

ON QUASI-COMPACT MARKOV OPERATORS¹

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Let P be a conservative Markov operator on $L_\infty(X, \Sigma, m)$. The following conditions are proved to be equivalent: (i) P is ergodic and quasi-compact. (ii) P is ergodic and $(I - P)L_\infty$ is closed. (iii) For every $u \in L_1$ with $\int u \, dm = 0$ the sequence $\{\sum_{n=0}^N uP^n\}$ is weakly sequentially compact in L_1 . (iv) There exists an invariant probability measure $\lambda \sim m$, and for every $u \in L_1$ with $\int u \, dm = 0$, $\sup_{N \geq 0} \|\sum_{n=0}^N uP^n\|_1 < \infty$. These conditions are used to study the quasi-compactness of the induced operators $P_f = \sum_{n=0}^\infty (PT_{1-f})^n PT_f$ in the case that P is Harris-recurrent: The following conditions are equivalent: (i) P_f is quasi-compact. (ii) f is "special" in the (modified) sense of Neveu. (iii) There exists a σ -finite measure λ , equivalent to m on $A = \{f > 0\}$, with $\int f \, d\lambda < \infty$, and $\int fg \, d\lambda = 0$ implies $\sup_{N \geq 0} \|\sum_{n=0}^N P^n(fg)\|_\infty < \infty$. Using this characterization certain limit theorems for P are obtained.

1. Introduction. Let (X, Σ, m) be a σ -finite measure space and let P be a positive linear contraction of $L_1(X, \Sigma, m)$, denoted by $u \rightarrow uP$. The adjoint of P , operating on $L_\infty(X, \Sigma, m)$, will be denoted by P and written to the left of its variable so that $\langle uP, f \rangle = \langle u, Pf \rangle = \int uPf \, dm$ for $u \in L_1$ and $f \in L_\infty$. P is called a *Markov operator*.

P is called *conservative* if $Pf \leq f$ for $f \in L_\infty$ implies $Pf = f$. It is *conservative and ergodic* if $Pf \leq f$ for $f \in L_\infty$ implies that f is constant a.e. (all inequalities are assumed a.e.).

A *Harris operator* is a conservative and ergodic Markov operator P such that for some $n > 0$ there is a function $k(x, y) \geq 0$, measurable in (x, y) , satisfying $P^n f(x) \geq \int k(x, y)f(y)m(dy)$ (and $k \not\equiv 0$). A bounded linear operator T in a Banach space is *quasi-compact* if $\|T^n - K\| < 1$ for some $n > 0$ and some compact linear operator $K \neq 0$. Quasi-compactness of a transition probability operator is discussed in Neveu [14] Section V. 3. It is equivalent to Dooblin's condition, which is treated in Doob [5].

Horowitz [10] studied the quasi-compactness of an ergodic and conservative Markov operator P and of certain induced Markov operators. Some additional information on quasi-compact Markov operators is obtained in Section 2, while Section 3 deals with the quasi-compactness of the induced operators

$$P_f = \sum_{n=0}^\infty (PT_{f'})^n T_f \quad (\text{where } T_f g = fg, f' = 1 - f)$$

for $0 \leq f \leq 1$, which is shown to be equivalent to f being "special" in the sense

Received April 2, 1973; revised July 11, 1973.

¹ Research supported by NSF grant GP 34118.

AMS 1970 subject classifications. Primary 60J05, 28A05; Secondary 47A35.

Key words and phrases. Quasi-compact Markov operators, Harris recurrence, Markov operators with closed range, invariant measures.

of Neveu [15], although in a somewhat different context. (Brunel and Revuz [2] have obtained independently and announced this result in the exact context of [15]. Their remark about “changing time” seems to indicate a different proof). The quasi-compactness of P_f is given a characterization in terms of P , and is used to obtain certain limit theorems for Harris operators (only for such Markov operators special functions exist), yielding as corollaries some results of Chung [4] about Markov chains.

2. Quasi-compact conservative Markov operators. In this section we obtain some necessary and sufficient conditions for a conservative and ergodic Markov operator to be quasi-compact.

By Theorems VI. 6.2 and VI. 6.4 of [6] we have that $L_1(I - P)$ is closed if and only if $(I - P)L_\infty$ is closed, and in that case $(I - P)L_\infty = \{f \in L_\infty : uP = u \Rightarrow \langle u, f \rangle = 0\}$. We now obtain a general result.

THEOREM 2.1. *Let P be a conservative Markov operator. If $(I - P)L_\infty$ is closed, then P has a finite invariant measure equivalent to m , and $N^{-1} \sum_{n=0}^N P^n f$ converges in L_∞ -norm for every $f \in L_\infty$.*

PROOF. By Theorem IV. E in [7], $X = C_0 + C_1$ where on C_1 there is a finite invariant measure $\lambda (\sim m_{|C_1})$, and every finite invariant measure vanishes on C_0 . If $uP = u$ then $u^\pm P = u^\pm$ and therefore $\langle u, 1_{C_0} \rangle = 0$. Hence $-1_{C_0} \in (I - P)L_\infty$, so that for some $f \in L_\infty$, $-1_{C_0} = (I - P)f$ or $Pf = f + 1_{C_0}$. Since P is conservative, $m(C_0) = 0$ and there is a finite measure $\lambda \sim m$, $\lambda P = \lambda$. Hence $f \rightarrow Pf$ is a contraction of $L_1(\lambda)$, with the adjoint $f \rightarrow P^*f$ also a contraction of $L_1(\lambda)$ and $L_\infty(\lambda)$. By the Radon-Nikodym theorem $L_1(\lambda) \sim L_1(m)$, so we assume $m = \lambda$. Then $P^*u = uP = u$ if and only if u is Σ_i -measurable, since P and P^* have the same invariant sets [7]. Denote by $Ef = E_\lambda(f | \Sigma_i)$ the conditional expectation projection, and $PEf = Ef$. For $f \in L_\infty$ write $f = Ef + f - Ef$. Then $E(f - Ef) = 0$, and if $uP = u$ then u is Σ_i -measurable and therefore $\langle u, f - Ef \rangle = 0$ and $f - Ef \in (I - P)L_\infty$. Thus we have $L_\infty = (I - P)L_\infty \oplus L_\infty(X, \Sigma_i, \lambda)$ and $\|N^{-1} \sum_{n=0}^{N-1} P^n f - Ef\|_\infty \rightarrow 0$ for every $f \in L_\infty$.

REMARKS. 1. If we assume only $P1 = 1$, we obtain (by restriction to the conservative part) an invariant measure supported on the conservative part.

2. P need not be ergodic ($P = I$ satisfies our conditions).

Theorem 2.1 yields the following improved version of Horowitz’ result [10].

THEOREM 2.2. *Let P be conservative and ergodic. Then the following conditions are equivalent:*

- (i) $(I - P)L_\infty$ is closed (in L_∞ -norm);
- (ii) $L_1(I - P)$ is closed (in L_1 -norm);
- (iii) P is quasi-compact.

REMARK. Without restricting the size of the invariant field Σ_i the theorem is not true. Instead of “ergodic” we put “ Σ_i finite” (and apply the theorem to

each of its atoms). Since if P is quasicompact $N^{-1} \sum_{n=1}^N P^n$ converges to a finite dimensional projection ([6] pages 709–711), Σ_i must be finite if (iii) holds.

THEOREM 2.3. *Let P be conservative. Then the following conditions are equivalent:*

- (i) *For every $u \in L_1$ with $\int u \, dm = 0$ the sequence $\{\sum_{n=0}^N uP^n\}$ is weakly sequentially compact in L_1 .*
- (ii) *There exists an invariant probability measure $\lambda \sim m$, and for every $u \in L_1$ with $\int u \, dm = 0$ we have $\sup_{N \geq 0} \|\sum_{n=0}^N uP^n\|_1 < \infty$.*
- (iii) *P is quasi-compact and ergodic.*

PROOF. (i) \Rightarrow (ii): Fix $u \in L_1$ with $\int u \, dm = 0$. Clearly $\sup_{N \geq 0} \|\sum_{n=0}^N uP^n\|_1 = K < \infty$, and we have to show the existence of a finite invariant measure. Define $v_n = \sum_{i=0}^{n-1} uP^i$. Then $v_n(I - P) = u - uP^n$. It follows easily from Theorem IV. 8.9, in [6] that also the sequence of averages $w_N = \sum_{n=1}^N v_n/N$ is weakly sequentially compact. If v is a weak limit of $\{w_N\}$, say $w_{N_j} \rightarrow_w v$, then for $f \in L_\infty$

$$\begin{aligned} \langle v(I - P), f \rangle &= \langle v, (I - P)f \rangle = \lim \langle w_{N_j}(I - P), f \rangle \\ &= \lim \langle u - N_j^{-1} \sum_{n=1}^{N_j} uP^n, f \rangle = \langle u, f \rangle \end{aligned}$$

and $v(I - P) = u$. By (i) P is ergodic (If $1 \not\equiv P1_A = 1_A \not\equiv 0$, we take $u_1 \geq 0$ supported in A and $u_2 \geq 0$ supported in $X - A$ with $\int u_2 = \int u_1$ and $u = u_1 - u_2$ will satisfy $\int u \, dm = 0$, but $\sum_{n=0}^N \langle uP^n, 1_A \rangle = N \langle u_1, 1_A \rangle \rightarrow \infty$ —a contradiction), so that

$$\{u \in L_1 : \int u \, dm = 0\} \subset L_1(I - P) \subset \text{cl}mL_1(I - P) = \{u \in L_1 : \int u \, dm = 0\}$$

and $L_1(I - P)$ is closed. Now Theorem 2.1 yields the existence of a finite invariant measure.

(ii) \Rightarrow (iii): We want to show first that if $g \in L_\infty$ satisfies $\int g \, d\lambda = 0$, then $\sup_{N \geq 0} \|\sum_{n=0}^N P^n g\|_\infty < \infty$. Denote $g_N = \sum_{n=0}^N P^n g$. For $v \in L_1$ define $u = v - (\int v \, dm)u_0$, where $u_0 = d\lambda/dm$. Then $\int u \, dm = 0$, and by our assumption $\sup \|\sum_{n=0}^N uP^n\|_1 = K_v < \infty$. But

$$\langle vP^n, g \rangle = \langle uP^n, g \rangle + \int v \, dm \langle u_0, g \rangle = \langle uP^n, g \rangle, \quad \text{yielding}$$

$$|\langle v, g_N \rangle| = |\sum_{n=0}^N \langle vP^n, g \rangle| = |\sum_{n=0}^N \langle uP^n, g \rangle| \leq K_v \|g\|_\infty.$$

By the Banach–Steinhaus theorem $\sup_N \|g_N\|_\infty < \infty$.

Now for $B \in \Sigma$ with $\lambda(B) > 0$ define $g = \lambda(B)1 - 1_B$. Then $\|\sum_{n=0}^N P^n g\|_\infty \leq K$ for $N = 0, 1, \dots$. If $R \geq (K + 1)/\lambda(B)$, then $\sum_{n=0}^R P^n 1_B \geq \lambda(B) \sum_{n=0}^R P^n 1 - K \geq 1$. By Horowitz [10] this condition implies that P is quasi-compact. (P is ergodic, as proved before).

(iii) \Rightarrow (i): By Theorem 2.2, $L_1(I - P)$ is closed. Thus, $L_1(I - P) = \{u \in L_1 : \int u \, dm = 0\}$. Hence $\int u \, dm = 0 \Rightarrow u = v(I - P)$ and $\sum_{n=0}^N uP^n = v - vP^{N+1}$. Since P is necessarily Harris, it has a period k , and necessarily vP^{nk+d}

converges weakly as $n \rightarrow \infty$, for fixed $0 \leq d \leq k$ (see [7] page 91. In fact, L_1 -norm convergence holds). Thus $\{vP^n\}$ is weakly sequentially-compact.

COROLLARY 2.1. *Let P be conservative. Then P is quasi-compact aperiodic if and only if for every $u \in L_1$ with $\int u \, dm = 0$ we have $\sum_{n=0}^{\infty} \|uP^n\|_1 < \infty$.*

PROOF. Assume that P is quasi-compact aperiodic. Then, in L_∞ , $\|P^n - E\| \rightarrow 0$, where $Ef = \int f \, d\lambda$ (see Neveu [14]. λ is the invariant probability). But $uE = (\int u \, dm)u_0$ ($u_0 = d\lambda/dm$), so that $\|P^n - E\| \rightarrow 0$ as L_1 operators, and $\|P^n - E\| \leq Kr^n$ with $0 < r < 1$. Hence, if $\int u \, dm = 0$, $\|uP^n\| \leq Kr^n\|u\|$, and $\sum_{n=0}^{\infty} \|uP^n\|_1 < \infty$.

For the converse, note that P is quasi-compact by Theorem 2.3, since (i) is clearly satisfied. $\sum_{n=0}^{\infty} \|uP^n\|_1 < \infty$ implies $\|uP^n\|_1 \rightarrow 0$ whenever $\int u \, dm = 0$, hence all the iterates P^k are ergodic.

COROLLARY 2.2. *If P is conservative, quasi-compact and aperiodic, then for every $u \in L_1$ we have $\lim_{n \rightarrow \infty} uP^n(x) = (\int u \, dm)u_0$, (where $u_0P = u_0$ and $\int u_0 \, dm = 1$).*

PROOF. $\sum_{n=0}^{\infty} |uP^n|(x)$ converges a.e. for $u \in L_1$ with $\int u \, dm = 0$. If $v \in L_1$ put $u = v - (\int v \, dm)u_0$ and $uP^n(x) \rightarrow 0$ a.e.

REMARK. If P is quasi-compact, P^* may fail to be quasi-compact. Take on $X = \{0, 1, 2, \dots\}$ $P_{i0} = \frac{1}{2}$, $P_{i,i+1} = \frac{1}{2}$, $P_{ij} = 0$ otherwise. Then $\sum_{n=0}^{j+1} P^n 1_{\{j\}}(i) \geq 2^{-j-1}$ for every i and P is quasi-compact, with invariant measure $\lambda\{i\} = 2^{-1-i}$ for $i \geq 0$. P^* is given by $P_{0j}^* = 2^{-j-1}$, $P_{i,i-1}^* = 1$ for $i \geq 1$, $P_{ij}^* = 0$ otherwise, and is not quasi-compact.

3. Application to Harris operators. In this section we obtain some results for Harris operators (for their properties see Foguel [7]), using the quasi-compactness of certain induced Markov operators.

We start by generalizing a result of Butzer and Westphal [3] to nonreflexive spaces (with a shorter proof).

THEOREM 3.1. *Let T be a linear operator on a Banach space Y satisfying $\sup_{n \geq 0} \|T^n\| = K < \infty$, and let $X = Y^*$. Then for $x \in X$, $\sup_{N \geq 0} \|\sum_{n=0}^N T^{*n}x\| < \infty$ if and only if $x \in (I^* - T^*)X$.*

PROOF. Let $x \in X$ satisfy $\sup_{N \geq 0} \|\sum_{n=0}^N T^{*n}x\| = c < \infty$. Then

$$\|N^{-1} \sum_{n=0}^{N-1} T^{*n}x\| \rightarrow 0.$$

Let $u_n = \sum_{i=0}^{n-1} T^{*i}x$ and let $z \in X$ be a weak-* limit point of $v_N = N^{-1} \sum_{i=1}^N u_n$. For $y \in Y$ there is a sequence N_j such that $\langle z, (I - T)y \rangle = \lim \langle v_{N_j}, (I - T)y \rangle$. But

$$\begin{aligned} \langle (I - T^*)z, y \rangle &= \langle z, (I - T)y \rangle = \lim \langle (I - T^*)v_{N_j}, y \rangle \\ &= \lim \langle x - N_j^{-1} \sum_{n=1}^{N_j} T^{*n}x, y \rangle = \langle x, y \rangle. \end{aligned}$$

Thus $x = (I^* - T^*)z \in (I^* - T^*)X$. The converse is obvious.

COROLLARY 3.1. [3]. *Let X be a reflexive Banach space and T a linear operator with $\sup_{n \geq 0} \|T^n\| < \infty$. Then $x \in X$ satisfies $\sup_N \|\sum_{n=0}^N T^n x\| < \infty$ if and only if $x \in (I - T)X$.*

In the remainder of this section P is a conservative and ergodic Markov operator. Let $0 \leq f \leq 1$ be in L_∞ with $f \not\equiv 0$, denote $f' = 1 - f$ and let T_f be the operator of multiplication by f . If $f = 1_A$ we put I_A instead of T_f .

LEMMA 3.1. (i) *The operator $P_f = \sum_{n=0}^\infty (PT_{f'})^n PT_f$ is a Markov operator with $P_f 1 = 1$.*

(ii) $\sum_{n=1}^N P^n(T_f - T_f P_f) = (I - P^N)P_f$.

(iii) *The conservative part of P_f is $A = \{x: f(x) > 0\}$ and $I_A P_f$ is conservative and ergodic.*

(iv) *If P_f has a finite invariant measure μ , then P has a σ -finite invariant measure λ and $\int f d\lambda < \infty$.*

(v) *If $\lambda \sim m$ is a σ -finite invariant measure for P with $\int f d\lambda < \infty$, then $\mu = \lambda T_f$ is a finite invariant measure for P_f .*

PROOF. Except for (iii), the lemma is proved in [8]. Clearly $L_1(X)P_f \subset L_1(A)$, so that $X - A$ must be in the dissipative part of P_f , and since $L_1(A)P_f = L_1(A)I_A P_f \subset L_1(A)$, we have to show that $I_A P_f$ is conservative (and ergodic).

Let $0 \leq g \in L_\infty(A)$ satisfy $I_A P_f g \leq g$. Then $0 \leq T_f g - T_f I_A P_f g = (T_f - T_f P_f)g$ and by (ii) $0 \leq P(T_f - T_f P_f)g = (I - P)P_f g$. Since P is conservative and ergodic, we have $P_f g = c$ (a constant) a.e. Hence $g \geq I_A P_f g \geq c1_A$, and

$$\begin{aligned} fg - cf &\geq 0 = c - cP_f 1 = c - cP_f 1_A = P_f(g - c1_A) \\ &\geq PT_f(g - c1_A) = P(fg - cf). \end{aligned}$$

Thus $P(fg - cf) = 0 = (fg - cf)$ and $g = c$ on A . Therefore $I_A P_f$ is ergodic and conservative.

REMARK. For the case $f = 1_A$ the lemma is well known.

We now investigate the quasi-compactness of $I_A P_f$. It is equivalent to the quasi-compactness of P_f , because $P_f^n = P_f(I_A P_f)^{n-1}$.

THEOREM 3.2. *Let P be conservative and ergodic and let $0 \leq f \leq 1$, with support $A = \{x: f(x) > 0\}$. Then the following conditions are equivalent:*

(i) *The operator $I_A P_f = I_A \sum_{n=0}^\infty (PT_{f'})^n PT_f$ is quasi-compact (on $L_\infty(A)$).*

(ii) *There exists a σ -finite measure λ , ($\lambda|_A \sim m|_A$) with $\int f d\lambda < \infty$, such that for every $g \in L_\infty(A)$ with $\int gf d\lambda = 0$ we have $\sup_{N \geq 0} \|\sum_{n=0}^N P^n(fg)\|_\infty < \infty$.*

PROOF. (i) \Rightarrow (ii): $I_A P_f$ is conservative, ergodic and quasi-compact, so it has a finite invariant measure μ , by Theorems 2.1 and 2.2. Thus P has a σ -finite invariant measure λ (see [8]), with $\int f d\lambda < \infty$, and $\mu = \lambda T_f$. Thus $0 = \int fg d\lambda = \int g d\mu$. By [10] $g \in (I_A - I_A P_f)L_\infty(A)$ i.e., $g = (I_A - I_A P_f)h$ with $h \in L_\infty(A)$. Then by Lemma 3.1 $P(fg) = P(T_f - T_f P_f)h = (I - P)P_f h$. This shows that $fg = (I - P)(P_f h - fg)$ and $\|\sum_{n=0}^N P^n(fg)\|_\infty \leq 2\|P_f h - fg\|_\infty$ for every $N \geq 0$.

(ii) \Rightarrow (i). Take $g \in L_\infty(A)$ with $\int fg d\lambda = 0$. By (ii) and Theorem 3.1 there is an $h \in L_\infty(X)$ satisfying $(I - P)h = fg$. Now $(I_A - I_A \sum_{n=0}^N (PT_{f'})^n PT_f)P = I_A P - I_A \sum_{n=0}^N (PT_{f'})^n P^2 + I_A \sum_{n=0}^N (PT_{f'})^n PT_{f'} P = I_A \sum_{n=0}^N (PT_{f'})^n P(I - P) + I_A (PT_{f'})^{N+1} P$. But $(I - P)h = fg$, and $(PT_{f'})^n 1 \rightarrow 0$ by Lemma 2.1 of [8], so

that $(I_A - I_A P_f)I_A P h = (I_A - I_A P_f)P h = \lim_{N \rightarrow \infty} I_A \sum_{n=0}^N (PT_{f'})^n P(I - P)h + \lim_{N \rightarrow \infty} I_A (PT_{f'})^{N+1} P h = \lim_N I_A \sum_{n=0}^N (PT_{f'})^n P(fg) = I_A P_f g$. Thus $I_A P_f g = (I_A - I_A P_f)I_A P h$ and $(I_A - I_A P_f)(I_A P h + g) = I_A g = g$. Since $\{g \in L_\infty(A) : \int fg \, d\lambda = 0\}$ is a closed subspace of $L_\infty(A)$, $(I_A - I_A P_f)L_\infty(A)$ has closed range, and by Theorem 2.2 $I_A P_f$ is quasi-compact, being conservative and ergodic by Lemma 3.1.

REMARKS. 1. (i) \Rightarrow (ii). For the case $f = 1_A$ was similarly proved first by Brunel [1] (using one of the equivalent conditions of quasi-compactness).

2. Horowitz [10] showed only that if (ii) holds for $f = 1_A$, then for certain subsets B of A , $I_B P_{1_B}$ is quasi-compact.

3. Following the method in [10] it can be shown that if f is "small" in the sense of Ornstein and Sucheston [17] (and $0 \leq f \leq 1$), then $I_A P_f$ is quasi-compact.

4. Let f satisfy (ii) in the theorem. If $E = \{x : f(x) \geq a\}$, then $\lambda(E) < \infty$, and $\lambda(E) > 0$ for a small enough. For $B \subset E$ define (on E) $g = f^{-1}(1_B - \lambda(B)/\lambda(E)1_E)$. Then $g \in L_\infty(A)$ and $\int fg \, d\lambda = 0$. Thus $\sup_{N \geq 0} \|\sum_{n=0}^N P^n(1_B - \lambda(B)/\lambda(E)1_E)\|_\infty < \infty$ for $B \subset E$ and by Horowitz [10] P is Harris. (By Theorem 3.2 $I_E P_{1_E}$ is quasi-compact). Thus Theorem 3.2 implicitly assumes that P is Harris.

DEFINITION. Let P be a conservative and ergodic Markov operator. A function $0 \leq f$ is called special if for every $0 \leq h \leq 1$ in L_∞ , $h \not\equiv 0$, we have $\sum_{n=0}^\infty (PT_{h'})^n P f \in L_\infty$. If 1_A is special then A is a special set.

Special functions were introduced in Neveu's remarkable paper [15], while special sets were treated by Brunel [1], who proved that A is special if and only if $L_1(I_A - I_A P_{1_A})$ is closed ($I_A P_{1_A}$ is quasi-compact). We extend this result to special functions, with a different proof, once we assume that P is Harris. Brunel's result is needed, however, to establish the fact that special functions exist only for Harris operators, which is done in the next lemma.

LEMMA 3.2. Let P be ergodic and conservative.

- (i) If there exists a special function $f \not\equiv 0$, then P is Harris.
- (ii) If P is Harris, with σ -finite invariant measure λ , then $\int f \, d\lambda < \infty$ for every special function f .

PROOF. (i) Let $f \geq 0$ be special. Then $P f$ is bounded, and $P f \not\equiv 0$. For $h \not\equiv 0$ with $0 \leq h \leq 1$ we have

$$\begin{aligned} \sum_{n=0}^\infty (PT_{h'})^n P(P f) &= \sum_{n=0}^\infty (PT_{h'})^n P T_{h'} P f + \sum_{n=0}^\infty (PT_{h'})^n P T_h P f \\ &\leq \sum_{n=0}^\infty (PT_{h'})^n P f + \|f\|_\infty P_h 1. \end{aligned}$$

Thus $P f \not\equiv 0$ is special and bounded. Clearly $A = \{x : P f(x) \geq \alpha\}$ is a special set. By Brunel [1] $I_A P_{1_A}$ is quasi-compact and thus P is Harris by Horowitz [10].

(ii) Let $0 \leq h \leq 1$ satisfy $\int h \, d\lambda < \infty$. Then

$$\begin{aligned} \|h\|_1 \|\sum_{n=0}^\infty (PT_{h'})^n Pf\|_\infty &\geq \int h \cdot \sum_{n=0}^\infty (PT_{h'})^n Pf \, d\lambda \\ &= \int f \cdot \sum_{n=0}^\infty (P^*T_{h'})^n P^*T_h 1 \, d\lambda = \int f \, d\lambda. \end{aligned}$$

(Lemma 3.1 is applied to the adjoint process P^*).

Some of Neveu's results [15] concerning special functions are collected in the following proposition.

PROPOSITION 3.1. *Let P be a Harris operator.*

(i) *There exists a function $0 < h \in L_\infty$ with $0 < h \leq 1$ a.e. and a measure $m_0 \sim m$, such that $\sum_{n=0}^\infty (PT_{h'})^n P1_A \geq m_0(A)$ a.e. for every $A \in \Sigma$.*

(ii) *A function $f \geq 0$ is special if and only if (for the function h in (i)) $\sum_{n=0}^\infty (PT_{h'})^n Pf \in L_\infty$.*

(iii) *If $\int g \, d\lambda = 0$ and $|g| \in L_\infty$ is special, then $g \in (I - P)L_\infty$.*

REMARK. Neveu's results are obtained for the operators on the space $B(X, \Sigma)$ of bounded measurable functions, induced by transition probabilities, with the probabilistic Harris condition: if $m(A) > 0$ then $\sum_{n=0}^\infty (PI_{A'})^n P1_A(x) = 1$ for every x , and with the assumption of separability of Σ . It does not seem that Doob's method of admissible sub- σ -algebras ([5] page 209) can be used to obtain the proof in the general case. We therefore give the reduction of Proposition 3.1 to Neveu's results.

PROOF. We may and do assume $m(X) = 1$. Let X' be the compact Hausdorff space such that $C(X')$ is isometrically isomorphic to $L_\infty(X, \Sigma, m)$, and it is known that order will also be preserved (see Horowitz [10], whose results we are now using). P is then represented by an operator P' on $C(X')$ which is induced by a transition probability $P'(x', A)$, $x' \in X'$, $A \in \Sigma' = \text{Baire } \sigma\text{-algebra of } X'$, and m is represented by a measure m' on (X', Σ') , such that $L_\infty(x, \Sigma, m)$ and $L_\infty(x', \Sigma', m')$ are isometrically and order-preserving isomorphic, and pointwise convergence is also preserved under this isomorphism.

Since P is Harris, by Theorem 3.2 of [10] there exists a set $N \in \Sigma'$ with $m'(N) = 0$ such that if $m'(A) > 0$ then $\sum_{n=0}^\infty (P'I_A)^n P'1_A(x') = 1$ for $x' \notin N$. From theorem 6 of Moy [13] it follows that P' is a Harris operator on (X', Σ', m') (separability is not necessary to obtain the existence of a kernel). Let $q'_n(x', y')$ be the maximal kernel of P'^n , with corresponding integral operator Q'_n . Then $R'_n = P'^n - Q'_n$ is given by a transitive probability, and for a.e. x' $R'_n(x', \cdot)$ is singular with respect to m' [7].

Define the measures $U_\theta(x'A) = \sum_{n=1}^\infty (1 - \theta)^{n-1} P'^n(x', A)$ on (X', Σ') . Then

$$U_\theta(x', A) = \sum_{n=1}^\infty (1 - \theta)^{n-1} Q'_n(x', A) + \sum_{n=1}^\infty (1 - \theta)^{n-1} R'_n(x', A).$$

Also $R'_n(x', X') \rightarrow 0$ m' -a.e. which shows that $Q'_n(x', X') \rightarrow 1$ a.e. and if

$$p_\theta(x', y') = \sum_{n=1}^\infty (1 - \theta)^{n-1} q'_n(x', y'),$$

then for a.e. x' , $p_\theta(x', y') > 0$ m' -a.e.

Let $N_0 = \{x' : m'\{y' : p_\theta(x', y') = 0\} > 0\}$, and let $N_1 = N_0 \cup N$. Define $N_2 = \{x' : \sum_{n=0}^\infty P'^n(x', N_1) > 0\}$. Since $m'(N_0) = m'(N) = 0$, also $m'(N_1) = m'(N_2) = 0$ (P' is a Markov operator on $L_\infty(X', \Sigma', m')$). Let $E = X' - N_2$. Then $m'(E) = 1$ and also $P'(y', E) = 1$ for $y' \in E$. On $(E, \Sigma' \cap E)$ we have a transition probability P'' satisfying the Harris condition and also $\sum_{n=1}^\infty (1 - \theta)^{n-1} P''^n(x', A) \geq \int_A P_\theta(x', y') m'(dy')$ for every $x' \in E$, with $p_\theta(x', y') > 0$ on $E \times E$. These are exactly the conditions in [15] to obtain all the results stated in Proposition 3.1 (with equalities everywhere). Going back to P' on (X', Σ') we have that Proposition 3.1 is true for P' on $L_\infty(X', \Sigma', m')$ and therefore for P on $L_\infty(X, \Sigma, m)$.

THEOREM 3.3. *Let P be an ergodic and conservative Markov operator with σ -finite invariant measure $\lambda \sim m$. Let $0 \leq f \leq 1$ have support $A = \{x : f(x) > 0\}$. Then f is special if and only if $I_A P_f$ is quasi-compact.*

PROOF. If $\lambda(A) = 0$ both conditions are trivially satisfied, so we assume $\lambda(A) > 0$.

Let $I_A P_f$ be quasi-compact. Without loss of generality we may assume $\int f d\lambda = 1$, by Lemma 3.1. Let $0 \leq h \leq 1$ and $0 \neq h \in L_\infty(X)$. Then $g = h1_A - (\int hf d\lambda)1_A \in L_\infty(A)$ and $\int gf d\lambda = 0$. By Theorem 3.2 there exists an $h_1 \in L_\infty$ with $(I_A - I_A P_f)h_1 = g$ (since $d\mu = f d\lambda$ is the invariant measure of $I_A P_f$). Thus $PT_f g = P(T_f - T_f P_f)h_1 = (I - P)P_f h_1$ by Lemma 3.1 and

$$\begin{aligned} hf - (\int hf d\lambda)f &= fg = T_f g = (I - P)(P_f h_1 + fg) \\ &= (I - T_h P - T_{h'} P)(P_f h_1 + fg). \end{aligned}$$

Denote $h_2 = P_f h_1 + fg$. Then $(\int hf d\lambda)f = T_h(f + Ph_2) + T_{h'} Ph_2 - h_2$.

$$\int hf d\lambda \sum_{n=0}^\infty (PT_{h'})^n P f = P_h(f + Ph_2) + \lim_{N \rightarrow \infty} (PT_{h'})^N Ph_2 - Ph_2.$$

Since $(PT_{h'})^N 1 \downarrow 0$ a.e. we have that if $\int hf d\lambda \neq 0$, then $\sum_{n=0}^\infty (PT_{h'})^n P f \in L_\infty$. Take $0 < h \leq 1$ which is defined in the proposition, and by (ii) of that proposition f is special.

For the converse, assume $0 \leq f \leq 1$ special. The invariant measure λ satisfies $\int f d\lambda < \infty$. Let $g \in L_\infty(A)$ satisfy $\int gf d\lambda = 0$. Then $|gf|$ is special, and by part (iii) of Proposition 3.1 $gf \in (I - P)L_\infty$. Thus $\sup_{N \geq 0} \|\sum_{n=0}^N P^n(fg)\|_\infty < \infty$. By Theorem 3.2 $I_A P_f$ is quasi-compact.

COROLLARY 3.2. *Let $0 \leq f \leq 1$. If there exists a measure m_0 with $m_0 I_A \sim m I_A$ such that for every $0 \leq g \in L_\infty(A)$ we have $\sum_{n=0}^\infty (PT_{f'})^n P g \geq \int g dm_0$, then f is special.*

PROOF. For $0 \leq h \in L_\infty(A)$ we have

$$I_A P_f h = I_A \sum_{n=0}^\infty (PT_{f'})^n P(fh) \geq I_A \int fh dm_0 = c I_A (c > 0, \text{ if } h \neq 0)$$

and by Horowitz [10] $I_A P_f$ is quasi-compact, and by Theorem 3.3, f is special.

REMARKS. 1. Neveu's Corollary 4.8 [15] is the particular case when $A = X$.

2. If $0 \leq f$ is special with $\|f\|_\infty < 1$, then an m_0 as in the corollary must exist [15].

THEOREM 3.4. *Let P be a Harris process. Then for every two probability measures $\mu, \nu \ll \lambda$ and for every two bounded special functions f and g , we have*

$$\lim_{N \rightarrow \infty} \frac{\sum_{n=0}^N \langle \mu P^n, f \rangle}{\sum_{n=0}^N \langle \mu P^n, g \rangle} = \int f d\lambda / \int g d\lambda .$$

($f, g \in L_1(\lambda)$ by Lemma 3.2).

PROOF. Define $h = f - (\int f d\lambda / \int g d\lambda)g$. Then h is a special function with $\int h d\lambda = 0$. By Proposition 3.1 $h \in (I - P)L_\infty$. Thus for every probability measure $\mu \ll \lambda$,

$$\begin{aligned} \sup_{N \geq 0} \left| \frac{\sum_{n=0}^N \langle \mu P^n, f \rangle}{\sum_{n=0}^N \langle \mu P^n, g \rangle} - \frac{\int f d\lambda}{\int g d\lambda} \right| &= \sup_{N \geq 0} \left| \frac{\sum_{n=0}^N \langle \mu P^n, h \rangle}{\sum_{n=0}^N \langle \mu P^n, g \rangle} \right| \\ &\leq \sup_N \| \sum_{n=0}^N P^n h \|_\infty < \infty . \end{aligned}$$

Thus $\frac{\sum_{n=0}^N \langle \mu P^n, f \rangle}{\sum_{n=0}^N \langle \mu P^n, g \rangle} \rightarrow \int f d\lambda / \int g d\lambda$. By Lin [11] there exists a special set A such that $\frac{\sum_{n=0}^N \mu P^n(A)}{\sum_{n=0}^N \nu P^n(A)} \rightarrow 1$. The theorem follows by applying our result first to μ, f and 1_A and then to $\nu, 1_A$ and g .

REMARK. When P is given by a transition probability with the Harris condition, the theorem was proved (probabilistically) by Metivier [12]. However, using the previously known ratio limit theorem his result can be obtained as in our proof. (Note that his proof is based on Neveu's work [15], so separability of (X, Σ) was implicitly assumed.)

COROLLARY 3.3. *Let P be Harris, with σ -finite invariant measure λ . Then there exists a probability measure $\mu \sim \lambda$, such that $\lambda(A) < \infty$ if and only if for every $B \subset A$; $\frac{\sum_{n=0}^N \mu P^n(B)}{\sum_{n=0}^N \mu P^n(A)}$ converges.*

PROOF. Since P is Harris, so is P^* (see [7]). By Proposition 3.1 there exists a bounded function $h > 0$ a.e. which is special for P^* . Let $d\mu/d\lambda = h$ (we may and do assume $\int h d\lambda = 1$). Then if $\lambda(A) < \infty$, we have for $B \subset A$ that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\sum_{n=0}^N \mu P^n(B)}{\sum_{n=0}^N \mu P^n(A)} &= \lim_{N \rightarrow \infty} \frac{\sum_{n=0}^N \langle 1_B, P^{*n} h \rangle}{\sum_{n=0}^N \langle 1_A, P^{*n} h \rangle} = \lambda(B) / \lambda(A) \end{aligned}$$

by applying the previous theorem to P^* .

For the converse, it can be proved that

$$\nu(B) = \lim_{N \rightarrow \infty} \frac{\sum_{n=0}^N \mu P^n(B)}{\sum_{n=0}^N \mu P^n(A)}$$

(which is a measure by the Vitali-Hahn-Saks theorem) is invariant for $I_A P_A$ (similar to the proof in [9]).

REMARK. In [8] it is indicated that not every $\mu \sim \lambda$ will necessarily satisfy the conclusion of the corollary (when $\lambda(X) = \infty$).

The remainder of this section deals with some results which use the special functions.

THEOREM 3.5. *Let P be Harris with invariant measure λ , and let $f \in L_\infty$ satisfy $\int f d\lambda = 0$ and $|f|$ special.*

(i) If $\lambda(X) = 1$ and P has period d , then $\lim_{N \rightarrow \infty} \sum_{n=0}^{Nd+r} P^n f(x)$ exists a.e., for fixed $0 \leq r < d$.

(ii) If $\lambda(x) = \infty$, then for every $v \in L_1(m)$ with $\int v dm = 0$ the limit $\lim_{N \rightarrow \infty} \sum_{n=0}^N \langle v, P^n f \rangle$ exists.

PROOF. (i) By Proposition 3.1 $f = (I - P)g$ for some $g \in L_\infty$, hence $\sum_{n=0}^{Nd+r} P^n f = g - P^{Nd+r}g$. Since $g \in L_\infty$, we have that $P^{Nd+r}g$ converges a.e. [7].

(ii) Again $f = (I - P)g$ and $\|\sum_{n=0}^N P^n f\|_\infty \leq 2\|g\|_\infty$ for every N . Fix $v \in L_1(m)$ with $\int v dm = 0$. Let N_i be a sub-sequence such that $\sum_{n=0}^{N_i} \langle v, P^n f \rangle$ converges. Define a linear functional on $L_1(m)$ by a Banach limit (and represent it by $h \in L_\infty$): $\langle u, h \rangle = \text{LIM} \{ \sum_{n=0}^{N_i} \langle u, P^n f \rangle \}$. Then

$$\begin{aligned} \langle u, (I - P)h \rangle &= \langle u(I - P), h \rangle = \text{LIM} \{ \sum_{n=0}^{N_i} \langle u(I - P), P^n f \rangle \} \\ &= \text{LIM} \{ \langle u, f \rangle - \langle u, P^{N_i+1}f \rangle \} = \langle u, f \rangle. \end{aligned}$$

Since $P^n f(x) \rightarrow 0$ a.e. (by [7] we have $P^n 1_A(x) \rightarrow 0$ a.e. when $\lambda(A) < \infty$. But $|f| \in L_\infty \cap L_1(\lambda)$ so that for every $\epsilon > 0$, $A = \{x: |f|(x) \geq \epsilon\}$ has finite λ measure, and $P^n |f| \leq \epsilon + \|f\|_\infty P^n 1_A \rightarrow \epsilon$. Hence $|P^n f| \leq P^n |f| \rightarrow 0$ a.e.).

Thus $(I - P)h = f$ and $h = g + \text{const}$. Since $\int v dm = 0$, we have

$$\lim_{N_i} \sum_{n=0}^{N_i} \langle v, P^n f \rangle = \langle v, h \rangle = \langle v, g \rangle,$$

so that the limit does not depend on $\{N_i\}$, and the convergence is proved.

COROLLARY 3.4. Let P be Harris, and $\lambda(X) = \infty$. Let $f \in L_\infty$ with $|f|$ special satisfy $\int f d\lambda = 0$. Then $\sum_{n=0}^N P^n f(x)$ converges a.e. if and only if for some $0 \leq u \in L_1(m)$ (with $\int u dm > 0$) $\sum_{n=0}^N \langle u, P^n f \rangle$ converges.

PROOF. (i) Assume $\sum_{n=0}^N P^n f(x)$ converges a.e. Since $\|\sum_{n=0}^N P^n f(x)\|_\infty \leq K$ for every N , we can apply the bounded convergence theorem.

(ii) Assume $\sum_{n=0}^N \langle u, P^n f \rangle$ converges for some $u \in L_1$, $\int u dm \neq 0$. Then from the theorem we have that $\sum_{n=0}^N \langle u, P^n f \rangle$ converges for every $u \in L_1(m)$. The fact that $\sum_{n=0}^N P^n f(x)$ is a Cauchy sequence a.e. can now be proved using the (standard) technique of ([11] page 364).

COROLLARY 3.5. Let $\{p_{ij}^{(n)}\}$ be the n -step transition probabilities of an irreducible chain.

(i) If the chain is positive-recurrent with period d , and invariant probability $\{\lambda_i\}$, then for every four points i, j, k, t the limit $\lim_{N \rightarrow \infty} \sum_{n=0}^{Nd+r} \{\lambda_k p_{ij}^{(n)} - \lambda_j p_{ik}^{(n)}\}$ exists ($0 \leq r < d$).

(ii) If the chain is null-recurrent with invariant measure $\{\lambda_i\}$ then for every four points i, j, k, t the sequence $\sum_{n=0}^N \{\lambda_k (p_{ij}^{(n)} - p_{ij}^{(n)}) - \lambda_j (p_{ik}^{(n)} - p_{ik}^{(n)})\}$ converges.

PROOF. The operator P induced by $p_{ij}^{(1)}$ is Harris, and so is P^* , and finite sets are special. (i) follows from applying Theorem 3.5 both to P and P^* (which have the same period). (ii) follows from Theorem 3.5.

REMARKS. 1. (i) is proved by Orey [16] using generating functions (even absolute convergence is shown). See also Chung ([4] pages 66-69).

2. (ii) is proved by Chung ([4] page 69) only with $t = j$ and with certain restrictions, but there the limit is also identified. Orey [16] shows that even for a chain the conditions of Corollary 3.4 may not be satisfied.

THEOREM 3.6. *Let P be conservative and ergodic with invariant probability measure λ . If the space $\{V \in \text{Ba}(X, \Sigma, m) : \langle V, Pf \rangle = \langle V, f \rangle \text{ for } f \in L_\infty\}$ of finitely additive invariant measures (charges) is separable, then P is Harris.*

PROOF. Given a pure charge V , we can find sets A_ε with $V(A_\varepsilon) = 0$ and $\lambda(A_\varepsilon) > 1 - \varepsilon$, for every $\varepsilon > 0$ (see [7]). If V_i is a dense sequence in the space of invariant charges, then $V_i = \alpha_i \lambda + V_{0i}$ with V_{0i} a pure charge. Find A_i such that $V_{0i}(A_i) = 0$ and $\lambda(A_i) \geq 1 - \varepsilon 2^{-i}$. Let $A = \bigcap_{i=1}^\infty A_i$. Then $\lambda(A) \geq 1 - \varepsilon$ and $V_{0i}(A) = 0$ for every i . For $B \subset A$ we put $f = 1_B - (\lambda(B)/\lambda(A))1_A$. Then $V_i(f) = 0$ for every i and therefore $V(f) = 0$ for every $V \in \text{Ba}(X, \Sigma, m)$ with $VP = V$. Thus $\|N^{-1} \sum_{n=0}^{N-1} P^n f\|_\infty \rightarrow 0$ (by the Hahn-Banach theorem). By the ergodic theorem $N^{-1} \sum_{n=0}^{N-1} P^n 1_A(x) \rightarrow \lambda(A)$ a.e. and by Egorov's theorem there is a set $A_1 \subset A$ such that this convergence is uniform. Let $B \subset A_1 \subset A$. For $x \in A_1$ we have $N^{-1} \sum_{n=0}^{N-1} P^n 1_B(x) \geq (\lambda(B)/\lambda(A))N^{-1} \sum_{n=0}^{N-1} P^n 1_A(x) - \delta \geq \lambda(B)(1 - \delta) - \delta$ if $N \geq N_0(\delta)$. If δ is small enough, $c = \lambda(B)(1 - \delta) - \delta > 0$ and $\sum_{n=0}^{N-1} P^n 1_B \geq Nc1_{A_1}$ for $B \subset A_1$. By Horowitz [10] ($I_{A_1} P_{A_1}$ is quasi-compact and) P is Harris.

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