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A NOTE ON SEPARABLE STOCHASTIC PROCESSES[1]

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Some sets L of sample paths have the desirable property that if there exists a process with given finite-dimensional distributions and with paths in L (with probability 1), then every separable process with these finite-dimensional distributions has paths in L. A class of such sets is constructed.

Suppose L is a subset of the space R^T of all real functions on $T = [0, 1]$. Certain sets L have the following property, in which $X = [X(t): t \in T]$ and $X' = [X'(t): t \in T]$ are stochastic processes, on spaces $(\Omega, \mathscr{B}, P)$ and $(\Omega', \mathscr{B}', P')$, with sample paths $X(\cdot, \omega)$ and $X'(\cdot, \omega')$.

PROPERTY ρ. For all X and X', if X and X' have the same finite-dimensional distributions, if $[\omega: X(\cdot, \omega) \in L]$ lies in $\mathscr{B}$ and has P-measure 1, and if X' is separable, then $[\omega': X'(\cdot, \omega') \in L]$ lies in $\mathscr{B}'$ and has P'-measure 1.

This property of L can be restated: If there *exists* a process with given finite-dimensional distributions and with paths in L (with probability 1), then *every separable* process with these finite-dimensional distributions has paths in L. Or, loosely: If a process with given finite-dimensional distributions *should* have paths in L, it *will* have paths in L if it is separable.

In [1] it is shown that the set of continuous functions (page 66) and the set of functions with discontinuities of the first kind (page 136) have Property ρ. The purpose of this note is to extend the property to a broad class of path-sets.

If D is a countable, dense subset of T, let S_D be the set of functions x in R^T that are separable with respect to D; that is, $x \in S_D$ if and only if for each t in T there is a sequence $\{t_n\}$ in D such that $t_n \to t$ and $x(t_n) \to x(t)$. In this terminology, the process X' on $(\Omega', \mathscr{B}', P')$ is separable with respect to D if $(\Omega', \mathscr{B}', P')$ is complete and if the set $[\omega': X'(\cdot, \omega') \in S_D]$ lies in $\mathscr{B}'$ and has P'-measure 1; X' is separable if it is separable with respect to some countable, dense D (see [2] page 86).

Let $\mathscr{R}^T$ be the σ-field in R^T generated by the sets of the form $[x: x(t) \leqq a]$. Let $\mathscr{L}$ consist of those L in R^T such that, for each countable, dense D in T, there exists in R^T a set $\bar{L}_D$ such that

$$(1) \qquad \bar{L}_D \in \mathscr{R}^T, \qquad \bar{L}_D \supset L, \qquad \bar{L}_D - L \subset R^T - S_D.$$

THEOREM. *Each L in $\mathscr{L}$ has Property ρ.*

PROOF. Consider the processes X and X' involved in the definition of Property

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ρ, and suppose X' is separable with respect to D. Since $[\omega': X'(\cdot, \omega') \in \bar{L}_D] - [\omega': X'(\cdot, \omega') \in L]$ is a subset of $[\omega': X'(\cdot, \omega') \in R^T - S_D]$, it lies in $\mathscr{B}'$ and has P'-measure 0. Hence, by (1) and the fact that X and X' have the same finite-dimensional distributions,

$$P'[\omega': X'(\cdot, \omega') \in L] = P'[\omega': X'(\cdot, \omega') \in \bar{L}_D] \\ = P[\omega: X(\cdot, \omega) \in \bar{L}_D] \geqq P[\omega: X(\cdot, \omega) \in L] = 1 .$$

Thus $[\omega': X'(\cdot, \omega') \in L]$ lies in $\mathscr{B}'$ and has P'-measure 1, which completes the proof.

Here follow some general observations and some applications.

1°. The class $\mathscr{L}$ is closed under the formation of countable unions and intersections.

2°. If $L \in \mathscr{R}^T$, then of course $L \in \mathscr{L}$.

3°. A natural candidate for $\bar{L}_D$ is

$$\alpha_D(L) = [x: x \text{ agrees on } D \text{ with some } y \text{ in } L] . \tag{2}$$

This set will satisfy the requirements (1) if

$$\alpha_D(L) \in \mathscr{R}^T , \qquad \alpha_D(L) \cap S_D \subset L . \tag{3}$$

To prove that $\alpha_D(L)$ lies in $\mathscr{R}^T$, one usually proves that it lies in the σ-field $\mathscr{R}^D$ generated by the sets of the form $[x: x(t) \leqq a]$ with t in D.

4°. From 3° it follows that

$$E(I, F) = \bigcap_{t \in I \cap T} [x: x(t) \in F] \tag{4}$$

lies in $\mathscr{L}$ if I is open and F is closed.

5°. Let C be the class of continuous functions on T. If $x \in S_D$ and $y \in C$, and if x and y agree on D, then $x = y$. It follows by 3° that $\alpha_D(L)$ will satisfy (1) if

$$L \subset C , \qquad \alpha_D(L) \in \mathscr{R}^T . \tag{5}$$

6°. Clearly

$$\alpha_D(C) = \bigcap_\varepsilon \bigcup_\delta \bigcap [x: |x(s) - x(t)| < \varepsilon] , \tag{6}$$

where ε and δ range over the positive rationals (as in all that follows) and the inner intersection extends over the pairs, s, t in D with $|s - t| < \delta$. Since (6) lies in $\mathscr{R}^D$, $C \in \mathscr{L}$ follows from 5°. Alternatively, $C \in \mathscr{L}$ follows from 1°, 4°, and the representation

$$C = \bigcap_\varepsilon \bigcup \bigcap_{i=1}^{k-1} E\left(\left(\frac{i-1}{k}, \frac{i+1}{k}\right), [\beta_i, \beta_i + \varepsilon]\right) ,$$

in which E is defined by (4) and the union extends over the systems $(k; \beta_1, \cdots, \beta_k)$, where k is an integer and the β_i are rational.

7°. Let J consist of those x in R^T with discontinuities of at most the first kind; $x \in J$ if and only if $x(t+) = \lim_{s \downarrow t} x(s)$ exists for $0 \leqq t < 1$, $x(t-) = \lim_{s \uparrow t} x(s)$ exists for $0 < t \leqq 1$, and $x(t)$ lies between $x(t+)$ and $x(t-)$ for

$0 < t < 1$. Consider the general system $V = (k; \beta_1, \cdots, \beta_k; r_1, \cdots, r_k; s_1, \cdots, s_k)$, where k is an integer, the β_i, r_i, and s_i are all rational, and

$$r_1 < 0 < s_1 < r_2 < s_2 < \cdots < r_k < 1 < s_k . \tag{7}$$

Put

$$\begin{aligned} G_V = \textstyle\bigcap_{i=1}^k E((r_i, s_i), [\beta_i, \beta_i + \varepsilon]) \\ \cap \textstyle\bigcap_{i=2}^k E((s_{i-1}, r_i), [\min\{\beta_{i-1}, \beta_i\}, \max\{\beta_{i-1}, \beta_i\} + \varepsilon]) . \end{aligned}$$

As pointed out in [1], page 135, $J = \bigcap_\varepsilon \bigcup_k \bigcap_\delta \bigcup G_V$, where k ranges over the positive integers and the inner union extends over those systems V having a fixed value of k and satisfying

$$r_i - s_{i-1} < \delta , \qquad i = 2, \cdots, k . \tag{8}$$

It follows by 1° and 4° that $J \in \mathscr{L}$.

8°. For open I, $\mathscr{L}$ contains the set $H(I)$ of x that are constant over I, as follows from 3°. Let J' be the set of x that are step functions in the sense that, for some $\{t_i\}$ with $0 = t_0 < t_1 < \cdots < t_k = 1$, x is constant over each (t_{i-1}, t_i). Then $J' \in \mathscr{L}$ because $J' = \bigcup_k \bigcap_\delta \bigcup H(r_i, s_i)$, the inner union extending over systems $(r_1, \cdots, r_k; s_1, \cdots, s_k)$ of rationals satisfying (7) and (8).

9°. Let C_k consist of the functions with k continuous derivatives and, for $x \in C_k$, let $D_k(x, t)$ be the kth derivative at t; here $C_0 = C$ and $D_0(x, t) = x(t)$. We shall prove inductively that $\alpha_D(C_k) \in \mathscr{R}^D$ and $D_k(\cdot, t)$ is measurable $\mathscr{R}^D$ on C_k. This is true for $k = 0$ by 6°. The induction step follows from

$$\begin{aligned} \alpha_D(C_{k+1}) = \alpha_D(C_k) \\ \cap \textstyle\bigcap_\varepsilon \bigcup_\delta \bigcap \left[x : \left| \frac{D_k(x, s) - D_k(x, t)}{s - t} - \frac{D_k(x, u) - D_k(x, v)}{u - v} \right| < \varepsilon \right], \end{aligned}$$

where the final intersection extends over sets s, t, u, v ($s \neq t$, $u \neq v$) of points in D within δ of one another, together with the fact that, if $x \in C_{k+1}$, then $D_{k+1}(x, t) = \lim (D_k(x, s) - D_k(x, s'))/(s - s')$, where in the limit s and then s' approach t through values in D.

Thus all the C_k lie in $\mathscr{L}$, and so does $C_\infty = \bigcap_k C_k$. If A is the set of functions having power series expansions about 0, then

$$\alpha_D(A) = \alpha_D(C_\infty) \cap \textstyle\bigcap_{t \in D} [x : x(t) = \sum_{k=0}^\infty D_k(x, 0) t^k / k!] .$$

Therefore $A \in \mathscr{L}$. We can put other restrictions on the derivatives in a measurable way, and it follows that $\mathscr{L}$ contains the class of completely monotonic functions, the class of analytic functions, the class of polynomials, etc.

10°. If $L \subset C$ and L is a Borel set in C, then $\alpha_D(L)$ lies in $\mathscr{R}^D$, so that $L \in \mathscr{L}$. To see this, observe first that, if L is the closed sphere in C with radius r and center $z \in C$, then $\alpha_D(L)$ is the set (6) intersected with $\bigcap_{t \in D} [x : |x(t) - z(t)| \leqq r]$, and hence $\alpha_D(L) \in \mathscr{R}^D$. Therefore, it is enough to show that $[L : L \subset C, \alpha_D(L) \in \mathscr{R}^D]$ is a σ-field in C. That this class is closed under the formation of countable unions is obvious; that it is closed under the formation of complements (in C)

follows from the fact that, if an x in R^T agrees on a dense D with a y in C, then that y is unique.

11°. For a set that does not have Property ρ, and hence does not lie in $\mathscr{L}$, consider the set of functions continuous from the right. With θ uniformly distributed over (0, 1), let $X(t) = I_{[0,\theta)}(t)$ and $X'(t) = I_{[0,\theta]}(t)$; then X and X' have the same finite-dimensional distributions and are separable, but X has right-continuous paths whereas X' does not. Another set that does not have Property ρ is $R^T - C$. It might be interesting to have an example of a set that has Property ρ but lies outside $\mathscr{L}$.

Most of the arguments above hold if $T = [0, 1]$ is replaced by an arbitrary subset of the line.

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