The Annals of Probability
1974, Vol. 2, No. 2, 270-276

DENSITY VERSIONS OF THE UNIVARIATE CENTRAL LIMIT THEOREM[1]

BY SUJIT K. BASU

University of North Carolina

Let $\{X_n\}$ be a sequence of independent random variables each with a finite expectation and a finite variance. Write Z_n for the standardized sum of $X_1, X_2, \cdots, X_n$ and suppose that for all large n, Z_n has a probability density function which we denote by $h_n(x)$. It is well known that the usual assumptions of the Central Limit Theorem do not necessarily imply the convergence of $h_n(x)$ to the standard normal density $\phi(x)$. In this study, we find a set of sufficient conditions under which the relation

$$\lim_{n\to\infty} |x|^k |h_n(x) - \phi(x)| = 0$$

holds uniformly with respect to $x \in (-\infty, +\infty)$, k being an integer greater than or equal to 2.

1. Introduction. Let $X_1, X_2, \cdots,$ *ad inf* be a sequence of independent random variables such that $\mathscr{E}X_j = 0$ and $\mathscr{E}X_j^2 = \sigma_j^2 < \infty$, for all j. We write $s_n^2 = \sum_{j=1}^n \sigma_j^2$, $Z_n = s_n^{-1} \sum_{j=1}^n X_j$ and $H_n(x) = \Pr\{Z_n \leqq x\}$. Lindeberg's form of the Central Limit Theorem for independent random variables asserts that if for every $\varepsilon > 0$,

$$\sum_{j=1}^n \mathscr{E} I_{[|x_j| > \varepsilon s_n]} |X_j|^2 = o(s_n^2) \tag{1.1}$$

as $n \to \infty$ (and hence $s_n \to \infty$, $\sup_{1 \leqq k \leqq n} \sigma_k / s_n \to 0$, as $n \to \infty$) then Z_n tends, in distribution, as $n \to \infty$ to the normal distribution with probability density function (pdf) $\phi(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)$. In some situations it is desirable to be able to claim that the pdf of Z_n tends to $\phi(x)$, a conclusion that is, in general, false. This is, however, true in the case where the X's (i.i.d. case) have the same distribution provided the characteristic function (ch.f.) of X_1 is integrable in rth power for some integer $r \geqq 1$ (see Feller (1966), Gnedenko and Kolmogorov (1968) and Smith (1953)). The i.i.d. case was later studied in greater detail by Petrov (1964), (1972). He showed that if, in addition to the integrability of the ch.f. (or of the rth power of it) of X_1, there exists an integer $k \geqq 3$ such that $\mathscr{E}|X_1|^k < \infty$, then $H_n(x)$ is absolutely continuous with a pdf $h_n(x)$ such that the relation

$$(1 + |x|^k)[h_n(x) - \varphi(x) - \textstyle\sum_{j=1}^{k-2} P_j(-\varphi)/n^{j/2}] = o(n^{-(k-2)/2}) \tag{1.2}$$

holds uniformly with respect to (w.r.t) x, as $n \to \infty$. Here

$$P_j(-\varphi) = q_{3j}(x)\varphi(x)$$

Received May 19, 1972; revised June 15, 1973.

[1] Research supported by the Office of Naval Research under contract N00014-67A-0321-0002.

AMS 1970 *subject classifications.* Primary 60F05; Secondary 60E05, 62E15.

Key words and phrases. Central Limit Theorem, smoothing subsequence, Lindeberg condition of order k.

where $q_{3j}(x)$ is the well-known polynomial in x of degree $3j$ with coefficients which depend only on the moments of X_1 up to order $j+2$.

No such general result is known when the X's have different distributions. Petrov (1956) provided an estimate of $(h_n(x) - \phi(x))$ under a certain condition on the integrability of the ch.f.'s of the X's and the assumption that $\mathscr{E}|X_j|^k < \infty$, where $k = 2 + \delta$, $\delta > 0$, and $h_n(x)$ is a pdf of Z_n. He also provided an expansion of $h_n(x)$ in the case where $\delta \geqq 1$ and some other stronger conditions hold. This expansion is somewhat similar to (1.2); but unlike (1.2), it does not contain $(1 + |x|^k)$ as a factor. Smith (1953), on the other hand, provided a different set of conditions under which $(1 + x^2)(h_n(x) - \varphi(x))$ converges to zero, uniformly w.r.t x, as $n \to \infty$. In order to motivate our present problem, we shall now briefly discuss Smith's findings.

Let us denote the characteristic function (ch.f.) of X_j by $\omega_j(t)$, $j = 1, 2, \cdots$. We shall write $\phi(x)$ for the standard normal pdf.

DEFINTION 1.1 [Smith]. The sequence of random variables $\{X_n\}$ will be said to contain a "smoothing subsequence" if there exists a subsequence $\omega_{n_1}(t)$, $\omega_{n_2}(t), \cdots$ such that for some positive R, A, α we have

$$|\omega_{n_j}(t)| \leqq A/|t|^\alpha$$

for real t with $|t| \geqq R$.

Denote by n^* the number of members of the smoothing subsequence in $X_1, X_2, \cdots, X_n$. He then proved:

THEOREM 1.1 [Smith]. *If the conditions of Lindeberg's theorem hold and if a smoothing subsequence exists which satisfies.*

Condition A: $$\liminf_{n\to\infty} n^*/s_n^2 > 0$$

then for all sufficiently large n, $H_n(x)$ is absolutely continuous with a pdf *$h_n(x)$ such that*

$$\lim_{n\to\infty} h_n(x) = \phi(x) \tag{1.3}$$

uniformly with respect to x in the interval $(-\infty < x < \infty)$. If further,

Condition B: $$\limsup_{n\to\infty} \sum_{j=1}^n s_n^{-2} \int_{-\infty}^{+\infty} |x| F_j(dx) < \infty$$

holds, then

$$\lim_{n\to\infty} x^2 h_n(x) = x^2 \phi(x) \tag{1.4}$$

uniformly with respect to x in the interval $(-\infty < x < \infty)$.

The purpose of this present note is to prove a theorem which embraces the ideas of Theorem 1.1 and even allows us to introduce x^k in place of x^2 in the expression (1.4), where $k \geqq 2$ is an integer. We shall prove that this is possible if we replace Condition B in Theorem 1.1 by a condition L_k (to be defined in Section 2). As will be seen, this condition L_k reduces to the usual Lindeberg condition (1.1) if $h = 2$; since we assume the latter anyway, this renders condition B in the above theorem redundant. It will be clear from Corollary 2.1.1 that such an assumption may not be avoided in the present situation. Similar results in the case where the X's are identically distributed can be found in Smith and Basu (1973); there it was possible to go one step further and prove a similar theorem even when r is not necessarily an integer.

We shall need additional notation. The ch.f. of Z_n will be denoted by $\Omega_n(t)$ so that

$$\Omega_n(t) = \prod_{j=1}^{n} \omega_j(t/s_n) .$$

Whenever the rth order derivatives exists, we shall write

$$\omega_j^{(r)}(t) = (d/dt)^r \omega_j(t) , \qquad j = 1, 2, \cdots$$

and

$$\Omega_n^{(r)}(t) = (d/dt)^r \Omega_n(t) .$$

Also if the rth order moments exist, we shall use

$$\mu_{jr} = \mathscr{E}|X_j|^r , \qquad j = 1, 2, \cdots$$

while we shall write

$$\phi_r = \int_{-\infty}^{+\infty} |x|^r \phi(x)\, dx .$$

$N(t)$ will represent ch.f. of the standard normal distribution and we shall use $N^{(r)}(t)$ to represent its rth derivative.

2. Some necessary lemmas. Here we shall discuss a few lemmas that will be necessary for our later purposes.

LEMMA 2.1. *For an integer k, $\Omega_n^{(k)}(t)$, whenever it exists, is given by*

$$\text{(2.1)} \qquad \Omega_n^{(k)}(t) = s_n^{-k} \sum_{Q_k} (k_1, k_2^k, \cdots, k_l) \sum_{1 \leqq j_1 < j_2 < \cdots < j_l \leqq n} \prod_{i=1}^{l} \omega_{j_i}^{(k_i)}(t/s_n) \times \prod_{j=1, j \neq j_1, j_2, \cdots, j_l}^{n} \omega_j(t/s_n)$$

where $(k_1, k_2^k, \cdots, k_l) = (k!)(k_1!\, k_2! \cdots k_l!)^{-1}$ and

$$Q_k = \bigcup_{l=1}^{k} \{(k_1, k_2, \cdots, k_l) : \textit{each } k_j \textit{ is an integer } \geqq 1,\ k_1 + k_2 + \cdots + k_l = k\} .$$

PROOF. The proof follows by an application of Leibnitz rule and then a rearrangement of terms.

LEMMA 2.2. *With notations introduced in the previous section, assume that the conditions of Lindeberg's theorem hold and that $\mu_{jk} < \infty$ for some integer $k \geqq 2$ and all j. Then $\Omega_n^{(k)}(t)$ exists for all t and*

(i) $\lim_{n\to\infty} \mathscr{E}|Z_n|^k = \phi_k$, *implies*
(ii) $\lim_{n\to\infty} \Omega_n^{(k)}(t) = N^{(k)}(t)$, *for all t.*

PROOF. That $\Omega_n^{(k)}(t)$ exists for all t is obvious. Next we note that

$$\Omega_n^{(k)}(t) - N^{(k)}(t) = D_{1n} + D_{2n} + D_{3n}$$

where

$$D_{1n} = \int_{-M}^{M} (ix)^k \exp(itx) H_n(dx) - \int_{-M}^{M} (ix)^k \exp(itx)\phi(x)\, dx ,$$
$$D_{2n} = \int_{|x|>M} (ix)^k \exp(itx) H_n(dx) ,$$
$$D_{3n} = -\int_{|x|>M} (ix)^k \exp(itx)\phi(x)\, dx ,$$

M being a positive number to be suitably chosen later. Since Lindeberg condition (1.1) holds, it is easy to see that for any fixed M, D_{1n} converges to zero as $n \to \infty$.

Also,

$$|D_{2n}| \leqq \int_{|x|>M} |x|^k H_n(dx)$$

which, because of Lindeberg condition and condition (i) can be made as small as we please by a suitable choice of M. D_{3n} can obviously be made small by choosing a large M. This completes the proof of the lemma.

DEFINITION 2.1. A sequence of independent random variables $\{X_n\}$ with $\mathscr{E}X_n = 0$, $S_n = X_1 + X_2 + \cdots + X_n$, $s_n^2 = \mathscr{E}S_n^2$, $n = 1, 2, \cdots$ is said to obey a Lindeberg condition of order $\nu \geqq 2$ (i.e. L_ν holds) if

(2.2) $$\prod_{j=1}^{n} \mathscr{E} I_{[|x_j| \geqq \varepsilon s_n]} |X_j|^\nu = o(s_n^\nu)$$

as $n \to \infty$ for all $\varepsilon > 0$.

When $\nu = 2$, this is the well-known Lindeberg condition which we mentioned in Section 1. It has been shown by Brown (1969) that when $\nu > 2$ the condition (2.2) is equivalent to

(2.3) $$\sum_{j=1}^{n} \mathscr{E} |X_j|^\nu = o(s_n^\nu)$$

as $n \to \infty$ and that L_ν implies L_α for $2 \leqq \alpha \leqq \nu$. We shall reproduce the following theorem from Brown (1970).

THEOREM 2.1 [Brown]. *For all $\nu > 2$, L_ν is both necessary and sufficient for the Central Limit Theorem and*

$$\lim_{n\to\infty} \mathscr{E} |S_n/s_n|^\nu = \phi_\nu .$$

COROLLARY 2.1.1. *Suppose $k \geqq 2$ is an integer. Whenever $\Omega_n^{(k)}(t)$ exists, a sufficient condition for the Central Limit Theorem and*

$$\lim_{n\to\infty} \Omega_n^{(k)}(t) = N^{(k)}(t)$$

(for all t) to hold is that L_k holds.

PROOF. Immediate from Lemma 2.2 and Theorem 2.1.

LEMMA 2.3 [Smith]. *Suppose the sequence $\{\Omega_n(t)\}$ of* ch.f.'*s contains a smoothing subsequence. Then there exists a positive function $\Omega(t)$ such that*

$$\Omega(t) = A/|t|^\alpha \qquad for \quad |t| \geqq R$$
$$= 1 - \gamma t^2 \qquad for \quad |t| < R$$

where γ is some positive number, and if $\omega_{n_j}(t)$ belongs to a smoothing subsequence, then for all t

$$|\omega_{n_j}(t)| \leqq \Omega(t) .$$

It may be observed that, without any loss of generality, R may be assumed large enough for $AR^{-\alpha} < 1$.

3. The main theorem. With this much of introduction, we are now in a position to state and prove the following theorem:

THEOREM 3.1. *Let $\{X_n\}$ be a sequence of independent random variables with* df'*s* $F_1, F_2, \cdots$ *and* ch.f.'*s* $\omega_1, \omega_2, \cdots$. *Suppose*

(i) $\mathscr{E}X_j = 0$, $\mathscr{E}X_j^2 = \sigma_j^2 < \infty$, $j = 1, 2, \cdots$,
(ii) *for some integer* $k \geqq 2$, L_k *holds,*
(iii) *a smoothing subsequence exists which satisfies*

Condition A: $$\liminf_{n\to\infty} n^*/s_n^2 > 0 .$$

Then for all sufficiently large n, $H_n(x)$ is absolutely continuous with a pdf $h_n(x)$ *such that the relation*

$$\lim_{n\to\infty} (1 + |x|^k)[h_n(x) - \phi(x)] = 0 \tag{3.1}$$

holds uniformly with respect to x in the interval $(-\infty < x < \infty)$.

PROOF. Since $\omega_j(t)$ is bounded by unity for all j and a smoothing subsequence exists, $\Omega_n(t)$ is (absolutely) integrable for all n greater than or equal to some n_0. Hence $H_n(x)$ is absolutely continuous for all $n \geqq n_0$. In view of Theorem 1.1, it is sufficient to prove that $|x|^k[h_n(x) - \phi(x)] \to 0$ as $n \to \infty$, uniformly in x. Next, we note that L_k implies L_2 and hence

$$\lim_{n\to\infty} H_n(x) = \Phi(x) \qquad \text{for all } x, \text{ and} \tag{3.2}$$

$$s_n \to \infty \qquad \text{as } n \to \infty . \tag{3.3}$$

Since $|\omega_j^{(r)}(t)| \leqq \mu_{jr} < \infty$ for $0 \leqq r \leqq k$ and $j = 1, 2, \cdots$, from condition A, it follows that $\Omega_n^{(k)}(t)$ is (absolutely) integrable for all large n. Thus for all large n,

$$\begin{aligned} |x|^k|h_n(x) - \phi(x)| &= (2\pi)^{-1}|\textstyle\int_{-\infty}^{+\infty} \{\Omega_n^{(k)}(t) - N^{(k)}(t)\} \exp(-itx)\, dt| \\ &\leqq (2\pi)^{-1}[I_{1n} + I_{2n} + I_{3n} + I_{4n}] \end{aligned}$$

where

$$\begin{aligned} I_{1n} &= \textstyle\int_{|t|\leqq Q} |\Omega_n^{(k)}(t) - N^{(k)}(t)|\, dt , \\ I_{2n} &= \textstyle\int_{|t|>Q} |N^{(k)}(t)|\, dt , \\ I_{3n} &= \textstyle\int_{Q<|t|<Rs_n} |\Omega_n^{(k)}(t)|\, dt , \\ I_{4n} &= \textstyle\int_{|t|>Rs_n} |\Omega_n^{(k)}(t)|\, dt , \end{aligned}$$

where $R > 0$ is as in Lemma 2.3 and $Q > 0$ will be chosen later.

Since L_k holds, Corollary 2.1.1 applies so that

$$\lim_{n\to\infty} \Omega_n^{(k)}(t) = N^{(k)}(t)$$

for all t. Also, by Theorem 2.1, there exists a finite number $B > 0$ such that $|\Omega_n^{(k)}(t)| \leqq B$. Hence whatever the constant $Q > 0$ may be, I_{1n} converges to zero as n tends to infinity. By choosing Q sufficiently large, I_{2n} can be made as small as we wish. By Lemmas 2.1 and 2.3

$$|\Omega_n^{(k)}(t)| \leqq (k!/s_n^k)\{\sum_{Q_k} \sum_{1\leqq j_1<j_2<\cdots<j_l\leqq n} \prod_{i=1}^{l} |\omega_{j_i}^{(k_i)}(t/s_n)|\}\{\Omega(t/s_n)\}^{n^*-k}$$

since $\Omega(t) \leqq 1$. But this implies

$$|\Omega_n^{(k)}(t) \leqq k! \sum_{Q_k} \{s_n^{-k_1} \sum_{j=1}^{n} |\omega_j^{(k_1)}(t/s_n)|\}\{s_n^{-k_2} \sum_{j=1}^{n} |\omega_j^{(k_2)}(t/s_n)|\} \cdots \{s_n^{-k_l} \sum_{j=1}^{n} |\omega_j^{(k_l)}(t/s_n)|\}\{\Omega(t/s_n)\}^{n^*-k} . \tag{3.4}$$

Now for each $j = 1, 2, \cdots, \omega_j^{(1)}(t)$ is differentiable and since the mean is zero, $\omega_j^{(1)}(0) = 0$. Thus

$$\omega_j^{(1)}(t/s_n) = (t/s_n)\omega_j^{(2)}(\xi_j t/s_n) , \qquad 0 < \xi_j < 1 ,$$

so that

$$|\omega_j^{(1)}(t/s_n)| \leqq (|t|\sigma_j^2)/s_n , \qquad j = 1, 2, \cdots .$$

Therefore,

$$s_n^{-1} \sum_{j=1}^{n} |\omega_j^{(1)}(t/s_n)| \leqq |t| . \tag{3.5}$$

Also since L_k holds, and L_k implies L_r, $r = 2, \cdots, k$, by (2.3) it follows that there exists constants $W_r > 0$ such that

$$s_n^{-r} \sum_{j=1}^{n} |\omega_j^{(r)}(t/s_n)| \leqq s_n^{-r} \sum_{j=1}^{n} \mathscr{E}|X_j|^r \leqq W_r , \tag{3.6}$$

for $r = 2, 3, \cdots, k$. Combining (3.4), (3.5) and (3.6) we can then say that

$$|\Omega_n^{(k)}(t)| \leqq \{\sum_{l=0}^{k} a_l|t|^l\} \times \{\Omega(t/s_n)\}^{n^*-k} ,$$

where a_l's are nonnegative constant coefficients. Also, because of condition A, there exists a $\mu > 0$ such that for all sufficiently large n, $n^* > \mu s_n^2$. Since $\Omega(t) \leqq 1$, for all large n, we must have

$$|\Omega_n^{(k)}(t)| \leqq \sum_{l=0}^{k} a_l|t|^l\{\Omega(t/s_n)\}^{\mu s_n^2-k} . \tag{3.7}$$

Thus, for all large n

$$\begin{aligned} I_{3n} &\leqq \sum_{l=0}^{k} a_l \int_{Q\leqq|t|\leqq Rs_n} |t|^l\{\Omega(t/s_n)\}^{\mu s_n^2-k} dt \\ &\leqq \sum_{l=0}^{k} a_l \int_{|t|\geqq Q} |t|^l \exp\{-\gamma t^2(\mu s_n^2 - k)/s_n^2\} dt \\ &\leqq \varepsilon , \end{aligned}$$

for large enough Q.

Finally, for all large n,

$$\begin{aligned} I_{4n} &\leqq \sum_{l=0}^{k} a_l \int_{|t|\geqq Rs_n} |t|^l\{\Omega(t/s_n)\}^{\mu s_n^2-k} dt \\ &\leqq \sum_{l=0}^{k} a_l \int_{|t|\geqq Rs_n} |t|^l(As_n^\alpha|t|^{-\alpha})^{\mu s_n^2-k} dt \end{aligned} \tag{3.8}$$

which converges to zero as $n \to \infty$ by simple calculations, since $AR^{-\alpha} < 1$. This completes the proof of the theorem.

Acknowledgments. I wish to express my thanks to the referee for his helpful suggestions.

REFERENCES

[1] BROWN, B. M. (1969). Moments of stopping rule related to the Central Limit Theorem. *Ann. Math. Statist.* **40** 1236–1249.

[2] BROWN, B. M. (1970). Characteristic functions, moments and the Central Limit Theorem. *Ann. Math. Statist.* **41** 658–666.

[3] FELLER, W. (1966). *An Introduction to Probability Theory and its Applications* **2**. Wiley, New York.

[4] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1968). *Limit distributions for sums of independent random variables* rev. ed. Addison-Wesley, Reading.

[5] PETROV, V. V. (1956). A local theorem for densities of sums of independent random variables. *Theor. Probability Appl.* **1** 316–322.

[6] PETROV, V. V. (1964). On local limit theorems for sums of independent random variables. *Theor. Probability Appl.* **9** 312–320.

[7] PETROV, V. V. (1972). *Sums of Independent Random Variables.* Theory of Probability and Mathematical Statistics Series, Nauka (in Russian).

[8] SMITH, W. L. (1953). A frequency function form of the Central Limit Theorem. *Proc. Cambridge Philos. Soc.* **49** 462–472.

[9] SMITH, W. L. and BASU, SUJIT K. (1973). General moment functions and a density version of the Central Limit Theorem. To be published in *Proc. Cambridge Philos. Soc.*

RESEARCH AND TRAINING SCHOOL
INDIAN STATISTICAL INSTITUTE
203, BARRACKPORE TRUNK ROAD
CALCUTTA-70035, INDIA