\gamma + \frac{1}{2}) < -1$ and $u(x) \geqq 0$ on $[a, b]$.

Moreover because $u(x) \in c^3[a, b]$ we have by a previous remark that $\lim_{n\to\infty} \hat{G}_n \Pi_n u(x) = Gu(x) < -1$, uniformly for $x \in I_n \cap (a, b)$. Applying the operator $\hat{G}_n$ to the function $\Pi_n(\psi(x) + A + u(x)||\phi||)$ yields $\hat{G}_n \Pi_n(\psi(x) + A + u(x)||\phi||) < \phi(x) - ||\phi|| \leqq 0$. In addition $\psi(a) + A + u(a)||\phi|| \geqq 0$, $\psi(b) + A + u(b)||\phi|| \geqq 0$ and therefore by the previous lemma we deduce the estimate

$\psi(x) \geqq -(A + u(x)||\phi||)$. A similar argument using $\psi(x) - (A + u(x)||\phi||)$ produces the inequality $\psi(x) \leqq A + u(x)||\phi||$. Now $\sup_{a \leqq x \leqq b} u(x) = b^2 - a^2$ and therefore $||\psi|| \leqq A + (b^2 - a^2)||\phi||$. This completes the proof of (3.7) with $K = x_2^2 - x_1^2$.

4. A martingale method with an application to the computation of some expected first passage times. In this section of the paper we show that

$$E_0(\tau(x)) = x^2/(2\gamma + 1)\,, \tag{4.1}$$

where $\tau(x)$ denotes the first passage time of the Markov process $X(t)$ discussed in Section 2. We use the well known and easily verified fact that the stochastic process

$$Z(t, w) = f(X(t, w)) - \int_0^t Gf(X(u, w))\, du \tag{4.2}$$

is a martingale relative to the sigma fields $\mathscr{F}(t) = \mathscr{F}\{X(u, w) : 0 \leqq u \leqq t\}$ provided f belongs to the domain of the strong infinitesimal generator G. We begin by establishing a more general result of which (4.1) is a particular consequence. Suppose then that $X(t, w)$ is an arbitrary one dimensional strong Markov process with continuous paths and state space $I = (r_0, r_1)$ on which it is regular in the sense of Dynkin (1965) page 125. In addition suppose that r_0 is either a regular reflecting boundary or an extrance boundary. Under either of these hypotheses it follows that the integral $\int_{r_0}^b (m(s) - m(r_0))\, dp(s)$ is finite for all b, $r_0 \leqq b < r_1$ where $p(s)$, $m(s)$ denote the scale and speed measures respectively of the process $X(t, w)$. Let $\tau(y)$ denote the first time that the process $X(t, w)$ hits the point y where $r_0 \leqq a < y$.

THEOREM. *Let r_0 be either a regular reflecting boundary or an entrance boundary of the process $X(t, w)$. Then*

$$E_a(\tau(y)) = \int_a^y (m(s) - m(r_0))\, dp(s)\,. \tag{4.3}$$

Before proving this result let us derive (4.1) as a consequence. An easy computation yields that $m(s) = s^{1+2\gamma}/(\gamma + \frac{1}{2})$, $dp(s) = s^{-2\gamma}\, ds$ and so $m(0) = 0$. Hence $E_0(\tau(y)) = (\int_0^y s\, ds)/(\gamma + \frac{1}{2}) = y^2/(2\gamma + 1)$. To prove (4.3) we put G into the Feller form: $Gf(x) = D_m D_p^+ f(x)$ (we refer the reader to Mandl (1968) for a particularly nice account of the Feller theory of generalized second order differential operators). Let $f(x) = \int_x^b (m(s) - m(r_0))\, dp(s)$ and note that $D_m D_p^+ f(x) = -1$, $D_p^+ f(r_0) = 0$, $f(b) = 0$—the reflecting and absorbing barrier conditions at r_0 and b respectively. So f is in the domain of $D_m D_p^+$, where the domain is characterized by the two boundary conditions just noted. By formula (4.2) we get $f(x(t)) + t$ is a martingale. If $X(0) = a$ then $E_a(f(x(t)) + t) = f(a)$ for all $t > 0$ and in particular for the stopping time $\tau(y)$ where $a < y < b$ we obtain, using Doob's optional stopping theorem, that $E_a(f(X(\tau(y))) + \tau(y)) = f(a)$ for $E_a(\tau(y)) = f(a) - f(y)$ since $X(\tau(y)) \equiv y$. A routine computation shows that $f(a) - f(y) = \int_a^y (m(s) - m(r_0))\, dp(s)$. □

It is to be observed that the martingale method used here goes back to Doob (1955). Our computations are simpler however because we have the Feller form of the infinitesimal generator at our disposal. The use of the Feller form of the generator G together with the martingale methods of Doob allow us to compute expected first passage times of arbitrary one dimensional continuous strong Markov processes satisfying various conditions at the boundary and without the use of cumbersome LaPlace transform techniques. For the more traditional approach the reader is advised to read Feller (1966) **2**.

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