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SUBADDITIVE ERGODIC THEORY[1]

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It is now ten years since Hammersley and Welsh discovered (or invented) subadditive stochastic processes. Since then the theory has developed and deepened, new fields of application have been explored, and further challenging problems have arisen. This paper is a progress report on the last decade.

1. Theory.

1.1. *Subadditive processes.* An interesting class of stochastic processes was isolated by Hammersley and Welsh in their wide-ranging survey [4] of the problems of percolation theory. They observed that certain of these problems could be formulated in terms of families of random variables x_{st} $(s < t)$, where the indices s and t run over the set T of nonnegative integers, which satisfy three conditions:

S$_1$. Whenever $s < t < u$,

$$x_{su} \leqq x_{st} + x_{tu} . \tag{1.1.1}$$

S$_2'$. The distribution of x_{st} depends only on $t - s$.

S$_3$. The expectation

$$g_t = \mathbf{E}(x_{0t}) \tag{1.1.2}$$

exists, and satisfies

$$g_t \geqq -At \tag{1.1.3}$$

for some constant A and all $t > 1$.

From **S**$_2'$, $\mathbf{E}(x_{st}) = g_{t-s}$, and taking expectations in (1.1.1) shows that $g_{u-s} \leqq g_{t-s} + g_{u-t}$, so that

$$g_{m+n} \leqq g_m + g_n \qquad (m, n \geqq 1) .$$

The familiar theory of subadditive functions [5] then implies that

$$\lim_{t\to\infty} g_t/t = \gamma , \tag{1.1.4}$$

where

$$\gamma = \inf_{t\geqq 1} g_t/t \tag{1.1.5}$$

is finite because of (1.1.3).

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Although Hammersley and Welsh were able to deduce some consequences from S_1, S_2' and S_3, these were fragmentary and incomplete, partly because the stationarity requirement S_2' is not strong enough to bring into play the powerful tools of ergodic theory. A more complete theory is possible if S_2' is strengthened to

S_2. The joint distributions of the process $(x_{s+1,t+1})$ are the same as those of (x_{st}).

This is strictly stronger than S_2', in the sense that it is possible to construct processes satisfying S_2' but not S_2. But these examples are highly artificial, and it seems that, for all the applications so far envisaged, there is no advantage in using the weaker condition S_2'.

A family $\mathbf{x} = (x_{st};\, s, t \in T, s < t)$ of random variables x_{st} satisfying S_1, S_2 and S_3 is called a *subadditive process*, and the constant γ associated with it will be denoted where necessary by $\gamma(\mathbf{x})$.

If the random variables of a subadditive process $\mathbf{x}$ are all degenerate, then S_2 shows that $x_{st} = x_{0,t-s} = g_{t-s}$, so that (1.1.4) implies that

$$\lim_{t\to\infty} x_{0t}/t = \gamma \,.$$

On the other hand, suppose that (1.1.1) is strengthened to

$$x_{su} = x_{st} + x_{tu} \tag{1.1.6}$$

(in which case $\mathbf{x}$ is called an additive process). Then the process $(y_t;\, t \geqq 1)$ defined by

$$y_t = x_{t-1,t} \tag{1.1.7}$$

is stationary, and

$$x_{0t} = \textstyle\sum_{r=1}^{t} y_r \,. \tag{1.1.8}$$

Hence Birkhoff's ergodic theorem shows that the limit

$$\lim_{t\to\infty} x_{0t}/t \tag{1.1.9}$$

exists with probability one.

These two special cases led Hammersley and Welsh to conjecture that the limit (1.1.9) might exist for all subadditive processes $\mathbf{x}$, an assertion which would generalise simultaneously the limit theorem for subadditive functions and the Birkhoff ergodic theorem. Such a result was proved in [8], and will be described in Sub-section 1.2. In Sub-sections 1.3–1.4 various ramifications and reformulations are discussed, and in Sub-sections 2.1–2.4 some typical applications of the theory are described. The paper ends with a list of unsolved problems which stand as a challenge to future research.

1.2. *The ergodic theorem.* The fundamental theorem proved in [8] is a complete generalization to subadditive processes of Birkhoff's theorem, when the latter is formulated in terms of the additive process (1.1.8).

THEOREM 1. *If* $\mathbf{x}$ *is a subadditive process, then the finite limit*

$$\xi = \lim_{t\to\infty} x_{0t}/t \tag{1.2.1}$$

exists with probability one and in mean, and

$$\mathrm{E}(\xi) = \gamma . \tag{1.2.2}$$

In general, the random variable ξ may be non-degenerate. If $\mathscr{I}$ denotes the σ-field of events defined in terms of $\mathbf{x}$ and invariant under the shift $\mathbf{x} \to \mathbf{x}'$, where

$$x'_{st} = x_{s+1,t+1} , \tag{1.2.3}$$

then ξ is an $\mathscr{I}$-measurable random variable, with the explicit representation

$$\xi = \lim_{t\to\infty} t^{-1}\mathrm{E}(x_{0t} \mid \mathscr{I}) . \tag{1.2.4}$$

Thus, if $\mathscr{I}$ is trivial (as may in many particular cases be proved by a zero-one law), we have $\xi = \gamma$ almost surely, so that ξ may be replaced by the constant γ in (1.2.1).

The proofs of these results can be found in [8], as can that of a "maximal ergodic theorem":

$$\mathrm{E}(x_{01} \mid x_{0t} \geqq 0 \ \text{ for some } \ t \geqq 1) \geqq 0 . \tag{1.2.5}$$

Unfortunately, the asymmetry of (1.1.1) means that the maximal ergodic theorem does not at once imply (as it does in the additive case) the more important Theorem 1. The proof of that theorem depends rather on a decomposition

$$x_{st} = y_{st} + z_{st} , \tag{1.2.6}$$

in which $\mathbf{y}$ is an additive process with $\mathrm{E}(y_{01}) = \gamma$, and $\mathbf{z}$ is a nonnegative subadditive process with $\gamma(\mathbf{z}) = 0$.

It is useful to note that the condition S_3 can for some purposes be weakened to:

S_3'. $\quad E(x_{01}^+) < \infty$.

Note that by (1.1.1) this implies that

$$\mathrm{E}(x_{st}^+) < \infty \tag{1.2.7}$$

for all $s < t$. Weakening S_3 to S_3' admits two new possibilities, first by that g_t is finite for all t but that

$$g_t/t \to -\infty \qquad (t \to \infty) \tag{1.2.8}$$

and secondly that

$$g_t = -\infty \tag{1.2.9}$$

for some (and then for all sufficiently large) t.

THEOREM 2. *If* $\mathbf{x}$ *satisfies* S_1, S_2 *and* S_3' *but not* S_3, *then the limit* (1.2.1) *exists with probability one in* $-\infty \leqq \xi < \infty$, *and*

$$\mathrm{E}(\xi) = -\infty . \tag{1.2.10}$$

PROOF. It is trivial to check that $\mathbf{x}^{(N)}$ is a subadditive process, where

$$x_{st}^{(N)} = \max(x_{st}, -N(t-s))$$

and N is any positive integer. Hence Theorem 1 shows that the finite limit

$$\xi^{(N)} = \lim_{t\to\infty} x_{0t}^{(N)}/t$$

exists for all N, with probability one. Since

$$x_{0t}^{(N)}/t = \max(x_{0t}/t, -N),$$

this means that the limit (1.2.1) exists, and that

$$\xi^{(N)} = \max(\xi, -N).$$

Hence

$$\begin{aligned}\mathbf{E}(\xi) &\leqq \mathbf{E}(\xi^{(N)}) = \gamma(\mathbf{x}^{(N)}) \\ &= \inf_{t\geqq 1} t^{-1}\mathbf{E}(x_{0t}^{(N)}) \\ &\leqq t^{-1}\mathbf{E}(x_{0t}^{(N)})\end{aligned}$$

for any N, t. Letting $N\to\infty$,

$$\mathbf{E}(\xi) \leqq t^{-1}\mathbf{E}(x_{0t}) = g_t/t,$$

and letting $t\to\infty$,

$$\mathbf{E}(\xi) \leqq -\infty,$$

which completes the proof.

1.3. *Other formulations.* Theorem 1 can, of course, be formulated in the language of ergodic theory in an obvious way. Thus, without loss of generality (cf. [1]), we may suppose that the probability space $(\Omega, \mathscr{F}, \mathbf{P})$ on which the subadditive process $\mathbf{x}$ is defined is sufficiently rich to admit a measure-preserving transformation $\theta: \Omega\to\Omega$ such that

$$x_{s+1,t+1}(\omega) = x_{st}(\theta\omega) \tag{1.3.1}$$

for all $\omega\in\Omega$, $s, t\in T$, $s<t$. Then $\mathbf{S}_2$ is automatically satisfied, and $\mathbf{S}_1$ and $\mathbf{S}_3$ can be expressed by the assumption:

$\mathbf{S}^E$. The functions $f_n = x_{0n}: \Omega\to R$ belong to $L_1(\Omega)$ and satisfy

$$f_{m+n}(\omega) \leqq f_m(\omega) + f_n(\theta^m\omega) \tag{1.3.2}$$

for all $m, n\geqq 1$ and all $\omega\in\Omega$, and

$$\int f_n\, d\mathbf{P} \geqq -An \tag{1.3.3}$$

for some constant A and all $n\geqq 1$.

Then according to Theorem 1, $\mathbf{S}^E$ implies that

$$\xi(\omega) = \lim_{n\to\infty} f_n(\omega)/n \tag{1.3.4}$$

exists for almost all ω, and that

$$\int_\Omega \left|\frac{f_n(\omega)}{n} - \xi(\omega)\right| d\mathbf{P} \to 0 \qquad (n\to\infty). \tag{1.3.5}$$

Now let Σ be the set of all functions

$$\varphi : \{(s, t); s, t = 0, 1, 2, \cdots; s < t\} \to R$$

which satisfy

$$\varphi(s, u) \leqq \varphi(s, t) + \varphi(t, u) \tag{1.3.6}$$

whenever $s < t < u$, and make Σ a measurable space by means of the smallest σ-field with respect to which the coordinate maps $\varphi \to \varphi(s, t)$ are measurable. Then, if Ω, $\mathscr{F}$, $\mathbf{P}$, θ, f_n satisfy $\mathbf{S}^E$; there is a measurable function $\mathbf{f}: \Omega \to \Sigma$ defined by

$$[\mathbf{f}(\omega)](s, t) = x_{st}(\omega) = f_{t-s}(\theta^s\omega) , \tag{1.3.7}$$

and $\mathbf{f}$ induces a probability measure

$$\Pi = \mathbf{P}\mathbf{f}^{-1} \tag{1.3.8}$$

on Σ.

If $\sigma : \Sigma \to \Sigma$ is the shift

$$(\sigma\varphi)(s, t) = \varphi(s + 1, t + 1) . \tag{1.3.9}$$

then Π is invariant under σ. Moreover, for some constant A,

$$-An \leqq \int_\Sigma \varphi(0, n)\Pi(d\varphi) < \infty \tag{1.3.10}$$

for all n. Hence the theory of subadditve processes can be regarded as the theory of probability measures on Σ, invariant under σ, which satisfy (1.3.10). Theorem 1 (whose proof in [8] depends on taking just this point of view) implies that any such measure Π is concentrated on the subset

$$\Sigma_0 = \{\varphi \in \Sigma; \lim_{n\to\infty} n^{-1}\varphi(0, n) \text{ exists}\} . \tag{1.3.11}$$

1.4. *Continuous-parameter processes.* There is no reason why the parameter set T should always consist of the nonnegative integers. If T is any subset of the real line which is closed under addition, we can define a *subadditive process on T* to be a family $\mathbf{x} = (x_{st}; s, t \in T, s < t)$ of random variables x_{st} satisfying:

$\mathbf{S}_1$. The inequality (1.1.1) holds whenever $s, t, u \in T$, $s < t < u$.

$\mathbf{S}_2$. For all $\tau \in T$, the joint distributions of $\mathbf{x}^{(\tau)} = (x_{s+\tau, t+\tau})$ are the same as those of $\mathbf{x}$.

$\mathbf{S}_3$. For all positive $t \in T$, the expectation (1.1.2) exists and satisfies (1.1.3).

Exactly as before, the finite limit

$$\gamma = \lim_{t\to\infty, t \in T} g_t/t = \inf_{t \in T, t>0} g_t/t \tag{1.4.1}$$

will exist.

Apart from $T = \{0, 1, 2, \cdots\}$, the most interesting case is that in which T consists of all nonnegative real numbers, and we therefore assume in this section that $T = [0, \infty)$. In this case $\mathbf{x}$ is called a *continuous-parameter subadditive process.*

If (1.1.1) is replaced by the corresponding equality (1.1.6), so that $\mathbf{x}$ is a continuous-parameter *additive* process, we have

$$x_{st} = x_{0t} - x_{0s} .$$

Then the process $(x_{0t};\, t > 0)$ is, by $\mathbf{S}_2$, a process with stationary increments. Reversing this argument, we can construct large classes of additive processes. For example, if $(y_t;\, t \geqq 0)$ is any stationary process with finite expectation, it is immediate that

$$x_{st} = y_t - y_s \tag{1.4.2}$$

defines a process satisfying $\mathbf{S}_1$ (with equality), $\mathbf{S}_2$ and $\mathbf{S}_3$. This example would seem to rule out the possibility of any general theorems about the *local* behavior of continuous-parameter subadditive processes, and interest therefore centres on the behavior of x_{0t} for large values of t.

It would be nice if Theorem 1 could be generalized to continuous-parameter subadditive processes, to yield the assertion that

$$\xi = \lim_{t\to\infty} x_{0t}/t \tag{1.4.3}$$

exists for all such processes. Such an assertion would obviously require an assumption of separability, but might perhaps be expected to be true without any further conditions. If true, it would have as a special case (when $\mathbf{x}$ is additive) the corollary that, for any separable process $(y_t;\, t \geqq 0)$ with stationary increments and finite expectations, the finite limit $\lim_{t\to\infty} y_t/t$ exists with probability one. Such a theorem does not (as far as I know) appear in the literature, for the excellent reason that the conclusion is false, and in a very strong sense.

THEOREM 3. *Let $\Gamma(t)$ $(t > 0)$ be any positive increasing function. Then there exists a process $(y_t;\, t \geqq 0)$ with stationary increments and finite expectations, whose sample functions have derivatives of all orders, such that*

$$\mathbf{P}\{|y_t| \leqq \Gamma(t) \text{ for all sufficiently large } t\} = 0 . \tag{1.4.4}$$

PROOF. Choose a C^∞ function ψ on R which vanishes outside the interval $(\frac{1}{4}, \frac{3}{4})$ and satisfies $0 \leqq \psi(x) \leqq \psi(\frac{1}{2}) = 1$. Let $\eta, \nu_0, \nu_1, \nu_2, \cdots$ be independent random variables, η having a uniform distribution on (0, 1), and the ν_r having the same distribution, which attaches mass $[n(n+1)]^{-1}$ to the integer next above $\Gamma(n+\frac{1}{2})$ for $n = 1, 2, \cdots$. Define

$$Y_t = \nu_n \psi[\nu_n(t-n)] \qquad (n \leqq t < n+1;\, n = 0, 1, 2, \cdots),$$
$$y_t = Y_{t+\eta} . \tag{1.4.5}$$

Then (y_t) is clearly stationary, and therefore has stationary increments, and its sample functions are of class C^∞. Moreover,

$$\begin{aligned} \mathbf{E}(|y_t|) &= \mathbf{E}(y_0) = \textstyle\int_0^1 \mathbf{E}(Y_s)\, ds \\ &= \mathbf{E} \textstyle\int_0^1 \nu_0 \psi(\nu_0 s)\, ds \\ &= \mathbf{E} \textstyle\int_0^{\nu_0} \psi(u)\, du = \int_0^1 \psi(u)\, du < \infty . \end{aligned}$$

However,

$$\begin{aligned}\mathbf{P}\{|y_t| &\leqq \Gamma(t) \text{ for all sufficiently large } t\} \\ &\leqq \mathbf{P}\{|y(n+\tfrac{1}{2}\nu_n^{-1}-\eta)| \leqq \Gamma(n+\tfrac{1}{2}\nu_n^{-1}-\eta) \text{ for all sufficiently large } n\} \\ &\leqq \mathbf{P}\{\nu_n \leqq \Gamma(n+\tfrac{1}{2}) \text{ for all sufficiently large } n\} \\ &= 0\end{aligned}$$

by the second Borel–Cantelli lemma, since the ν_n are independent and

$$\sum_{n=1}^{\infty} \mathbf{P}\{\nu_n > \Gamma(n+\tfrac{1}{2})\} \geqq \sum_{n=1}^{\infty} \sum_{r=n}^{\infty} \frac{1}{r(r+1)} = \infty .$$

Hence (1.4.4) is established, and the proof is complete.

To state a condition under which Theorem 1 can be generalized, define the oscillation of the subadditive process $\mathbf{x}$ on an interval $I \subset T$ as

$$\Omega_I = \sup_{s<t,s,t\in I} |x_{st}| . \tag{1.4.6}$$

It is easy to see that, if

$$\mathbf{E}(\Omega_I) < \infty \tag{1.4.7}$$

is true for any non-degenerate interval I, it is true for every bounded interval.

THEOREM 4. *If* $\mathbf{x}$ *is a separable continuous-parameter subadditive process satisfying* (1.4.7), *then the limit* (1.4.3) *exists with probability one and in mean, and* $\mathbf{E}(\xi) = \gamma$.

PROOF. Without loss of generality take $I = [0, 1]$, and consider the discrete-parameter subadditive process

$$\mathbf{x}^{\mathrm{D}} = (x_{st};\ s, t = 0, 1, 2, \cdots, s < t) .$$

Then $\gamma(\mathbf{x}^{\mathrm{D}}) = \lim_{n\to\infty} n^{-1}\mathbf{E}(x_{0n}) = \gamma$, and Theorem 1 shows that the limit

$$\xi = \lim_{n\to\infty} n^{-1}x_{0n} \tag{1.4.8}$$

(as n increases through the integers) exists with probability one and in mean, and that $\mathbf{E}(\xi) = \gamma$. For any t, let n be the integer part of t, and note that

$$x_{0,n+1} - x_{t,n+1} \leqq x_{0t} \leqq x_{0n} + x_{nt} ,$$

so that

$$x_{0,n+1} - \Omega_{[n,n+1]} \leqq x_{0t} < x_{0n} + \Omega_{[n,n+1]} .$$

Then (1.4.3) will follow from (1.4.8) if we can show that

$$\lim_{n\to\infty} n^{-1}\Omega_{[n,n+1]} = 0 \tag{1.4.9}$$

with probability one and in mean.

Using $\mathbf{S}_2$ and separability, the distribution of $\Omega_{[n,n+1]}$ is the same as that of Ω_I, so that

$$\mathbf{E}(n^{-1}\Omega_{[n,n+1]}) = \mathbf{E}(n^{-1}\Omega_I) \to 0$$

and (1.4.9) holds in mean. For any $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbf{P}(\Omega_{[n,n+1]} > \varepsilon n) &= \sum_{n=1}^{\infty} \mathbf{P}(\Omega_I > \varepsilon n) \\ &\leqq \varepsilon^{-1}\mathbf{E}(\Omega_I) < \infty ,\end{aligned}$$

and the Borel–Cantelli lemma proves (1.4.9) with probability one. Hence the proof is complete.

Comparing the proof with the counterexample of Theorem 3, we see that the process (1.4.5), divided by t, is an instance of the phenomenon discovered in [7]; a process with continuous sample functions whose discrete skeletons all converge to zero with probability one, but which does not itself converge to zero.

The continuous-parameter ergodic theorem is usually stated (for instance in [1]) for additive processes of the form

$$x_{st} = \int_s^t u(\tau)\, d\tau \,, \tag{1.4.10}$$

where u is a measurable stationary process with finite expectation. For such a process (1.4.7) is immediate, since

$$\Omega_I \leqq \int_I |u(\tau)|\, d\tau \,,$$

so that the ergodic theorem for such processes is a corollary of Theorem 4. Notice however that (1.4.7) does not necessarily imply continuity of x_{st} in s or t (since Ω_I is not required to be small) and the theorem applies to many discontinuous subadditive and additive processes.

2. Applications.

2.1. *Percolation problems.* The situation which led Hammersley and Welsh to formulate the notion of a subadditive process was of the following type. Let $\mathscr{G}$ be a connected graph, and suppose that with each edge $e = (v, v')$ of the graph there is associated a positive random variable $u_e = u(v, v')$, with finite mean, the u_e for distinct edges e being independent. If v and v' are any two vertices, and $p = (v_0 = v, v_1, v_2, \cdots, v_k = v')$ is a path in $\mathscr{G}$ from v to v', define

$$U_p = \sum_{r=1}^k u(v_{r-1}, v_r) \,, \tag{2.1.1}$$

and denote by $U(v, v')$ the infimum of U_p over all paths p from v to v'. The properties of the random variable $U(v, v')$ are of great importance in various problems of percolation theory (for details of which we refer to [4]), and their calculation is usually of great difficulty.

The graphs of interest usually have a certain regularity of structure, at least to the extent of possessing interesting isomorphisms. By an isomorphism, we mean a function φ from the set of vertices of $\mathscr{G}$ onto itself, such that $\varphi(e) = (\varphi(v), \varphi(v'))$ is an edge of $\mathscr{G}$ if and only if $e = (v, v')$ is. If φ has this property, and if v_0 is any vertex, define a sequence (v_n) of vertices recursively by

$$v_n = \varphi(v_{n-1}) \qquad (n = 1, 2, \cdots) \,. \tag{2.1.2}$$

Then the random variables

$$x_{st} = U(v_s, v_t) \qquad (s, t = 0, 1, 2, \cdots ; s < t) \tag{2.1.3}$$

clearly satisfy (1.1.1). If the random variables are such that $u_{\varphi(e)}$ has the same distribution as u_e, then

$$(x_{s+1,t+1}) = (U(\varphi(v_s), \varphi(v_t)))$$

has the same distributions as (x_{st}), so that $\mathbf{S}_1$ and $\mathbf{S}_2$ are both satisfied. Moreover, $\mathbf{S}_3$ holds, since for any path p from v_0 to v_n,

$$0 \leqq \mathbf{E}(x_{0n}) \leqq \mathbf{E}(U_p) \leqq \sum_{e \in p} \mathbf{E}(u_e) < \infty .$$

Hence the theory of Sub-section 1.1 is directly applicable, and we can conclude that the limit

$$\xi = \lim_{n\to\infty} n^{-1} U(v_0, v_n) \tag{2.1.4}$$

exists with probability one and in mean. The fact that $\mathbf{E}(\xi) = \gamma$ is of rather little use, since there seems in general to be no way of calculating the constant γ exactly. However, as was pointed out in a special case in [8], it is often possible to replace ξ by γ in (2.1.4) by invoking a zero-one law. For example, if φ is such that, for every edge e, and any finite set F of edges of $\mathscr{G}$, the sequence (e_n) defined by

$$e_0 = e , \qquad e_n = \varphi(e_{n-1})$$

remains ultimately outside F, then the invariant σ-field $\mathscr{I}$ will be trivial, so that $\mathbf{P}(\xi = \gamma) = 1$.

2.2. *Products of random matrices.* A problem attacked by a number of authors in different contexts is the following. Let $Y_1, Y_2, \cdots$ be random $(k \times k)$ matrices, and define

$$X_n = Y_1 Y_2 \cdots Y_n . \tag{2.2.1}$$

What can be said about the product X_n for large values of n? A profound contribution was made by Furstenberg and Kesten [2] who proved among other results (a slight variant of) the following theorem. As will be seen from the proof given here, the result is a simple corollary of Theorem 1.

THEOREM 5. *Suppose that the elements of the matrices Y_n are strictly positive, and that their logarithms have finite expectations. Suppose also that the sequence (Y_n) is stationary. Then the finite limit*

$$\lambda = \lim_{n\to\infty} n^{-1} \log [X_n]_{ij} \tag{2.2.2}$$

exists with probability one and in mean, and does not depend on i or j.

(The notation $[\cdot]_{ij}$ stands for the (i, j)th element of a matrix.)

PROOF. If

$$z_{st} = [Y_{s+1} Y_{s+2} \cdots Y_t]_{11}$$

then, for $s < t < u$,

$$z_{su} = \sum_{j=1}^{k} [Y_{s+1} Y_{s+2} \cdots Y_t]_{1j} [Y_{t+1} Y_{t+2} \cdots Y_u]_{j1} \geqq z_{st} z_{tu} ,$$

so that

$$x_{st} = -\log z_{st}$$

satisfies $\mathbf{S}_1$ of Sub-section 1.1. Since (Y_n) is stationary, $\mathbf{S}_2$ holds, and by assumption x_{st} has finite expectation g_{t-s}.

For any matrix A, write

$$||A|| = \max_i \sum_j |[A]_{ij}|$$

for the ℓ_1-norm. Then

$$\begin{aligned} \mathbf{E}\{\log ||Y_1||\} &= \mathbf{E}\{\max_i \log \sum_j [Y_1]_{ij}\} \\ &\leqq \mathbf{E}\{\max_{i,j} \log k[Y_1]_{ij}\} \\ &\leqq \mathbf{E}\{\textstyle\sum_{i,j} (\log [Y_1]_{ij})^+\} + \log k < \infty , \end{aligned}$$

and hence

$$\begin{aligned} -g_n = \mathbf{E}\{\log z_{0n}\} &\leqq \mathbf{E}\{\log ||Y_1 Y_2 \cdots Y_n||\} \\ &\leqq \textstyle\sum_{r=1}^n \mathbf{E}\{\log ||Y_1||\} , \end{aligned}$$

so that

$$\inf g_n/n \geqq -\mathbf{E}\{\log ||Y_1||\} > -\infty ,$$

and $\mathbf{S}_3$ is satisfied.

Theorem 1 may therefore be applied to $\mathbf{x}$ to show that the limit (2.2.2) exists when $i = j = 1$. To deal with the other values of (i, j), note first that

$$[X_n]_{ij} \geqq [Y_1]_{i1} z_{1,n-1} [Y_n]_{1j} ,$$

so that with probability one

$$\begin{aligned} \liminf_{n\to\infty} n^{-1} \log [X_n]_{ij} \geqq \liminf_{n\to\infty} n^{-1} \log [Y_1]_{i1} + \lambda \\ + \liminf_{n\to\infty} n^{-1} \log [Y_n]_{1j} . \end{aligned}$$

The finiteness of $\mathbf{E}\{\log [Y_n]_{1j}\}$ shows (as in the proof of Theorem 4) that the final term is zero, so that

$$\mathbf{P}\{\liminf n^{-1} \log [X_n]_{ij} \geqq \lambda\} = 1 . \tag{2.2.3}$$

Similarly, the inequality

$$z_{0,n+2} \geqq [Y_1]_{1i} [X_n']_{ij} [Y_{n+2}]_{j1} ,$$

where $X_n' = Y_2 Y_3 \cdots Y_{n+1}$, shows that

$$\mathbf{P}\{\limsup n^{-1} \log [X_n']_{ij} \leqq \lambda\} = 1 ,$$

and stationarity allows us to deduce that

$$\mathbf{P}\{\limsup n^{-1} \log [X_n]_{ij} \leqq \lambda\} = 1 .$$

Comparing this with (2.2.3) shows that (2.2.2) holds with probability one. The proof of convergence in mean is exactly similar, and the theorem is therefore proved.

It is not difficult to extend the analysis to cover various cases in which some of the elements of the matrices Y_n may vanish, or (using Theorem 2) where some of the expectations of logarithms are infinite.

If the matrices Y_n are of infinite order, the argument carries through except that the manipulations of norms may not be available to show that $\gamma > -\infty$. But in many problems this can be proved in other ways. For example, if the Y_n are infinite stochastic matrices, $z_{st} \leqq 1$, so that $\gamma \geqq 0$.

2.3. *Random products in Banach algebras.* At the core of the argument of the last section is the supermultiplicative property of the diagonal elements of positive matrices:

$$[AB]_{11} \geqq [A]_{11}[B]_{11} . \tag{2.3.1}$$

A dual analysis can be based on inequalities expressing subadditivity of matrix norms:

$$||AB|| \leqq ||A|| \cdot ||B|| . \tag{2.3.2}$$

Such an analysis carries through in much more general contexts, as the next theorem illustrates.

THEOREM 6. *Let (Y_n) be a stationary sequence taking values in a (real or complex) Banach algebra $\mathscr{B}$, and suppose that*

$$\mathbf{E}\{(\log ||Y_1||)^+\} < \infty . \tag{2.3.3}$$

Then

$$\xi = \lim_{n\to\infty} n^{-1} \log ||Y_1 Y_2 \cdots Y_n|| \tag{2.3.4}$$

exists in $-\infty \leqq \xi < \infty$ with probability one, and

$$\mathbf{E}(\xi) = \lim_{n\to\infty} n^{-1}\mathbf{E}\{\log ||Y_1 Y_2 \cdots Y_n||\} . \tag{2.3.5}$$

PROOF. If

$$x_{st} = \log ||Y_{s+1} Y_{s+2} \cdots Y_t|| ,$$

then (2.3.2) implies that $\mathbf{x} = (x_{st})$ satisfies $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3'$. The result then follows from Theorems 1 and 2.

If the Y_n were all equal to a nonrandom element y of $\mathscr{B}$, then e^ξ would be the spectral radius of y. Thus e^ξ could be regarded as a stochastic spectral radius for the sequence (Y_n). In particular, if the Y_n are independent, and if π denotes their common probability distribution, then e^ξ is a constant $\rho(\pi)$ depending only on π. Hence we have a way of defining a "spectral radius" $\rho(\pi)$ for a probability measure π on $\mathscr{B}$.

If $\mathscr{B}$ is the algebra of $(k \times k)$ matrices, endowed with any matrix norm, then Theorem 6 reduces to another result of Furstenberg and Kesten [2]. In particular, if the elements of Y_n are positive, the limit ξ can be identified with the limit λ of Theorem 5. This identification depends critically, however, on the finite order of the matrices. For example, if $\mathscr{B}$ is the algebra of infinite matrices with finite ℓ_1-norm, then it will be true that $\lambda \leqq \xi$, but strict inequality can occur (cf. [6]).

2.4. *Random permutations.* A quite different application of subadditive ergodic theory has been made by Hammersley [3] to a problem of Ulam. Let $\mathscr{S}_n$ be the group of permutations of $\{1, 2, \cdots, n\}$, and define $l(\pi)$ for $\pi \in \mathscr{S}_n$ to be the length of the longest ascending sequence in π; the largest integer k for which there exist integers $i_1, i_2, \cdots, i_k$ with

$$1 \leqq i_1 < i_2 < \cdots < i_k \leqq n\,, \qquad \pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)\,.$$

It is required to discuss the distribution of $l(\pi)$ when π is drawn at random from a uniform distribution on $\mathscr{S}_n$, and to do this especially when n is large.

Hammersley attacks this problem by a most ingenious device. He first constructs a Poisson process Π of unit rate in the plane, and defines l_{st} for $0 < s < t$ to be the length of the longest ascending path with vertices in Π lying in the square R_{st} with vertices (s, s), (s, t), (t, s), (t, t). More explicitly, l_{st} is the largest integer k for which there exist points (x_i, y_i) $(i = 1, 2, \cdots, k)$ of Π with

$$s < x_1 < x_2 < \cdots < x_k \leqq t\,, \qquad s < y_1 < y_2 < \cdots < y_k \leqq t\,.$$

Then it is immediate that

$$l_{su} \geqq l_{st} + l_{tu} \qquad (s < t < u)\,,$$

so that $-\mathbf{l} = (-l_{st})$ satisfies (1.1.1). It is in fact easy to check that $-\mathbf{l}$ is a continuous-parameter subadditive process, that it satisfies (1.4.7), and that its invariant σ-field $\mathscr{I}$ is trivial (since the shift of Π along the diagonal is ergodic). Hence Theorem 4 implies the existence of an absolute constant c such that

$$\lim_{t\to\infty} l_{0t}/t = c\,, \tag{2.4.1}$$

with probability one.

Now let $\tau(n)$ be the smallest value of t for which the square R_{0t} contains n points of Π. By the strong law for the Poisson process,

$$\lim_{n\to\infty} n/\tau(n)^2 = 1$$

with probability one, so that (2.4.1) implies that

$$\lim_{n\to\infty} n^{-\frac{1}{2}} l_{0\tau(n)} = c \tag{2.4.2}$$

with probability one.

In terms of Π, we can define a random element π_n of $\mathscr{S}_n$ as follows. Let the n points of Π in $R_{0\tau(n)}$ be written as (x_i, y_i), where $0 < x_1 < x_2 < \cdots < x_n \leqq \tau(n)$. Then π_n is to be the unique $\pi \in \mathscr{S}_n$ such that

$$y_{\pi(1)} < y_{\pi(2)} < \cdots < y_{\pi(n)}\,.$$

It is clear that

$$l_{0\tau(n)} = l(\pi_n)\,,$$

and hence (2.4.2) shows that

$$\lim_{n\to\infty} n^{-\frac{1}{2}} l(\pi_n) = c \tag{2.4.3}$$

with probability one.

The properties of the Poisson process imply that, for each n, the distribution of π_n is uniform over $\mathscr{S}_n$. Since convergence with probability one implies convergence in probability, and since this latter property is an attribute of the marginal distributions of the π_n alone, it follows that (2.4.3) holds in probability whenever a sequence (π_n) with $\pi_n \in \mathscr{S}_n$ has the property that each π_n is uniformly distributed.

THEOREM 7 (Hammersley). *Let π_n be a random permutation uniformly distributed over $\mathscr{S}_n$. Then, as $n \to \infty$, $n^{-\frac{1}{2}} l(\pi_n)$ converges in probability to the absolute constant c.*

The obvious question is: what is the value of c? Hammersley in [3] shows that c must satisfy $\frac{1}{2}\pi \leqq c \leqq e$, and his arguments may be refined to give the following bounds. (Dr. Hammersley tells me that he now has a quite different technique which lends support to the hypothesis that $c = 2$.)

THEOREM 8. *The constant c in (2.4.3) satisfies*

$$(8/\pi)^{\frac{1}{2}} \leqq c \leqq \beta \,, \tag{2.4.4}$$

where $\beta = \delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}}$, and δ is the unique positive root of

$$\log(1 + \delta) = \frac{2\delta}{1 + \delta} \,.$$

Thus

$$1.59 < c < 2.49 \,. \tag{2.4.5}$$

PROOF. Choose a sequence (x_r, y_r) of points of Π as follows: (x_1, y_1) is the point of Π with the smallest value of $x + y$ subject to $x \geqq 0$, $y \geqq 0$, and for $r \geqq 2$, (x_r, y_r) is the point of Π with the smallest value of $x + y$ subject to $x > x_{r-1}$, $y > y_{r-1}$. Then

$$0 < x_1 < x_2 < \cdots \,, \qquad 0 < y_1 < y_2 < \cdots \,,$$

so that, if $t(n) = \max(x_n, y_n)$,

$$l_{0t(n)} \geqq n \,.$$

Now the differences $x_r - x_{r-1}$ are independent and identically distributed, with mean

$$\int_0^\infty \int_0^\infty x e^{-\frac{1}{2}(x+y)^2} \, dx \, dy = (\pi/8)^{\frac{1}{2}} \,.$$

Hence by the strong law,

$$\lim_{n\to\infty} x_n/n = (\pi/8)^{\frac{1}{2}} \,,$$

and likewise

$$\lim_{n\to\infty} y_n/n = (\pi/8)^{\frac{1}{2}} \,,$$

so that

$$\lim_{n\to\infty} t(n)/n = (\pi/8)^{\frac{1}{2}} \,,$$

with probability one. Therefore

$$c = \lim_{n\to\infty} l_{0t(n)}/t(n) \geqq \lim_{n\to\infty} n/t(n) = (8/\pi)^{\frac{1}{2}} > 1.59 \,.$$

To prove the reverse inequality, let k be any positive integer, let π be drawn from a uniform distribution on $\mathscr{S}_n$, and let ν be the number of sequences $i_1 < i_2 < \cdots < i_k \leqq n$ with $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. Then

$$\begin{aligned} \mathbf{E}(\nu) &= \textstyle\sum_{i_1<i_2<\cdots<i_k} \mathbf{P}\{\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)\} \\ &= \tbinom{n}{k}(k!)^{-1} . \end{aligned}$$

However, if $l(\pi) \geqq k$, there is an ascending sequence of length $l(\pi)$, and each subsequence of length k is ascending, so that

$$\nu \geqq \tbinom{l(\pi)}{k} .$$

In particular, since $\binom{\cdot}{k}$ is non-decreasing,

$$\mathbf{P}\{l(\pi) \geqq r\} \leqq \tbinom{n}{k}[k!\tbinom{r}{k}]^{-1} , \qquad (r \geqq k) . \tag{2.4.7}$$

Now let $n \to \infty$, $k \to \infty$, $r \to \infty$ in such a way that

$$k \sim \alpha n^{\frac{1}{2}} , \qquad r \sim \beta n^{\frac{1}{2}}$$

where $0 < \alpha < \beta < \infty$.

Then Stirling's formula shows easily that

$$\mathbf{P}\{l(\pi) \geqq \beta n^{\frac{1}{2}}\} \to 0$$

as $n \to \infty$ if

$$2\alpha + (\beta - \alpha) \log (\beta - \alpha) - \alpha \log \alpha - \beta \log \beta < 0 . \tag{2.4.8}$$

Hence $c \leqq \beta$ if α can be chosen to satisfy (2.4.8). It is easy to check that the infimum of the values of β with this property is the number β defined in the statement of the theorem, and the proof is complete.

If the random permutations π_n happen to be defined on the same probability space, the question arises whether Theorem 7 can be strengthened to yield convergence with probability one. Equation (2.4.3) shows that this can be done when the π_n are constructed from the Poisson process Π, but this is by no means the only way in which they might arise,

To take one example, let z_n ($n = 1, 2, 3, \cdots$) be independent random variables with a common continuous distribution (whose form does not affect the problem). Let π_n be the unique permutation $\pi \in \mathscr{S}_n$ such that

$$z_{\pi(1)} < z_{\pi(2)} < \cdots < z_{\pi(n)} . \tag{2.4.9}$$

Then π_n is obviously uniformly distributed over $\mathscr{S}_n$, so that $n^{-\frac{1}{2}}l(\pi_n) \to c$ in probability; it is not known whether convergence takes place with probability one.

The obvious ways of defining the π_n on a common probability space all have the property that $L(n) = l(\pi_n)$ is non-decreasing, and when this is true one way of arguing might be as follows. Let κ be any positive constant, and let $r = r(n)$ be the largest integer with $r^\kappa \leqq n$. Then

$$L(r^\kappa) \leqq L(n) \leqq L((r + 1)^\kappa) ,$$

where L is defined for nonintegral arguments by linear interpolation, so that

$$\limsup L(r^\kappa)r^{-\frac{1}{2}\kappa} \leqq \limsup L(n)n^{-\frac{1}{2}} \leqq \limsup L((r + 1)^\kappa)(r + 1)^{-\frac{1}{2}\kappa} ,$$

and the same with "lim sup" replaced by "lim inf." Since

$$(r+1)^{-\frac{1}{2}\kappa} \sim r^{-\frac{1}{2}\kappa},$$

this shows that

$$\mathbf{P}\{\lim_{n\to\infty} L(n)n^{-\frac{1}{2}} = c\} = 1 \tag{2.4.10}$$

will follow from

$$\mathbf{P}\{\lim_{r\to\infty} L(r^\kappa)r^{-\frac{1}{2}\kappa} = c\} = 1 .$$

Hence, by the Borel–Cantelli lemma, (2.4.10) will follow if we can prove that

$$\sum_{r=1}^{\infty} \mathbf{P}\{|L(r^\kappa) - cr^{\frac{1}{2}\kappa}| > \varepsilon r^{\frac{1}{2}\kappa}\} < \infty .$$

In particular, if

$$\mathbf{P}\{|L(n) - cn^{\frac{1}{2}}| > \varepsilon n^{\frac{1}{2}}\} \leqq An^{-\delta} \tag{2.4.11}$$

for positive constants A, δ (perhaps depending on ε), then almost sure convergence will follow (taking $\kappa > \delta$).

Exactly similar arguments yield one-sided inequalities. For example, the explicit bounds which follow from (2.4.7) by Stirling's formula show that, so long as the sequence (π_n) is defined so that $l(\pi_n) \leqq l(\pi_{n+1})$, then

$$P\{\limsup_{n\to\infty} n^{-\frac{1}{2}}l(\pi_n) \leqq \beta < 2.49\} = 1 . \tag{2.4.12}$$

3. Problems.

3.1. *The constant* γ. Pride of place among the unsolved problems of subadditive ergodic theory must go to the calculation of the constant γ (which in the presence of a zero-one law is the same as the limit ξ). In none of the applications described here is there an obvious mechanism for obtaining an exact numerical value, and indeed this usually seems to be a problem of some depth.

Of course, it may well be possible (as in the argument of Sub-section 2.4) to derive bounds for γ. For example, (1.1.5) shows that

$$\gamma \leqq g_n/n \tag{3.1.1}$$

for any n, and for small values of n it may be possible explicitly to compute this bound. In the other direction, it should be noted that, if $\mathbf{x}$ is a given subadditive process, and if we can construct an additive process with

$$x_{st} \geqq y_{st}, \tag{3.1.2}$$

then

$$\gamma(\mathbf{x}) \geqq \gamma(\mathbf{y}) = \mathbf{E}(y_{01}) . \tag{3.1.3}$$

The representation (1.2.6) shows that this lower bound is potentially sharp, though it may not be possible explicitly to compute an additive process $\mathbf{y}$ attaining equality in (3.1.3).

3.2. *Rate of convergence.* How fast does the convergence of x_{0t}/t to ξ take place? In the additive case it is of course possible to assert, under suitable conditions, that

$$t^{-\frac{1}{2}}(x_{0t} - \xi t)$$

has an asymptotically normal distribution and is of order $(\log \log t)^{\frac{1}{2}}$ as $t \to \infty$. Presumably such conclusions hold also for subadditive processes which do not deviate too far from additivity. No significant general results in this direction are known, though there some first steps in [4].

3.3. *Characterization problems.* An interesting feature of Theorem 1 is that it derives a result about the one-parameter process (x_{0t}) from assumptions about the two-parameter process (x_{st}). This suggests the question: given a process $(x_t; t \geqq 1)$, under what conditions does there exist a subadditive process (x_{st}) with $x_t = x_{0t}$ for all t?

A similar question is: given a general process (x_{st}), under what conditions are there subadditive processes $\mathbf{x}'$ and $\mathbf{x}''$ with $x_{st} = x'_{st} - x''_{st}$? Likewise, given a one-parameter process (x_t), when do there exist subadditive processes $\mathbf{x}'$ and $\mathbf{x}''$ such that $x_t = x'_{0t} - x''_{0t}$?

3.4. *Structure problems.* If, for any i in an arbitrary index set I, we have an additive process $\mathbf{a}^i$ (defined on a probability space independent of i), then

$$x_{st} = \sup_{i \in I} a^i_{st} \tag{3.4.1}$$

defines a subadditive process so long as $\mathbf{E}(x_{01}) < \infty$. Conversely, if $\mathbf{x}$ is a subadditive process, does it admit a representation (3.4.1) as a supremum of additive processes? More concretely, if $\{\mathbf{a}^i\}$ consists of all the additive processes with $a_{st} \leqq x_{st}$ for all $s < t$, is (3.4.1) true? (Certainly, if $\hat{\mathbf{x}}_{st}$ denotes the right hand side of (3.4.1), $\hat{\mathbf{x}}$ is a subadditive process with $\hat{\mathbf{x}} \leqq \mathbf{x}$, $\gamma(\hat{\mathbf{x}}) = \gamma(\mathbf{x})$.) Questions like this are crucial to an understanding, for instance, of the non-uniqueness of the decomposition (1.2.6) (cf. [8]).

When studying such problems from the point of view of ergodic theory, one encounters questions like the following. With each point t of the unit interval is associated (in a reasonably measurable way) a subset E_t of $[0, 1]$. When does there exist a measure-preserving transformation T of $[0, 1]$ with $Tt \in E_t$ for almost all t? An obvious necessary condition is that, for any Borel subset B of $[0, 1]$, the measure of $\bigcup \{E_t; t \in B\}$ should not be less than that of B. By analogy with Hall's theorem on distinct representatives, one might expect this condition also to be sufficient; I am indebted to Professor W. Parry for pointing out that life is more complicated than I had expected.

3.5. *Independent subadditive processes.* Hammersley and Welsh noted that some significant subadditive processes have the property that, for $0 = t_0 < t_1 < t_2 < \cdots < t_n$, the random variables

$$x_{t_{r-1}t_r} \qquad (r = 1, 2, \cdots, n) \tag{3.5.1}$$

are independent. For example, in the problems of Sub-sections 2.2 and 2.3, this will be the case when the elements Y_n are independent. An immediate consequence is that $\mathscr{I}$ is trivial, so that $\xi = \gamma$. But there ought surely to be more weighty consequences of such a strong condition.

Let **x** be a nonnegative subadditive process for which the variables (3.5.1) are independent. For any $\theta > 0$, $s < t < u$,

$$\begin{aligned} \mathrm{E}\{\exp(-\theta x_{su})\} &\geqq \mathrm{E}\{\exp(-\theta x_{st})\exp(-\theta x_{tu})\} \\ &= \mathrm{E}\{\exp(-\theta x_{st})\}\mathrm{E}\{\exp(-\theta x_{tu})\}\,. \end{aligned}$$

Moreover, $\mathrm{E}\{\exp(-\theta x_{st})\}$ depends only on $(t - s)$, whence it follows that

$$\Psi(\theta) = \lim_{n\to\infty} n^{-1} \log \mathrm{E}\{\exp(-\theta x_{0n})\} \tag{3.5.2}$$

exists. What properties has the function Ψ, and to what extend does it enshrine information about the process **x**?

These are only a few of the questions which one might ask about the theory described in this paper. Other problems may be found in [4], and yet others will occur to the reader.

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