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WEAK CONVERGENCE OF MULTIDIMENSIONAL EMPIRICAL PROCESSES FOR STATIONARY ϕ-MIXING PROCESSES[1]

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For a stationary ϕ-mixing sequence of stochastic $p(\geqq 1)$-vectors, weak convergence of the empirical process (in the J_1-topology on $D^p[0, 1]$) to an appropriate Gaussian process is established under a simple condition on the mixing constants $\{\phi_n\}$. Weak convergence for random number of stochastic vectors is also studied. Tail probability inequalities for Kolmogorov-Smirnov statistics are provided.

1. Introduction. Let $\{\mathbf{X}_i = (X_{i1}, \cdots, X_{ip})', -\infty < i < \infty\}$ be a stationary ϕ-mixing sequence of stochastic vectors defined on a probability space $(\Omega, \mathscr{A}, P)$ with each $\mathbf{X}_i$ having (marginally) a continuous distribution function (df) $F(\mathbf{x})$, $\mathbf{x} \in R^p$, the $p(\geqq 1)$ dimensional Euclidean space. Thus, if $\mathscr{M}_{-\infty}^k$ and $\mathscr{M}_{k+n}^{\infty}$ be respectively the σ-fields generated by $\{\mathbf{X}_i, i \leqq k\}$ and $\{\mathbf{X}_i, i \geqq k+n\}$, and if, $A \in \mathscr{M}_{-\infty}^k$ and $B \in \mathscr{M}_{k+n}^{\infty}$, then for all k: $-\infty < k < \infty$,

$$|P(A \cap B) - P(A)P(B)| \leqq \phi_n P(A)\,, \qquad \phi_n \geqq 0, \text{ for all nonnegative } n\,, \tag{1.1}$$

where ϕ_n is $\downarrow$ in n and $\lim_{n\to\infty} \phi_n = 0$. We denote the marginal df of X_{ij} by $F_{[j]}$, let $Y_{ij} = F_{[j]}(X_{ij}), j = 1, \cdots, p$; $\mathbf{Y}_i = (Y_{i1}, \cdots, Y_{ip})'$, $-\infty < i < \infty$, and denote the df of $\mathbf{Y}_i$ by

$$G(\mathbf{t}) = P\{\mathbf{Y}_i \leqq \mathbf{t}\}\,, \qquad \mathbf{t} \in E^p \quad (\text{so that } G_{[j]}(t) = P\{Y_{ij} \leqq t\} = t; \; 0 \leqq t \leqq 1; j = 1, \cdots, p)\,, \tag{1.2}$$

where $E^p = \{\mathbf{t}: \mathbf{0} \leqq \mathbf{t} \leqq \mathbf{1}\}$ is the p-dimensional unit cube, $\mathbf{0} = (0, \cdots, 0)$, $\mathbf{1} = (1, \cdots, 1)$ and $\mathbf{a} \leqq \mathbf{b}$ means that $a_j \leqq b_j$, $1 \leqq j \leqq p$. Note that $G(\mathbf{t}) = 0$ if at least one coordinate of $\mathbf{t}$ is 0. For a sample $\mathbf{X}_1, \cdots, \mathbf{X}_n$ of size n, the empirical df for $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ is defined by

$$G_n(\mathbf{t}) = n^{-1} \textstyle\sum_{i=1}^n c(\mathbf{t} - \mathbf{Y}_i)\,, \qquad \mathbf{t} \in E^p,\ n \geqq 1\,, \tag{1.3}$$

where $c(\mathbf{u}) = 1$ iff $\mathbf{u} \geqq \mathbf{0}$, and 0, otherwise. Also, $G_n(\mathbf{t}) = 0$, when at least one coordinate of $\mathbf{t}$ is 0. The empirical process $W_n = \{W_n(\mathbf{t}), \mathbf{t} \in E^p\}$ is then defined by

$$W_n(\mathbf{t}) = n^{\frac{1}{2}}[G_n(\mathbf{t}) - G(\mathbf{t})]\,, \qquad \mathbf{t} \in E^p,\ n \geqq 1\,. \tag{1.4}$$

For every $n \geqq 1$, the process W_n belongs to the space $D^p[0, 1]$ of all real valued

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functions on E^p with no discontinuities of the second kind, and with $D^p[0, 1]$, we associate the (extended) Skorokhod J_1-topology. For excellant expositions of weak convergence of processes on $D^p[0, 1]$, we may refer to [1], [5], [12]. Also, for $p = 1$, a detailed account is given in Billingsley (1968).

When the $\mathbf{X}_i$ are independent and identically distributed (i.i.d.), i.e., $\phi_n = 0$ for $n \geqq 1$, W_n converges in distribution (in the J_1-topology on $D^p[0, 1]$) to an appropriate Gaussian process ([1], [5]) (for $p = 1$, Brownian bridge). For ϕ-mixing processes and $p = 1$, weak convergence of W_n to an appropriate Gaussian function has been studied by Billingsley ((1968) page 197) and Sen (1971). Our first objective is to show that for general $p \geqq 1$, the weak convergence of W_n to an appropriate Gaussian function holds under identical conditions.

For i.i.d. random variables, Pyke (1968) has studied the weak convergence of empirical processes for random sample sizes. Related results for multidimensional empirical processes are treated in [1], [11], [13]. Our second objective is to extend these results for ϕ-mixing processes.

For i.i.d. random vectors, Kiefer (1961) has obtained an exponential bound for the tail probability of the Kolmogorov–Smirnov statistics. Bounds for the tail probability of the Kolmogorov–Smirnov statistics for multivariate ϕ-mixing processes are studied in the last section of the paper.

2. Weak convergence of empirical processes. Let us write

$$A_k(\phi) = \textstyle\sum_{n=0}^{\infty} (n+1)^k \phi_n^{\frac{1}{2}} \quad \text{and} \quad A_k(\phi^2) = \textstyle\sum_{n=0}^{\infty} (n+1)^k \phi_n, \quad k \geqq 0. \tag{2.1}$$

Then, $[A_k(\phi) < \infty] \Rightarrow [A_q(\phi) < \infty]$, $\forall q \leqq k$, and $[A_q(\phi^2) < \infty]$, $\forall q \leqq 2k$. Consider now a p-dimensional Gaussian process $W = \{W(\mathbf{t}), \mathbf{t} \in E^p\}$, where $EW(\mathbf{t}) = 0$, $\mathbf{t} \in E^p$, and for every $\mathbf{s}, \mathbf{t} \in E^p$,

$$\begin{aligned} \gamma(\mathbf{s}, \mathbf{t}) = E[W(\mathbf{s})W(\mathbf{t})] &= E\{[c(\mathbf{s} - \mathbf{Y}_1)c(\mathbf{t} - \mathbf{Y}_1)] - G(\mathbf{s})G(\mathbf{t})\} \\ &\quad + \textstyle\sum_{k=2}^{\infty} E\{c(\mathbf{s} - \mathbf{Y}_1)c(\mathbf{t} - \mathbf{Y}_k) + c(\mathbf{s} - \mathbf{Y}_k)c(\mathbf{t} - \mathbf{Y}_1) \\ &\quad - 2G(\mathbf{s})G(\mathbf{t})\}. \end{aligned} \tag{2.2}$$

Note that by Theorem 20.1 of Billingsley ((1968) page 174), the series on the right hand side (rhs) of (2.2) converges when $A_0(\phi) < \infty$. The following theorem extends Theorem 22.1 of Billingsley ((1968) page 197) to the case of $p \geqq 1$ under the conditions of Sen (1971).

THEOREM 2.1. *Under* (1.1) *and* $A_1(\phi) < \infty$, W_n *converges in law* (*in the Skorokhod* J_1-*topology on* $D^p[0, 1]$) *to a Gaussian process* W *for which* (2.2) *holds.*

PROOF. We need to establish

(a) the convergence of the finite dimensional distributions of $\{W_n\}$ to those of W, and

(b) the tightness of $\{W_n\}$.

Under $A_0(\phi) < \infty$ and (1.1), the proof of (a) follows along the same line as in

Billingsley ((1968) page 197), so that in the remaining of the proof, we only establish the tightness of $\{W_n\}$. Also, it is understood that $p \geqq 2$. Since, the usual techniques for proving the tightness of multiparameter stochastic processes [viz., Neuhaus (1971) and Bickel and Wichura (1971)] are not directly applicable for ϕ-mixing processes, we adapt a modified approach.

For general ϕ-mixing processes, it has been observed in Sen (1972a) that empirical processes behave quite smoothly for large n. This, along with the treatment of the univariate case in Billingsley ((1968) pages 198–199), suggests the following approach. First, using the basic inequality between the modulii of continuity for $C^p[0, 1]$ and $D^p[0, 1]$ spaces [cf. Billingsley (1968) page 110 and Neuhaus (1971) page 1288] and the fact that $W_n(\mathbf{t}) = 0$ when at least one coordinate of $\mathbf{t}$ is 0, it suffices to show that for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta > 0$ and an integer n_0, such that

$$P\{\omega_\delta(W_n) > \varepsilon\} < \eta\,, \qquad n \geqq n_0\,, \tag{2.3}$$

where for every $0 < \delta < 1$ and $n \geqq 1$,

$$\omega_\delta(W_n) = \sup\{|W_n(\mathbf{t}) - W_n(\mathbf{s})| : \mathbf{s}, \mathbf{t} \in E^p \text{ and } |\mathbf{t} - \mathbf{s}| < \delta\}\,. \tag{2.4}$$

Second, by a direct multiparameter extension of Theorem 8.3 of Billingsley ((1968) page 56), it suffices to show that for every $\mathbf{0} \leqq \mathbf{b}_0 \leqq \mathbf{1}$, $\varepsilon > 0$ and $\eta > 0$, there exist a δ: $0 < \delta < 1$ and an integer n_0, such that for $B = \{\mathbf{t} : \mathbf{b}_0 \leqq \mathbf{t} \leqq \mathbf{b}_0 + \delta\mathbf{1}\}$ and $n \geqq n_0$,

$$P\{\sup_{\mathbf{t}\in B} |W_n(\mathbf{t}) - W_n(\mathbf{b}_0)| > \varepsilon\} < \eta[\mu(B) + \delta^p]/2\,, \tag{2.5}$$

where $\mu(B) = P\{\mathbf{Y}_1 \in B\}$. [Note that $[\mu(A) + ||A||]/2$ (where $||A||$ is the Lebesgue measure of A) is $\leqq 1$ for every $A \in E^p$, and as $\mu(A)$ is bounded from above by any side of A, we have $\delta^p \leqq \mu(B) + \delta^p \leqq \delta + \delta^p \to 0$ as $\delta \to 0$.]

For a given $\varepsilon > 0$ and $\delta > 0$ (to be chosen later on), select n_0 so large that $\delta > n_0^{-\frac{1}{2}}\varepsilon/2p$. Let then (for $n \geqq n_0$),

$$\mathbf{b}_n(\mathbf{i}) = \mathbf{b}_0 + (n^{-\frac{1}{2}}\varepsilon/2p)\mathbf{i}\,, \qquad \mathbf{0} \leqq \mathbf{i} \leqq \mathbf{m}_n\,; \tag{2.6}$$

$$\mathbf{m}_n = m_n \cdot \mathbf{1} \quad \text{and} \quad m_n = [2p\delta n^{\frac{1}{2}}/\varepsilon] + 1\ (\geqq 1)\,. \tag{2.7}$$

Also, let

$$B(\mathbf{i}, n) = \{\mathbf{t} : \mathbf{b}_n(\mathbf{i}) \leqq \mathbf{t} \leqq \mathbf{b}_n(\mathbf{i} + \mathbf{1})\}\,, \qquad \mathbf{i} \geqq \mathbf{0}\,. \tag{2.8}$$

Note that by (1.2) and (2.6), for every $\mathbf{i} \geqq \mathbf{0}$,

$$\begin{aligned} n^{\frac{1}{2}}[G(\mathbf{b}_n(\mathbf{i}+\mathbf{1})) - G(\mathbf{b}_n(\mathbf{i}))] &\leqq \textstyle\sum_{j=1}^p n^{\frac{1}{2}}[G_{[j]}(i_j + 1) - G_{[j]}(i_j)] \\ &= \textstyle\sum_{j=1}^p (\varepsilon/2p) = \varepsilon/2\,. \end{aligned} \tag{2.9}$$

Hence, on using the fact that for $\mathbf{t} \in B(\mathbf{i}, n)$, $G_n(\mathbf{b}_n(\mathbf{i})) \leqq G_n(\mathbf{t}) \leqq G_n(\mathbf{b}_n(\mathbf{i} + \mathbf{1}))$ and $G(\mathbf{b}_n(\mathbf{i})) \leqq G(\mathbf{t}) \leqq G(b_n(\mathbf{i} + \mathbf{1}))$, we obtain by (1.4), (2.9) and a few routine steps [as in Billingsley (1968) page 199] that

$$\sup_{\mathbf{t}\in B} |W_n(\mathbf{t}) - W_n(\mathbf{b}_0)| \leqq \max_{\mathbf{0}\leqq\mathbf{i}\leqq\mathbf{m}_n} |W_n(\mathbf{b}_n(\mathbf{i})) - W_n(\mathbf{b}_0)| + \varepsilon/2\,. \tag{2.10}$$

Thus, it suffices to show that for every $\varepsilon > 0$ and $\eta > 0$, there exist a δ: $0 < \delta < 1$ and an integer n_0, such that for $n \geqq n_0$ and every $\mathbf{b}_0 \in E^p$.

$$(2.11)\qquad P\{\max_{\mathbf{0}\leqq\mathbf{i}\leqq\mathbf{m}_n} |W_n(\mathbf{b}_n(\mathbf{i})) - W_n(\mathbf{b}_0)| > \tfrac{1}{2}\varepsilon\} < \tfrac{1}{2}\eta[\mu(B) + \delta^p] .$$

Now, by (1.4) and (2.6), for every $\mathbf{i} \geqq \mathbf{0}$,

$$(2.12)\qquad W_n(\mathbf{b}_n(\mathbf{i} + \mathbf{1})) - W_n(\mathbf{b}_0) = \textstyle\sum_{\mathbf{0}\leqq\mathbf{j}\leqq\mathbf{i}} V_n(B(\mathbf{j}, n)) ;$$

$$(2.13)\qquad V_n(B) = n^{\frac{1}{2}}[\mu_n(B) - \mu(B)] ; \qquad \mu_n(B) = [\# \text{ of } \mathbf{Y}_i \in B, i = 1, \cdots, n]/n ,$$

for every $B \in E^p$, where by Lemma 2.1 of Sen (1971), under $A_1(\phi) < \infty$,

$$(2.14)\qquad E[V_n^4(B(\mathbf{i}, n))] \leqq K_\phi[\mu^2(B(\mathbf{i}, n)) + n^{-1}\mu(B(\mathbf{i}, n))] .$$

Unfortunately $\mu(B(\mathbf{i}, n))$, though bounded from above by $e/2n^{\frac{1}{2}}$, can be arbitrarily close to 0. For example, if $G(\mathbf{t})$ is degenerate on a lower dimensional space, then $\mu(B)$ may be equal to 0 for some $B \in E^p$. To overcome this difficulty, we define

$$(2.15)\qquad \lambda_n(\mathbf{i}) = \max\{\mu(B(\mathbf{i}, n)), (\varepsilon/2pn^{\frac{1}{2}})^p\} , \qquad \forall\, \mathbf{i} \geqq \mathbf{0} .$$

Now, (2.15) implies that $\lambda_n(\mathbf{i}) \geqq (\varepsilon/2pn^{\frac{1}{2}})^p$ i.e., $(2p/\varepsilon)^2\lambda_n^{2/p}(\mathbf{i}) \geqq n^{-\frac{1}{2}}$. Hence, from (2.14) we have under $A_1(\phi) < \infty$,

$$(2.16)\qquad E[V_n^4(B(\mathbf{i}, n))] \leqq K_\phi\{\lambda_n^2(\mathbf{i}) + (2p/\varepsilon)^2[\lambda_n(\mathbf{i})]^{1+2/p}\} \leqq K_{\phi,\varepsilon}[\lambda_n(\mathbf{i})]^\beta ,$$

where

$$(2.17)\qquad \beta = 1 + 2/p > 1 \quad\text{and}\quad K_{\phi,\varepsilon} \leqq K_\phi\{1 + (2p/\varepsilon)^2\} < \infty , \qquad \forall\, \varepsilon > 0 .$$

By virtue of (2.17) and Theorem 1 of Bickel and Wichura (1971), we obtain that for every $\varepsilon > 0$.

$$(2.18)\qquad P\{\max_{\mathbf{0}\leqq\mathbf{i}\leqq\mathbf{m}_n-\mathbf{1}} |S_n(\mathbf{i})| > \tfrac{1}{2}\varepsilon\} \leqq (16K^*_{\phi,\varepsilon}/\varepsilon^4)(\textstyle\sum_{\mathbf{0}\leqq\mathbf{i}\leqq\mathbf{m}_n-\mathbf{1}} \lambda_n(\mathbf{i}))^\beta ,$$

where $K^*_{\phi,\varepsilon}(< \infty)$ depends on ε through $K_{\phi,\varepsilon}$ in (2.17). Now, $\lambda_n(\mathbf{i}) \leqq \mu(B(\mathbf{i}, n)) + (\varepsilon/2pn^{\frac{1}{2}})^p$, $\forall\, \mathbf{i} \geqq 0$, so that the rhs of (2.18) is bounded by

$$(2.19)\qquad (16K^*_{\phi,\varepsilon}/\varepsilon^4)[\mu(B_n) + \delta_n^p]^\beta , \qquad \beta = 1 + 2/p > 1 ,$$

where $B_n = \{\mathbf{t} : \mathbf{b}_0 \leqq \mathbf{t} \leqq \mathbf{b}_0 + (\varepsilon/2pn^{\frac{1}{2}})\mathbf{m}_n\}$ and $\delta_n = m_n(\varepsilon/2pn^{\frac{1}{2}})$. By (2.6), $0 \leqq \mu(B_n) - \mu(B) \leqq \varepsilon/2n^{\frac{1}{2}}$ and $\delta \leqq \delta_n \leqq \delta + \varepsilon/2pn^{\frac{1}{2}}$. Thus, using the fact that $\delta_n^p \leqq \mu(B_n) + \delta_n^p \leqq \delta_n + \delta_n^p$, we obtain that for every $\eta > 0$, there exist a $\delta > 0$ and an n_0, such that

$$(2.20)\qquad \mu(B_n) + \delta_n^p \leqq \tfrac{3}{2}(\mu(B) + \delta^p) , \qquad n \geqq n_0 ,$$

$$(2.21)\qquad (16K^*_{\phi,\varepsilon}/\varepsilon^4)[\mu(B_n) + \delta_n^p]^{2/p} < \eta/3 , \qquad n \geqq n_0 ,$$

which completes the proof of (2.11). Hence the proof of Theorem 2.1 is complete.

Consider now a sequence of stochrstic processes $\{W_n^* = [W_n^*(\mathbf{t}, u), \mathbf{t} \in E^p, 0 \leqq u \leqq 1]; n \geqq 1\}$, defined on the $D^{p+1}[0, 1]$ space, where

$$(2.22)\qquad W_n^*(\mathbf{t}, u) = [nu]^{\frac{1}{2}} W_{[nu]}(\mathbf{t})/n^{\frac{1}{2}} , \qquad 0 \leqq u \leqq 1 \text{ and } \mathbf{t} \in E^p ,$$

$[s]$ being the largest integer contained in s. Also, let $W^* = [W^*(\mathbf{t}, u), \mathbf{t} \in E^p, 0 \leqq u \leqq 1]$ be a Gaussian function with $EW^*(\mathbf{t}, u) = 0$ and

$$E[W^*(\mathbf{s}, v)W^*(\mathbf{t}, u)] = \min(u, v)\gamma(\mathbf{s}, \mathbf{t}), \quad \text{for every } \mathbf{s}, \mathbf{t} \in E^p, \; 0 \leqq u, v \leqq 1, \tag{2.23}$$

where $\gamma(\mathbf{s}, \mathbf{t})$ is defined by (2.2). Then, we have the following theorem.

THEOREM 2.2. *Under* (1.1) *and* $A_1(\phi) < \infty$, W_n^* *converges in law* (*in the extended Skorokhod* J_1*-topology on* $D^{p+1}[0, 1]$ *space*) *to* W^* *for which* (2.23) *holds.*

PROOF. Here also, the convergence of the finite dimensional laws of W_n^* to those of W^* poses no problem, so we shall only prove the tightness of $\{W_n^*\}$. As in (2.3)—(2.5), it suffices to show that for every $\mathbf{b}_0 \in E^p$, $0 \leqq u_0 \leqq 1$, $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta: 0 < \delta < 1$ and an integer n_0, such that for $B^* = \{(\mathbf{t}, u): \mathbf{b}_0 \leqq \mathbf{t} \leqq (\mathbf{b}_0 + \delta\mathbf{1}) \wedge \mathbf{1}, u_0 \leqq u \leqq (u_0 + \delta) \wedge 1\}$ and $n \geqq n_0$,

$$P\{\sup_{(\mathbf{t},u)\in B^*} |W_n^*(\mathbf{t}, u) - W_n^*(\mathbf{b}_0, u_0)| > \varepsilon\} < \delta\eta[\mu(B) + \delta^p]/2, \tag{2.24}$$

where $B = \{\mathbf{t}: \mathbf{b}_0 \leqq \mathbf{t} \leqq (\mathbf{b}_0 + \delta\mathbf{1}) \wedge \mathbf{1}\}$, $\mu(B) = P\{\mathbf{Y}_1 \in B\}$ and $\mathbf{a} \wedge \mathbf{b} = (\min(a_1, b_1), \cdots, \min(a_p, b_p))$. Let now $k_n = [(\varepsilon/4)n^{\frac{1}{2}}] + 1$, and

$$b_n^*(\mathbf{i}, j) = (\mathbf{b}_0, u_0) + ((\varepsilon/4pn^{\frac{1}{2}})\mathbf{i}, n^{-\frac{1}{2}}jk_n), \quad \mathbf{0} \leqq \mathbf{i} \leqq \mathbf{m}_n^*, \; 0 \leqq j \leqq m_n^{**}, \tag{2.25}$$

where $\mathbf{m}_n^* = m_n^*\mathbf{1}$, $m_n^* = [4p\delta n^{\frac{1}{2}}/\varepsilon] + 1$ and $m_n^{**} = \max\{j: jn^{-1}k_n \leqq \delta\} + 1$. Then, by the same technique as in (2.10), we have by a few standard steps that

$$\begin{aligned} \sup_{(\mathbf{t},u)\in B^*} &|W_n^*(\mathbf{t}, u) - W_n^*(\mathbf{b}_0, u_0)| \\ &\leqq \max_{1 \leqq j \leqq m_n^{**}} \max_{\mathbf{0} \leqq \mathbf{i} \leqq \mathbf{m}_n^*} |W_n^*(b_n^*(\mathbf{i}, j)) - W_n^*(\mathbf{b}_0, u_0)| + \varepsilon/2, \end{aligned} \tag{2.26}$$

where by Lemma 2.1 of Sen (1971), under $A_1(\phi) < \infty$,

$$\begin{aligned} E[W_n^*(b_n^*(\mathbf{i}, j)) &- W_n^*(\mathbf{b}_0, u_0)]^4 \\ &\leqq K_\phi\{(n^{-1}jk_n)^2\mu^2(B(\mathbf{i}, n)) + n^{-2}jk_n\mu(B(\mathbf{i}, n))\}, \quad K_\phi < \infty, \end{aligned} \tag{2.27}$$

and $B(\mathbf{i}, n)$ is defined by (2.8). Repeating then the steps (2.14) through (2.21) and replacing $\lambda_n(\mathbf{i})$ by

$$\lambda_n(\mathbf{i}, j) = (n^{-1}jk_n)\lambda_n(\mathbf{i}) \quad \text{for} \quad 0 \leqq j \leqq m_n^{**}, \quad \mathbf{0} \leqq \mathbf{i} \leqq \mathbf{m}_n^*, \tag{2.28}$$

the proof of (2.24) follows along the same line as of the proof of (2.11). Hence, the details are omitted. □

Let now $\{N_\nu, \nu \geqq 1\}$ be a sequence of positive integer valued random variables such that as $\nu \to \infty$, $\nu^{-1}N_\nu \to \xi$, in probability, where ξ is a positive random variable defined on the same probability space $(\Omega, \mathscr{A}, P)$. Then, by virtue of Theorem 2.2, we conclude that $\{W_{N_\nu}\}$ converges weakly to W as $\nu \to \infty$.

3. Tail probability inequalities for Kolmogorov–Smirnov statistics.

For i.i.d. stochastic $p(\geqq 1)$ vectors, Kiefer (1961) has shown that for every $\varepsilon > 0$, there exists a positive $c_p(\varepsilon)$, such that for every $n > 1$ and $\lambda > 0$.

$$P\{\sup_{\mathbf{t}\in E^p} |W_n(\mathbf{t})| > \lambda\} \leqq c_p(\varepsilon)\{\exp[-(2 - \varepsilon)\lambda^2]\}. \tag{3.1}$$

For ϕ-mixing processes, such a strong result is not known. We provide here with certain alternative bounds depending on $A_k(\phi) < \infty$ for some positive k. For this, we consider first the following lemma which extends Lemma 2.1 of Sen (1971) to general $k \geqq 1$. Let $\{T_i = T(\mathbf{X}_i), -\infty < i < \infty\}$ be stationary mixing such that (1.1) holds and

$$ET = 0\,, \qquad ET_i^2 = \tau\,; \quad 0 \leqq \tau \leqq 1\,, \tag{3.2}$$
$$P\{|T_i| > 1\} = 0 \qquad \text{and} \qquad E|T_i| \leqq c\tau\,, \qquad 0 < c < \infty\,.$$

Let then, $S_n = T_1 + \cdots + T_n$ for $n \geqq 1$.

LEMMA 3.1. *Under* (1.1) *and* (3.2) *if* $A_k(\phi) < \infty$ *for some* $k(\geqq 1)$, *then for every* $n \geqq 1$,

$$E(S_n^{2(k+1)}) \leqq K_\phi\{n\tau + \cdots + (n\tau)^{k+1}\}\,, \qquad K_\phi < \infty\,, \tag{3.3}$$

where K_ϕ *depends only on* $\{\phi_n\}$ *and* c.

PROOF. Note that for $k \geqq 0$,

$$E(S_n^{2(k+1)}) \leqq [(2k+2)!]n \textstyle\sum_{n,2k+1} |E(T_1 T_{i_1} \cdots T_{i_{2k+1}})| \tag{3.4}$$

where the summation $\sum_{n,2k+1}$ extends over all $1 \leqq i_1 \leqq \cdots \leqq i_{2k+1} \leqq n$. Also, note that if ξ and η be $\mathscr{M}_{-\infty}^k$ and $\mathscr{M}_{k+n}^\infty$ measurable, $E|\xi| < \infty$ and $P\{|\eta| > 1\} = 0$, then [cf. Billingsley (1968) page 171]

$$|E(\xi\eta) - E(\xi)E(\eta)| \leqq 2\phi_n E|\xi|\,, \qquad \forall n\,. \tag{3.5}$$

Proceeding as in the proof of Lemma 2.1 of Sen (1971) and using (3.5), we obtain that if $A_0(\phi) < \infty$, under (1.1) and (3.2),

$$n \textstyle\sum_{n,1} |E(T_1 T_{i_1})| \leqq [2cA_0(\phi^2)](n\tau)\,, \tag{3.6}$$

and if $A_1(\phi) < \infty$ $[\Rightarrow A_2(\phi^2) < \infty]$, then

$$n \textstyle\sum_{n,2} |E(T_1 T_{i_1} T_{i_2})] \leqq 6nc\tau \sum_{i=0}^{n-1} (i+1)^2\phi_i < [6cA_2(\phi^2)](n\tau)\,, \tag{3.7}$$

$$n \textstyle\sum_{n,3} |E(T_1 T_{i_1} \cdots T_{j_3})| \leqq K_\phi[n\tau + (n\tau)^2]\,, \qquad K_\phi < \infty\,, \tag{3.8}$$

Let us now assume that for $1 \leqq a \leqq 2k-1$, $k \geqq 1$, $n \geqq 1$,

$$n \textstyle\sum_{n,a} |E(T_1 T_{i_1} \cdots T_{i_a})| \leqq K_{\phi,a}\{n\tau + \cdots + (n\tau)^{a*}\}\,, \qquad K_{\phi,a} < \infty\,, \tag{3.9}$$

where $a^* = t$ for $a = 2t$ or $2t-1$, $t \geqq 1$. Then, we shall show that $A_k(\phi) < \infty$ implies that (3.9) also holds for $a = 2k$ and $2k+1$. We consider only the case of $a = 2k+1$ (as the other case follows similarly). For this, we let $i_0 = 1$, $i_j = i_{j-1} + r_j$, $r_j \geqq 0$, $1 \leqq j \leqq 2k+1$, and let $\sum_{n,2k+1}^{(j)}$ be the summation over all $1 \leqq i_1 \leqq \cdots \leqq i_{2k+1} \leqq n$ for which $r_j = \max\{r_1, \cdots, r_{2k+1}\}$, for $j = 1, \cdots, 2k+1$. Then

$$n \textstyle\sum_{n,2k+1} |E(T_1 T_{i_1} \cdots T_{i_{2k+1}})| \leqq \sum_{j=1}^{2k+1} \{n \sum_{n,2k+1}^{(j)} |E(T_1 \cdots T_{i_{2k+1}})|\}\,, \tag{3.10}$$

where by (3.2) and (3.5), for each j: $1 \leqq j \leqq 2k+1$,

$$\begin{aligned} n \textstyle\sum_{n,2k+1}^{(j)} |E(T_1 \cdots T_{i_{2k+1}})| &\leqq n \textstyle\sum_{n,2k+1}^{(j)} |E(T_1 \cdots T_{i_{j-1}})E(T_{i_j} \cdots T_{i_{2k+1}})| \\ &\quad + 2n \textstyle\sum_{n,2k+1}^{(j)} \phi_{r_j} E[T_1 \cdots T_{i_{j-1}}]\,, \end{aligned} \tag{3.11}$$

and the second term on the rhs of (3.11) is bounded by

$$(3.12)\qquad 2nE|T_1| \sum_{n,2k+1}^{(j)} \phi_{r_j} \leqq 2nc\tau \sum_{r_j=0}^{n-1} (r_j+1)^{2k}\phi_{r_j} < 2nc\tau(\sum_{i=0}^{\infty}(i+1)^{2k}\phi_i) = [2cA_{2k}(\phi^2)](n\tau)\,,$$

and $A_{2k}(\phi^2) < \infty$. For $j = 1$ or $2k+1$, the first term on the rhs of (3.11) vanishes [by (3.2)], while for $2 \leqq j \leqq 2k$, we have

$$(3.13)\qquad n \sum_{n,2k+1}^{(j)} |E(T_1 - T_{i_{j-1}})E(T_{i_j} \cdots T_{i_{2k+1}})| \leqq n \sum_{i_j=1}^{n} \{\sum_{i_j,j-1} |E(T_1 \cdots T_{i_{j-1}})|\} \times \{\sum_{n-i_j+1,2k+1-j} |E(T_{i_0'} \cdots T_{i'_{2k+1-j}})|\}\,,$$

where $i_l' = i_{j+l} - i_j + 1$, $l = 0, \cdots, 2k+1-j$. Since for $2 \leqq j \leqq 2k$, $1 \leqq j-1$, $2k+1-j \leqq 2k-1$, and by assumption, (3.9) holds for $a \leqq 2k-1$, we obtain from (3.9), (3.13) and the inequality that for $a \geqq 0$, $b \geqq 0$, $\sum_{i=1}^{n} i^a(n-i+1)^b \leqq c(n+1)^{a+b+1} \leqq c^* n^{a+b+1}$, $c^* < \infty$, that the rhs of (3.13) is bounded by

$$(3.14)\qquad K_{\phi,j}[n\tau + \cdots + (n\tau)^{k*}]\,, \qquad K_{\phi,j} < \infty\,; \quad k^* = k\,, \quad j = \text{odd}\,, \qquad = k+1\,, \quad j = \text{even}\,.$$

Thus, from (3.10) through (3.14), we conclude that (3.9) holds for $a = 2k+1$. Using then (3.6)—(3.8), the proof for general $k \geqq 1$ follows by the method of induction.

THEOREM 3.2. *Under* (1.1) *and* $A_k(\phi) < \infty$ *for some* $k \geqq 1$,

$$(3.15)\qquad \sup_n P\{\sup_{\mathbf{t}\in E} |W_n(\mathbf{t})| > \lambda\} \leqq C_\phi \lambda^{-2(k+1)}\,, \qquad \forall \lambda \geqq 1\,,$$

where C_ϕ $(< \infty)$ *depends on* $\{\phi_n\}$.

PROOF. Virtually, we repeat the steps in (2.5) through (2.19) with the following changes: (i) in (2.6), (2.7) and (2.15), we let $\varepsilon = \delta = 1$, (ii) in (2.14), with the aid of Lemma 3.1, we use the moment of order $2(k+1)$, and (iii) in (2.18), we take $\varepsilon = \lambda$, $\lambda \geqq 1$. Then, the corresponding bound in (2.19) appears to be

$$(3.16)\qquad [2^{2(k+1)}K_{\phi,1}^*][1 + (1 + 1/pn^{\frac{1}{2}})^q]^\beta \lambda^{-2(k+1)}\,, \quad K_{\phi,1}^* \leqq K_\phi[1 + 4p^2] < \infty\,,$$

where $1 < \beta \leqq \max(k+1, 1 + 2k/p)$, so that (3.15) follows. □

Note that (3.15) implies that under (1.1) and $A_k(\phi) < \infty$ for some $k \geqq 1$,

$$(3.17)\qquad \sup_n E\{\sup_{\mathbf{t}\in E^p} |W_n(\mathbf{t})|^{2k+\delta}\} < \infty\,, \qquad \forall 0 \leqq \delta < 1\,.$$

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