The Annals of Probability
1973, Vol. 1, No. 6, 1026–1034

LOCAL VARIATION OF DIFFUSION IN LOCAL TIME

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Let $X(t)$, $X(0) = 0$, be a nonsingular diffusion in the natural scale in a neighborhood of 0, and let $f(t, x)$ be its local time. The local behavior of $X(f^{-1}(t, 0))$ is studied, and used to obtain upper and lower functions of a new type for $X(t)$ at $t = 0$.

0. Introduction. Let $X(t) = X(t, w)$, $X(0) = 0$, be a nonsingular diffusion in the natural scale in a neighborhood of 0. The terminology is that of Itô and McKean [3]: the speed measure is $m(dx)$ and the local time is $f(t, x) = d^+/dm^+$ (Lebesgue measure $\{s < t: X(s) < x\}$). Both $X(t)$ and $f(t, x)$ are continuous for each path w in the variables shown. An interesting open problem (see [3] 4.11, Problem 12) is to find an analytic test for a function $0 = h(0) \leqq h(t)$, nondecreasing and continuous, to be of upper or lower class locally for $X(t)$, in the sense of Kolmogorov (h is "upper" of "lower" according to whether $P\{X(t) \leqq h(t), 0 < t < \varepsilon$, for some $\varepsilon > 0\}$ is 1 or 0). If $X(t)$ is Brownian motion such a test is well known (Petrowsky [7]).

In this paper we study the related problem in which t is replaced by the left-continuous inverse local time $f^{(-1)}(\alpha) = \inf\{t: f(t, 0) \geqq \alpha\}$. This much more tractable problem then affords the solution to a problem in ordinary time which is very similar, but not equivalent, to the above. For purposes of comparison we can state this result as

Theorem 0.[1] *Let $p(t, x, y)$ be the continuous transition density of $X(t)$ (with respect to $m(dx)$—see* ([3] 4.11)). *Then $P\{X(t) \leqq h(f^{(-1)}(f(t))), 0 < t < \varepsilon$ for some $\varepsilon > 0\}$ is 1 or 0 according as $\int_{0+} h^{-1}(t)p(t, 0, 0)\, dt$ is finite or infinite.*

Remarks. It is not hard to see from the fact that zeros of X are points of increase of f [1, V, (3.8)] that $f^{(-1)}(f(t)) = \sup\{s < t: X(s) = 0\}$. The conclusion of Theorem 0 is unchanged if $X(t)$ is replaced by $-X(t)$, in contrast to the Kolmogorov classes which may be different for $X(t)$ and $-X(t)$. Thus it represents a symmetric analogue of the latter. On the other hand, the result also holds if $X(t)$ is only defined on $[0, \delta)$ with 0 an instantaneous singular point, so that the generator G satisfies $(Gu(0))m(0) = u^+(0) - u(0)k(0)$, as in ([3] Section 4.1). Indeed, the proof below needs no change in this case.

1. Variation of $X(f^{(-1)}(\alpha))$. In the remainder of the paper we assume that $X(t)$ is *conservative* and *persistent*. Since Theorem 0 concerns only a local property at $t = 0$ this assumption evidently entails no loss of generality (we are free

Received November 22, 1972.

[1] We use the notations $f(t) = f(t, 0)$ and $h^{-1} = 1/h$ in distinction to the inverse function $h^{(-1)}$.

AMS 1970 *subject classification.* Primary 60J60.
Key words and phrases. Upper and lower classes, local time, Kolmogorov's test, local oscillation.

to set $k(dx) = 0$ and change $m(dx)$ outside a neighborhood of 0). In particular, $\lim_{t\to\infty} f(t) = \infty$ with probability 1. We now introduce two obvious concepts.

DEFINITION 1. Let $0 \leqq g(t)$ be continuous and non-decreasing, and let $M(t) = \max_{0\leqq s\leqq t} X(s)$. We say that

(i) $g(t)$ is upper (resp. lower) in local time if $P\{M(f^{(-1)}(\alpha)) \leqq g(\alpha),\ 0 < \alpha < \varepsilon$ for some $\varepsilon > 0\} = 1$ (resp. equals 0);

(ii) $g(t)$ is lower for M in local time if $P\{M(f^{(-1)}(\alpha)) > g(\alpha),\ 0 < \alpha < \varepsilon$ for some $\varepsilon > 0\} = 1$.

Since $\lim_{\alpha\to 0} f^{(-1)}(\alpha) = 0$ a.s., the Blumenthal Zero-One Law shows that in (i) every g is either upper or lower in local time. Investigation of these concepts is quite simple when based on certain known facts which hold independently of $m(dx)$, and which we state as

THEOREM 1. *For $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n$ the processes*

$$s_k(x, w) = f(f^{(-1)}(\alpha_k), x) - f(f^{(-1)}(\alpha_{k-1}), x)\ , \qquad 1 \leqq k \leqq n,\ x \geqq 0$$

are independent diffusions in the parameter x with the same generator $y(d^2/dy^2)$, absorbing barrier at $y = 0$, and initial values $\alpha_k - \alpha_{k-1}$ at $x = 0$ respectively. Letting $x(\alpha_{k-1}, \alpha_k) = \inf\{x\colon s_k(x, w) = 0\}$ denote the absorption times, we have

$$x(\alpha_{k-1}, \alpha_k) = \max_{f^{(-1)}(\alpha_{k-1})<t<f^{(-1)}(\alpha_k)} X(t) \quad \text{a.s.,} \qquad \textit{and}$$

$$P\{x(\alpha_{k-1}, \alpha_k) \leqq z\} = \exp -\frac{(\alpha_k - \alpha_{k-1})}{z}\ , \quad z > 0\ .$$

These results are from [4], or alternatively [8]. To obtain a rationale for their validity the reader can imagine $X(t)$ approximated by simple random walks of step size $\Delta > 0$ and of step duration at $x = n\Delta$ given by $\Delta m[x, x + \Delta)$, as in ([4] Section 4). $f^{(-1)}(\alpha)$ is approximated by the sum of the durations steps preceding the $[\alpha\Delta^{-1}]$th step from 0 to Δ, and since the number of such steps does not depend on $m(dx)$ the maximum of the random walk in this interval is likewise free of $m(dx)$. It is thus evident that the classes introduced in (i) and (ii) will not depend on m.

THEOREM 2. *In Definition 1, $g(t)$ is upper or lower in local time according as $\int_{0+} g^{-1}(t)\,dt$ is finite or infinite.*

PROOF. Using the continuity of g we have

$$\begin{aligned} P\{M(f^{(-1)}(\alpha)) &\leqq g(\alpha),\ 0 < \alpha < \varepsilon\} \\ &= P\{x(0, \alpha) \leqq g(\alpha),\ 0 < \alpha < \varepsilon\} \\ &= \lim_{\Delta\to 0+} P\{x(0, k\Delta) \leqq g(k\Delta),\ 0 < k < \varepsilon\Delta^{-1}\}\ . \end{aligned}$$

By the monotonicity of g and Theorem 1, this becomes

$$\begin{aligned} \lim_{\Delta\to 0+} P\{x((k-1)\Delta, k\Delta) &\leqq g(k\Delta),\ 0 < k < \varepsilon\Delta^{-1}\} \\ &= \lim_{\Delta\to 0+} \prod_{k<\varepsilon\Delta^{-1}} \exp -(\Delta g^{-1}(k\Delta)) \\ &= \lim_{\Delta\to 0+} \exp -\sum_{k<\varepsilon\Delta^{-1}} \Delta g^{-1}(k\Delta) \\ &= \exp -\int_{0+}^{\varepsilon} g^{-1}(t)\,dt\ . \end{aligned}$$

Letting $\varepsilon \to 0+$ we get 1 or 0 according as the integral converges or diverges, completing the proof.

REMARK. If g is upper (resp. lower) in local time then *conditional upon* the process $f(\cdot)$, with probability 1 $g(f(\cdot))$ is upper (resp. lower) in the usual sense. However, this does not mean that $g(f(\cdot))$ is with probability 1 upper (resp. lower), even if $X(t)$ is Brownian motion.

Turning to the second part of Definition 1 we will be content to obtain a function $g(\alpha)$ for which $P\{\liminf_{\alpha\to 0+} M(f^{(-1)}(\alpha))g^{-1}(\alpha) = 1\} = 1$. Then $(1+c)g(\alpha)$ is lower for M in local time if $-1 < c < 0$, but not for $c > 0$.

THEOREM 3.

$$P\{\liminf_{\alpha\to 0+} M(f^{(-1)}(\alpha))g^{-1}(\alpha) = 1\} = 1 \qquad \text{for} \quad g(\alpha) = \alpha(\log\log 1/\alpha)^{-1}.$$

PROOF. The proof is a simplified version of one given by P. Lévy [6] in a similar situation. For $\frac{1}{2} \leqq \beta < 1$ and fixed c we set

$$A_n = \{M(f^{(-1)}(\beta^n)) < (1+c)\beta^n(\log\log\beta^{-n})^{-1}\}$$

and obtain

$$\begin{aligned}\textstyle\sum_{n=1}^{\infty} P(A_n) &= \textstyle\sum_{n=1}^{\infty} \exp-((1+c)^{-1}\log\log\beta^{-n}) \\ &= \textstyle\sum_{n=1}^{\infty} (n\log\beta^{-1})^{-(1+c)^{-1}}\end{aligned}$$

which is finite for $-1 < c < 0$ and infinite for $c > 0$. By the first Borel–Cantelli lemma the corresponding lim inf along the sequence β^n is not less than 1 for $-1 < c < 0$. But for $\beta^n < \alpha < \beta^{n-1}$ we have

$$\frac{M(f^{(-1)}(\beta^n))}{g(\beta^{n-1})} < \frac{M(f^{(-1)}(\alpha))}{g(\alpha)} < \frac{M(f^{(-1)}(\beta^{n-1}))}{g(\beta^n)},$$

so that by considering $\beta_k = 1 - k^{-1}$ and using $\lim_{n\to\infty} g(\beta_k{}^n)g^{-1}(\beta_k{}^{n-1}) = 1 - k^{-1}$ we get the lower bound 1 for the unrestricted lim inf.

Conversely, for $c > 0$ we can show that with probability 1 infinitely many of the events A_n occur. It suffices for this that, for $m > 1$, $\sum_{n=m+1}^{\infty} P(A_n \mid A_k', m \leqq k < n) = \infty$ where A_k' is the complement of A_k. We have

$$\begin{aligned}&P(A_n \mid A_k', m \leqq k < n) \\ &\quad = P(A_n)\frac{P\{x(\beta^n, \beta^{n-1}) \geqq (1+c)g(\beta^{n-1}), \cdots, x(\beta^n, \beta^m) \geqq (1+c)g(\beta^m)\}}{P\{x(0, \beta^{n-1}) \geqq (1+c)g(\beta^{n-1}), \cdots, x(0, \beta^m) \geqq (1+c)g(\beta^m)\}}.\end{aligned}$$

The second factor on the right may be written in the form $I_1 I_2^{-1}$ where

$$I_1 = \int_\alpha^\infty \frac{(\beta^{n-1}-\beta^n)}{y^2}\exp-\left(\frac{\beta^{n-1}-\beta^n}{y}\right)p(y)\,dy$$

$$I_2 = \int_\alpha^\infty \frac{\beta^{n-1}}{y^2}\exp-\left(\frac{\beta^{n-1}}{y}\right)p(y)\,dy$$

in which $\alpha = (1+c)g(\beta^{n-1})$ and $p(y)$ denotes the conditional probability of the inequalities following the first, given $x(\beta^n, \beta^{n-1}) = y$ in the numerator or given

$x(0, \beta^{n-1}) = y$ in the denominator. Since the ratio of the integrands is $(1-\beta)\exp(\beta^n/y) > 1-\beta$, this second factor exceeds $1-\beta$ and the sum is infinite along with $\sum_{n=m+1}^{\infty} P(A_n)$. Consequently the lim inf along this sequence is at most 1, and the theorem is proved.

2. Variation of $X(t)$. In the present section we adapt the method of Section 1 to prove Theorem 0. We need a variety of lemmas. The first one enables us to "discretize" the problem.

LEMMA 2.1. *Let* $0 = h(0) \leqq h(t)$ *be non-decreasing and continuous. For* $\alpha > 0$ *the events* $\{X(t) \leqq h(f^{(-1)}f(t)),\ 0 < t < f^{(-1)}(\alpha)\}$ *and* $\lim_{\delta\to 0+} \bigcup_{n=1}^{\infty} \{x(0, k2^{-n}\alpha) \leqq h(f^{(-1)}((k-1)2^{-n}\alpha)) + \delta,\ 1 \leqq k \leqq 2^n\}$ *differ by at most a set of probability* 0.

PROOF. It is clear that $X(t) > h(f^{(-1)}f(t))$ implies $x(0, k2^{-n}\alpha) > h(f^{(-1)}((k-1)2^{-n}\alpha))$ for $f((k-1)2^{-n}\alpha) < t \leqq f(k2^{-n}\alpha)$, proving the inclusion from right to left. On the other hand, if $\delta > 0$ is fixed and for all n there is a k with $x(0, k2^{-n}\alpha) > h(f^{(-1)}((k-1)2^{-n}\alpha)) + \delta$ then for a subsequence k_n we can obtain $\lim_{n\to\infty} f^{(-1)}((k_n-1)2^{-n}\alpha) = t \leqq f^{(-1)}(\alpha)$. It follows that $X(t) = 0$ (since X is continuous), and either $\lim_{n\to\infty} x(0, (k_n-1)2^{-n}) = \lim_{n\to\infty} x(0, k_n 2^{-n}) = x(0, f(t))$, or both limits are $x(0, f(t)+)$, or else the first limit is $x(0, f(t))$ and the second is $x(0, f(t)+) > x(0, f(t))$. Since $f^{(-1)}f(t) \leqq t$ we have in all cases $x(0, f(t)+) \geqq h(f^{(-1)}f(t)) + \delta$. But either $x(0, f(t)+) = x(0, f(t)) = M(t)$, and then $M(t) \geqq h(f^{(-1)}f(t)) + \delta$, or else $x(0, f(t)+) > x(0, f(t))$ and there is a $t' > t$ with $f(t') = f(t)$ and $x(0, f(t)+) = M(t')$. In this case $M(t') \geqq h(f^{(-1)}f(t')) + \delta$, and we may assume $t' \leqq f^{(-1)}(\alpha)$ since the event $\{x(0, \alpha+) > x(0, \alpha)\}$ has probability 0. Reducing t if necessary, this implies in both cases a $t \leqq f^{(-1)}(\alpha)$ with $X(t) \geqq h(f^{(-1)}f(t)) + \delta$, establishing the reverse inclusion.

The central part of the proof is in

LEMMA 2.2. *If* $P\{X(t) \leqq h(f^{(-1)}f(t)),\ 0 < t < \varepsilon$ *for some* $\varepsilon > 0\} = 0$, *then* $E \int_{0+} h^{-1}(f^{(-1)}(\alpha))\, d\alpha = \infty$. *If, however, this event has probability* 1 *then* $\int_{0+} h^{-1}(f^{(-1)}(\alpha))\, d\alpha < \infty$ a.s.

PROOF. By the Blumenthal Zero-One Law the event in question has probability 0 or 1, as does also the convergence of the integral (it will be shown later that the expectation of the integral is infinite if and only if the integral diverges a.s.). Lemma 2.1 shows that if the probability is 0, so that for every $\varepsilon > 0$ there is a $0 < t < \varepsilon$ with $X(t) > h(f^{(-1)}f(t))$, then $1 = P\{\lim_{\delta\to 0} \liminf_{n\to\infty}$ (the number of $k \leqq 2^n$ for which $x(0, k2^{-n}\alpha) \geqq h(f^{(-1)}((k-1)2^{-n}\alpha)) + \delta) = \infty\}$. Indeed, the lemma is applied with $2^{-m}\alpha$ in place of α, for m arbitrarily large. Here we may replace $x(0, k2^{-n}\alpha)$ by $x((k-1)2^{-n}\alpha, k2^{-n}\alpha)$, as in Theorem 2. Now the first Borel–Cantelli Lemma (or more precisely its proof) shows that

$$\infty = \lim_{\delta\to 0} \liminf_{n\to\infty} \sum_{k=1}^{2^n} P\{x((k-1)2^{-n}\alpha, k2^{-n}\alpha) > h(f^{(-1)}((k-1)2^{-n}\alpha)) + \delta\}\,.$$

Using Theorem 1 the probabilities are

$$E(1 - \exp - [\alpha 2^{-n}(h(f^{(-1)}(k-1)2^{-n}\alpha) + \delta)^{-1}]) \leqq \alpha 2^{-n} E(h(f^{(-1)}(k-1)2^{-n}\alpha) + \delta)^{-1}.$$

The sum is thus not larger as $n \to \infty$ than $E \int_{0+}^{\alpha} (h(f^{(-1)}(\beta)) + \delta)^{-1} d\beta$, and as $\delta \to 0$ we obtain divergence of the expectation.

Conversely, suppose that the probability is 1, so that for small $\alpha > 0$ and small $\delta > 0$ we have $\lim_{n\to\infty} P(\bigcap_{k=1}^{2^n} S_k) > 0$ where $S_k = \{x((k-1)2^{-n}\alpha, k2^{-n}\alpha) \leqq h(f^{(-1)}((k-1)2^{-n}\alpha)) + \delta\}$. Letting $\mathscr{F}(f^{(-1)}(\beta))$ denote the usual σ-field of the past of $X(t)$ up to the stopping time $f^{(-1)}(\beta)$ and $I(S)$ the indicator of S we have

$$\begin{aligned} P(\textstyle\bigcap_{k=1}^{2^n} S_k) &= E[E(I(S_{2^n}) \mid \mathscr{F}(f^{(-1)}(\alpha - 2^{-n}\alpha)))I(\textstyle\bigcap_{k=1}^{2^n-1} S_k)] \\ &= E[E\{E(I(S_{2^n})|\mathscr{F}(f^{(-1)}(\alpha - 2^{-n}\alpha)))I(S_{2^n-1})| \\ &\qquad \mathscr{F}(f^{(-1)}(\alpha - 2^{-(n-1)}\alpha))\}I(\textstyle\bigcap_{k=1}^{2^n-2} S_k)] \\ &= E[E\{E(\cdots E(E(I(S_{2^n})|\mathscr{F}(f^{(-1)}(\alpha - 2^{-n}\alpha)))I(S_{2^n-1})| \\ &\qquad \mathscr{F}(f^{(-1)}(\alpha - 2^{-(n-1)}\alpha))) \cdots) \\ &\qquad I(S_3)|\mathscr{F}(f^{(-1)}(2^{-(n-1)}\alpha))I(S_2)|\mathscr{F}(f^{(-1)}(2^{-n}\alpha))\}I(S_1)]. \end{aligned}$$

The strong Markov property now permits us to bound the successive conditional expectations from above using the inequalities

(2.1)

$$\begin{aligned} &E\left\{\left(\textstyle\prod_{k=m+1}^{2^n} \exp \dfrac{-2^{-n}\alpha}{h(f^{(-1)}((k-1)2^{-n}\alpha)) + \delta}\right) I(S_m) \,\middle|\, \mathscr{F}(f^{(-1)}((m-1)2^{-n}\alpha))\right\} \\ &\qquad \leqq E\left\{\textstyle\prod_{k=m}^{2^n} \exp \dfrac{-2^{-n}\alpha}{h(f^{(-1)}((k-1)2^{-n}\alpha)) + \delta} \,\middle|\, \mathscr{F}(f^{(-1)}((m-1)2^{-n}\alpha))\right\} \end{aligned}$$

for $1 \leqq m < 2^n$. Granting (2.1) it follows by Theorem 1 that

$$\begin{aligned} P(\textstyle\bigcap_{k=1}^{2^n} S_k) &\leqq E\left[E\left(\textstyle\prod_{k=1}^{2^n} \exp \dfrac{-2^{-n}\alpha}{h(f^{(-1)}((k-1)2^{-n}\alpha)) + \delta} \,\middle|\, \mathscr{F}(0)\right)\right] \\ &= E \exp \textstyle\sum_{k=1}^{2^n} \dfrac{-2^{-n}\alpha}{h(f^{(-1)}((k-1)2^{-n}\alpha)) + \delta}. \end{aligned}$$

But as $n \to \infty$ the last expression becomes $E \exp - \int_0^{\alpha} (h(f^{(-1)}(\beta)) + \delta)^{-1} d\beta$, and letting $\delta \to 0$ this becomes 0 unless $\int_{0+} h^{-1}(f^{(-1)}(\beta))\, d\beta$ converges with probability 1, as was to be shown.

Returning to the proof of (2.1) we shall only write the case $m = 2^n - 1$ as the others are completely analogous. Setting $c = f^{(-1)}(\alpha - 2^{-(n-1)}\alpha)$ and $d = 2^{-n}\alpha$, and using $X(c) = 0$ together with the strong Markov property at time c, this reduces to showing that

$$\begin{aligned} &E\left(\exp - \left(\frac{d}{h(c + f^{(-1)}(d)) + \delta}\right) I_{\{x(0,d) < h(c) + \delta\}}\right) \\ &\qquad \leqq E \exp - \left(\frac{d}{h(c + f^{(-1)}(d)) + \delta} + \frac{d}{h(c) + \delta}\right) \end{aligned}$$

when c is given. The left side may be written

$$E\left(\exp - \frac{d}{h(c + f^{(-1)}(d)) + \delta} \,\middle|\, \{x(0, d) \leqq h(c) + \delta\}\right) P\{x(0, d) \leqq h(c) + \delta\}$$

and in view of Theorem 1 it is enough to prove the bound $E \exp -[d/(h(c + f^{(-1)}(d)) + \delta)]$ for the conditional probability. But the condition $\{x(0, d) \leqq h(c) + \delta\} = \{M(f^{(-1)}(d)) \leqq h(c) + \delta\}$ has the effect of increasing $P(f^{(-1)}(d) < t\}$ for each $t > 0$, as is easily seen.[2] Hence the result follows.

The next aim is to prove the following Lemma 2.5, but to do so we need the (purely analytical) Lemmas 2.3 and 2.4.

LEMMA 2.3. *If* $0 \leqq r(t)$ *is continuous and non-increasing on* $[0, c]$ *and if* $F(dA)$ *and* $G(dA)$ *are two (nonnegative, finite) measures on* $[0, c] \times [0, c]$ *satisfying, for each rectangle of the form* $Q = [0, a] \times [0, b]$, $0 \leqq a, b \leqq c$, *the inequality* $F(Q) \leqq G(Q)$, *then* $\int_0^c \int_0^c r(s)r(t)F(dA) \leqq \int_0^c \int_0^c r(s)r(t)G(dA)$.

PROOF. It suffices to prove this for the functions $r_n(t) = r(k_{2^{-n}})$ on $\{k_{2^{-n}} \leqq t < (k + 1)2^{-n}\}$, $0 \leqq k \leqq [2^n c]$ in place of $r(t)$, and then to let $n \to \infty$. For $r_n(t)$ the result can be written in the matrix form $\mathbf{r}_n(f_{i,j})\mathbf{r}_n' \leqq \mathbf{r}_n(g_{ij})\mathbf{r}_n'$ where $f_{i,j}$ and $g_{i,j}$ are the F and G-measures of the rectangle $[i2^{-n}, (i + 1)2^{-n}) \times [j2^{-n}, (j + 1)2^{-n})$. Setting $\mathbf{r}_n = \mathbf{r}_{n,1}$ we now use an inductive procedure in which, at the kth step, $\mathbf{r}_{n,k}$ is replaced by $\mathbf{r}_{n,k+1}$ for which the first $k + 1$ elements equal $r(k2^{-n})$ and the remaining are unchanged. It is shown that the inequality for $\mathbf{r}_{n,k+1}$ implies that for $\mathbf{r}_{n,k}$. Since in the case $k + 1 = [2^n c]$ the entries of $\mathbf{r}_{n,k+1}$ are equal and nonnegative, the inequality is obvious in that case and the others follow by the induction.

The induction step to be established is thus

$$\mathbf{r}_{n,k}(g_{ij} - f_{ij})\mathbf{r}'_{n,k} - \mathbf{r}_{n,k+1}(g_{ij} - f_{ij})\mathbf{r}'_{n,k+1} \geqq 0 .$$

The left side is seen to equal

$$\begin{aligned} &2(\mathbf{r}_n(k) - \mathbf{r}_n(k + 1)) \textstyle\sum_{j>k} \sum_{i=1}^{k} (g_{i,j} - f_{i,j})\mathbf{r}_n(j) \\ &\quad + (\mathbf{r}_n^2(k) - \mathbf{r}_n^2(k + 1)) \textstyle\sum_{i,j\leqq k} \sum (g_{i,j} - f_{i,j}) \\ &\qquad \geqq 2(\mathbf{r}_n(k) - \mathbf{r}_n(k + 1))[\textstyle\sum_{j>k} \sum_{i=1}^{k} (g_{i,j} - f_{i,j})\mathbf{r}_n(j) \\ &\qquad\qquad + \mathbf{r}_n(k + 1) \textstyle\sum_{j\leqq k} \sum_{i=1}^{k} (g_{i,j} - f_{i,j})] \\ &\qquad \geqq 2(\mathbf{r}_n(k) - \mathbf{r}_n(k + 1)) \textstyle\sum_{j=1}^{[2^n c]} \sum_{i=1}^{k} (g_{i,j} - f_{i,j})\mathbf{r}_n([2^n c]) \\ &\qquad \geqq 0 \end{aligned}$$

where the second inequality is proved by replacing inside the brackets $\mathbf{r}_n(k + 1)$ by $\mathbf{r}_n(k + 2)$, then by $\mathbf{r}_n(k + 3)$, and so forth to $\mathbf{r}_n([2^n c])$, decreasing the total

[2] A formal proof can be given, starting with $E \exp - \lambda f^{(-1)}(d) = \exp - d\psi(\lambda)$ where $\psi(\lambda) = m(0)\lambda + \int_{0+}^{\infty} (1 - e^{-\lambda y})n(dy)$, as in ([3] 6.2). The conditioning may be introduced by simply excising the excursions of $X(t)$ from 0 which reach $h(c) + \delta$, as in [5]. Then since these excursions do not contribute to $f(t)$ the conditional local time will have a similar representation but with a new measure $n_1(dy) \leqq n(dy)$.

at each step in view of $F(Q) \leqq G(Q)$ and $\mathbf{r}_n(k+1) \geqq \mathbf{r}_n(k+2) \geqq \cdots \geqq \mathbf{r}_n([2^n c])$. This proves Lemma 2.3.

The next lemma is essentially proved in ([2] page 131) in the case when $m(0) = 0$, so that $f^{(-1)}$ has no linear part (see footnote 2), but we do not need this added assumption.[3]

LEMMA 2.4. $Ef^2(t) \leqq 2(Ef(t))^2$, $0 \leqq t$.

PROOF. We show first that $\lim_{x\to\infty} x^k P\{f(t) > x\} = 0$ for $k > 0$ and $t > 0$. This follows from

$$\begin{aligned} P\{f(t) > x\} &= P\{f^{(-1)}(x) < t\} \\ &\leqq E \exp -\lambda(f^{(-1)}(x) - t) \\ &= \exp(-x\psi(\lambda) + \lambda t)\,, \qquad \psi(\lambda) > 0\,. \end{aligned}$$

We use this in integrating by parts below.

$$\begin{aligned} \int_0^\infty e^{-\lambda t} Ef(t)\, dt &= \int_0^\infty e^{-\lambda t} \int_0^\infty x d_x(P\{f(t) \leqq x\} - 1)\, dt \\ &= \int_0^\infty e^{-\lambda t} \int_0^\infty P\{f(t) > x\}\, dx\, dt \\ &= \int_0^\infty \left(\int_0^\infty e^{-\lambda t} P\{f^{(-1)}(x) < t\}\, dt\right) dx \\ &= \frac{1}{\lambda} \int_0^\infty \int_0^\infty e^{-\lambda t} d_t P\{f^{(-1)}(x) < t\}\, dx \\ &= \frac{1}{\lambda} \int_0^\infty \exp(-x\,\psi(\lambda))\, dx \\ &= (\lambda\psi(\lambda))^{-1}\,. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_0^\infty e^{-\lambda t} Ef^2(t)\, dt &= \int_0^\infty e^{-\lambda t} \int_0^\infty x d_x(P\{f(t) \leqq x^{\frac{1}{2}}\} - 1)\, dt \\ &= \int_0^\infty \int_0^\infty e^{-\lambda t} P\{f^{(-1)}(x^{\frac{1}{2}}) < t\}\, dt\, dx \\ &= \frac{1}{\lambda} \int_0^\infty \int_0^\infty e^{-\lambda t} d_t P\{f^{(-1)}(x^{\frac{1}{2}}) < t\}\, dx \\ &= \frac{1}{\lambda} \int_0^\infty \exp(-x^{\frac{1}{2}}\psi(\lambda))\, dx \\ &= 2\lambda^{-1}\psi(\lambda)^{-2}\,. \end{aligned}$$

Comparison of these results shows that

$$Ef^2(t) = 2\frac{d}{dt} \int_0^t Ef(s)Ef(t-s)\, ds\,.$$

Now by ([3] 5.4, 3) we have $(d/dt)Ef(t) = p(t, 0, 0)$, and together with $Ef(0) = 0$ this yields

$$Ef^2(t) = 2\int_0^t Ef(s)\frac{d}{dt} Ef(t-s)\, ds \leqq 2Ef(t) \int_0^t \frac{d}{dt} Ef(t-s)\, ds = 2(Ef(t))^2\,.$$

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[3] The author is informed that R. K. Getoor has given another proof (unpublished).

We can now prove

LEMMA 2.5. *$\int_{0+} h^{-1}(f^{(-1)}(\alpha))\,d\alpha$ converges or diverges with probability* 1 *according as $\int_{0+} h^{-1}(t)p(t, 0, 0)\,dt$ converges or diverges.*

PROOF. It is first to be observed that for $0 < \varepsilon' < \varepsilon$ $\int_{\varepsilon'}^{\varepsilon} h^{-1}(f^{(-1)}(\alpha))\,d\alpha = \int_{f^{(-1)}(\varepsilon')}^{f^{(-1)}(\varepsilon)} h^{-1}(t)\,df(t)$. To show this one can write the left side as

$$\lim_{\Delta\to 0} \Delta \sum_{[\varepsilon'\Delta^{-1}]\leqq k\leqq[\varepsilon\Delta^{-1}]} h^{-1}(f^{(-1)}(k\Delta))\,,$$

and choosing t_k so that $f(t_k) = k\Delta$ and $f^{-1}f(t_k) = t_k$ this becomes

$$\lim_{\Delta\to 0} \sum_k h^{-1}(t_k)(f(t_{k+1}) - f(t_k))\,.$$

Now as $\Delta \to 0$ the terms where $t_{k+1} - t_k > \delta$ for fixed $\delta > 0$ have a negligible effect on the sum, enabling us to insert additional t's in these intervals (t_k, t_{k+1}) to obtain $t_{k+1} - t_k < \delta$ for all k. Since δ is arbitrarily small the assertion follows. Now, letting $\varepsilon' \to 0$, we get the same identity with $\varepsilon' = 0+$. But $E\int_{0+}^{f^{(-1)}(\varepsilon)} h^{-1}(t)\,df(t)$ is finite if and only if $E\int_{0+}^{\varepsilon} h^{-1}(t)\,df(t)$ is finite, since if $Eh^{-1}(f^{(-1)}(\varepsilon)) = \infty$ then both expectations are clearly infinite, and otherwise the difference of the two expectations is finite. Next, by ([3] 5.4, 3), we have $E\int_{0+}^{\varepsilon} h^{-1}(t)\,df(t) = \int_{0+}^{\varepsilon} h^{-1}(t)p(t, 0, 0)\,dt$, so it only remains to prove that if this is infinite then $\int_{0+} h^{-1}(t)\,df(t)$ diverges with probability 1. By Lemma 2.4,

$$\begin{aligned} E(f(s)f(t)) &\leqq (Ef^2(s)Ef^2(t))^{\frac{1}{2}} \\ &\leqq 2Ef(s)Ef(t)\,, \end{aligned}$$

and setting

$$\begin{aligned} r(t) &= h^{-1}(t) && \text{if } \varepsilon' < t < \varepsilon \\ &= h^{-1}(\varepsilon') && \text{if } 0 < t \leqq \varepsilon' \qquad 0 < \varepsilon' < \varepsilon\,, \end{aligned}$$

we can apply Lemma 2.3 to obtain

$$\begin{aligned} E\left(\int_{0+}^{\varepsilon} r(t)\,df(t)\right)^2 &= \int_{0+}^{\varepsilon}\int_{0+}^{\varepsilon} r(s)r(t)\,dE(f(s)f(t)) \\ &\leqq 2\left(\int_{0+}^{\varepsilon} r(t)\,dEf(t)\right)^2\,. \end{aligned}$$

Consequently, setting $R(\varepsilon', \varepsilon) = \int_{0+}^{\varepsilon} r(t)\,df(t)$, we have

$$\text{Variance } R(\varepsilon', \varepsilon)(ER(\varepsilon', \varepsilon))^{-2} \leqq 1\,.$$

If, contrary to what we wish to show, $\int_{0+}^{\varepsilon} h^{-1}(t)\,df(t)$ is small with probability near to 1 for small ε while its expectation is infinite, then for small ε' and ε, $R(\varepsilon', \varepsilon)$ will also be small, while its expectation will be arbitrarily large. In this case one would have for $\delta > 0$, Variance $R(\varepsilon', \varepsilon)(ER(\varepsilon', \varepsilon))^{-2} \geqq 2 - \delta$, contradicting the previous inequality. This completes the proof of Lemma 2.5.

Combining Lemmas 2.2 and 2.5, Theorem 0 follows immediately.

REFERENCES

[1] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory*. Academic Press, New York.

[2] GETOOR, R. K. (1963). The asymptotic distribution of the number of zero free intervals of a stable process. *Trans. Amer. Math. Soc.* **106** 127–138.

[3] Itô, K. and McKean, H. P. (1965). *Diffusion Processes and Their Sample Paths*. Springer-Verlag, Berlin.

[4] Knight, F. B. (1963). Random walks and a sojourn density process of Brownian motion. *Trans. Amer. Math. Soc.* **109** 56-86.

[5] Knight, F. B. and Pittenger, A. O. (1972). Excision of a strong Markov process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **23** 114-120.

[6] Lévy, P. (1953). La measure de Hausdorff de la courbe de mouvement Brownien. *Giorn. Ist. Ital.* Attuari **16** 1-37.

[7] Petrowsky, I. (1935). Zur ersten randwertaufgabe der warmeleitungsgleichung. *Compositio Math.* **1** 383-419.

[8] Silverstein, M. L. (1968). A new approach to local time. *J. Math. Mech.* **17** 1023-1054.

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