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SENSITIVE DISCOUNT OPTIMALITY IN CONTROLLED ONE-DIMENSIONAL DIFFUSIONS[1]

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In this paper we consider the problem of optimally controlling a diffusion process on a compact interval in one-dimensional Euclidean Space. Under the assumptions that the action space is finite and the cost rate, drift and diffusion coefficients are piecewise analytic, we present a constructive proof that there exist piecewise constant n-discount optimal controls for all finite $n \geqq 1$ and measurable ∞-discount optimal controls. In addition we present a sequence of second order differential equations that characterize the coefficients of the Laurent series of the expected discounted cost of an n-discount optimal control.

1. Introduction. In this paper we consider the problem of optimally controlling a one-dimensional diffusion process on a compact interval, where the coefficients of the infinitesimal operator and the costs are piecewise analytic functions and the set of possible actions is finite. We extend Pliska's results [12] concerning the existence of piecewise constant optimal controls for the expected discounted cost criterion to the case where the discount rate is allowed to decrease to zero. The optimality criterion used for studying this problem is n-discount optimality. Our results and those of Pliska depend heavily on Mandl's work [10].

The criterion of n-discount optimality was introduced in the finite state and action Markovian decision problem in the case $n = 0$ and $n = +\infty$ by Blackwell [1] and extended to include all $n \geqq -1$ by Veinott [16]. In these papers it is proved that there exist stationary n-discount optimal policies for all n. In the paper by Veinott [16] it is shown that these results are also valid for the continuous time model. In [2] Denardo applies this criterion to Markov renewal programming. Jaquette [9] introduces a new optimality criterion, moment optimality and using expansions analogous to ours proves that there exists moment optimal policies for all small interest rates.

In order to prove the existence of n-discount optimal controls, we first prove the validity of the Laurent expansion of the expected discounted cost in powers of the discount rate. This approach was first used by Miller and Veinott [11] in the finite state and action case. In Section 3 we give a development of this theory. In Theorem 4 we present a recursive scheme for computing the terms

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of the Laurent expansion of the expected discounted cost of a controlled diffusion. This theorem is the essential tool in proving the results of Section 5.

In Section 4 we extend Veinott's definition of n-discount optimality to include controlled diffusions. In Section 5 we prove that there exist piecewise constant n-discount optimal controls for all finite n in both the conservative and nonconservative cases. Further, we present a quasi-linear second order differential equation for the nth term in the Laurent expansion of the expected discounted cost of an n-discount optimal control. These results appear as Theorem 5. Implicit in the proof of this theorem is an algorithm for computing n-discount optimal controls.

In Section 6 we give a nonconstructive proof that there exists a measurable ∞-discount optimal control.

2. Model description and known results. Consider a system that is observed continuously through time at all $t \in [0, \infty)$. At each time t, the system is found to be in a state x where $x \in I = [r_0, r_1]$, $-\infty < r_0 < r_1 < \infty$. We will call I the *state space* of the system and denote the interior of I by I°.

We will need the following preliminary definitions. Let J be an arbitrary finite set. We say that the set valued function $Z_x\colon I^\circ \to 2^J$ is *piecewise constant* in x on I° if $Z_x \neq \phi$ and $Z_x \supseteqq Z_{x+}$ for all $x \in I^\circ$, and I is the union of finitely many closed intervals with disjoint interiors such that Z_x is constant on the interior of each. Let $Z = \prod_{x \in I^\circ} Z_x$ be the set of all functions $\omega\colon I^\circ \to J$ with $\omega(x) \in Z_x$ for each $x \in I^\circ$. If $N \subseteqq Z$, define the *section* N_x of N at $x \in I^\circ$ by

$$N_x = \{z \in Z_x \colon z = \omega(x) \ \text{ for some } \ \omega \in N\}\,.$$

We say the N is *piecewise constant* if N_x is piecewise constant on I°.

A function $\omega \in Z$ is called *piecewise constant* if ω is right continuous on I° and I is the union of finitely many closed intervals with disjoint interiors such that ω is constant on the interior of each. Denote by N^p the set of all piecewise constant functions in $N \subseteqq Z$. Note that if N is piecewise constant, then $N^p \neq \phi$.

A function $f(x)$ will be called analytic on a closed interval I_1 if there exists an open interval $I_2 \supset I_1$ and a function $g(x)$ which is analytic on I_2 such that $f(x) = g(x)$ for all x in $I_1{}^\circ$. A function $f(x)$ is called *piecewise analytic* on I° if I is the union of a finite number of closed intervals with disjoint interiors on each of which f is analytic and if f is right continuous on I°.

When the system is observed in state x at time t, an action is chosen from the piecewise constant set valued function Z_x. The finite set, J, is called the *action set*, Z^p the *control set*, and a function $\omega \in Z^p$ a *control*. Corresponding to choosing $z \in Z_x$ at time t a cost $c(x, z)\,dt$ is incurred during the time interval $[t, t + dt)$. We require for each $z \in J$, that $c(\cdot, z)$ be a piecewise analytic function on I°. The function $c(x, z)$ is said to be the *continuous movement cost rate*.

We let $a(\cdot, z) > 0$ and $b(\cdot, z)$ be piecewise analytic functions on I° for each $z \in J$. If action z is chosen when the system is observed in state x, the behavior of the system is that of a diffusion process with its infinitesimal operator given by

$$a(x, z)\frac{d^2}{dx^2} + b(x, z)\frac{d}{dx}.$$

The boundaries of I, r_0 and r_1, will be assumed to be regular in the sense of Mandl ([10] page 25). At r_0 and r_1 the behavior of the system is determined by a boundary condition of the form $\Phi_i(f, g) = 0$ for $i = 0, 1$ where

$$\Phi_i(f, g) = (\theta_i + \kappa_i)f(r_i) - \theta_i \textstyle\int_I (f(x) + \nu_i(x))\, d\mu_i(x) - (-1)^i\pi_i f'(r_i) + \sigma_i(g(r_i) - c_i) - \kappa_i\gamma_i \tag{1}$$

for a given function g. Here $\theta_i, \kappa_i, \pi_i, \sigma_i \geqq 0$; $\theta_i + \kappa_i + \pi_i + \sigma_i > 0$; $\mu_i(\cdot)$ is a probability measure on I°; $f'(r_i)$ denotes the derivative of f evaluated at r_i and each of the constants corresponds to a particular kind of boundary behavior at r_i. The constant θ_i corresponds to jumps from the boundary into I° according to the measure $\mu_i(\cdot)$, κ_i to absorption, π_i to reflection, and σ_i to adhesion. A more detailed treatment of the boundary behavior can be found in Mandl ([10] pages 37–68), Feller [5], and Pliska [12].

In addition (1) includes a cost structure that corresponds to the boundary behavior. There is a cost $\gamma_i = \gamma(r_i)$, corresponding to the termination of the process at r_i, which is relevant only if $\kappa_i > 0$. There is a cost $\nu_i(x)$ associated with a jump into the interior from the boundary r_i, which is relevant only if $\theta_i > 0$. We require that $\nu_i(\cdot)$ be integrable with respect to the measure $\mu_i(\cdot)$. Last there is a cost c_i corresponding to adhesion at r_i, this cost being relevant only if $\sigma_i > 0$.

We will say that the process is *conservative* if $\kappa_0 + \kappa_1 = 0$ and both boundaries are not purely adhesive, i.e., $\theta_i + \pi_i > 0$ for $i = 0$ or 1. The problem will be formulated so that (1) is independent of the control ω and hence will be either conservative or nonconservative for all controls.

Corresponding to the control $\omega \in Z^p$, define the infinitesimal operator, A_ω, by

$$A_\omega = a_\omega(x)\frac{d^2}{dx^2} + b_\omega(x)\frac{d}{dx} \tag{2}$$

where $a_\omega(x) = a(x, \omega(x))$ and $b_\omega(x) = b(x, \omega(x))$.

The hypotheses on ω, $a(\cdot, \cdot)$, and $b(\cdot, \cdot)$ are sufficient to insure that there exists a diffusion process corresponding to $\omega \in Z^p$ and (1) in which all of the costs have been set to zero and $g = \lambda f$ for $\lambda > 0$ (Mandl [10] pages 37–47).

We define $P^\omega(t, x, G)$ for $t \in [0, \infty)$, $x \in I$, and G a Borel set in I to be the transition function of the diffusion process corresponding to ω and we define the family of linear operators $\{T_t^\omega;\ t \geqq 0\}$ by

$$T_t^\omega f(x) = \textstyle\int_I f(y)P^\omega(t, x, dy)$$

where $f \in C_I$, the Banach space of bounded, continuous functions on I with the norm of f given by

$$\|f\| = \sup_{x \in I} |f(x)|.$$

Let X_t^ω be the random function associated with the diffusion process determined by the control ω and for $x \in I$, let E_x denote the expectation corresponding to the process starting at x. We note that for $f \in C_I$, $E_x f(X_t^\omega) = T_t^\omega f(x)$. For $\lambda > 0$, the expected discounted cost starting from x and using control ω, $v_w(x, \lambda)$, is given by

$$v_w(x, \lambda) = E_x \textstyle\int_0^\infty e^{-\lambda t}\, dC_\omega(t) \tag{3}$$

where

$$C_w(t) = \textstyle\int_0^{\min(\xi,t)} c(X_s^\omega, \omega(X_s^\omega))\, ds + \sum_{i=0}^1 \int_I \nu_i(y) N_i^\omega(t, dy) + \chi_{\{\xi \leq t\}} \gamma(X_{\xi-}^\omega)\,. \tag{4}$$

Here the random variable $N_i^\omega(t, dy)$ represents the number of jumps from r_i into the interval $[y, y + dy)$ made by the trajectory up to time t and the nonnegative random variable ξ represent the termination time of the process. We note that $E_x\{\xi < \infty\}$ if and only if $\kappa_0 + \kappa_1 > 0$. The function $\chi_{\{\xi \leq t\}}$ is the characteristic function of the set $\{\xi \leqq t\}$.

Define the domain of the infinitesimal operator A_w for $\omega \in Z^p$ and given g by

$$\mathscr{D}(g) = \{f \in C_I \colon\ f'' \in C_I \ \text{ and } \ \Phi_i(f, g) = 0 \ \text{ for } \ i = 0, 1\}\,. \tag{5}$$

The following theorem of Mandl ([10] page 149) enables us to compute $v_\omega(x, \lambda)$.

THEOREM 1. *If $a_w > 0$, b_w, and c_w are piecewise analytic on I° and $\lambda > 0$, then $v = v_w(x, \lambda)$ is the unique piecewise analytic solution in $\mathscr{D}(\lambda v_\omega)$ of*

$$(A_\omega - \lambda I)v + c_\omega = 0\,. \tag{6}$$

The problem of minimizing the expected discounted cost can be stated as follows: choose an $\hat{\omega} \in Z$ such that

$$v_{\hat{\omega}}(x, \lambda) = \inf_{\omega \in Z^p} v_\omega(x, \lambda) \equiv \hat{v}(x, \lambda) \qquad \text{for all } \ x \in I\,. \tag{7}$$

In the nonconservative case, $v_\omega(x, 0)$ is the expected total cost. In the conservative case, where $v_\omega(x, \lambda)$ diverges as λ decreases to zero, the long run average cost criteria has been considered. This will be discussed below.

The following theorem is Pliska's [12] extension of a result of Mandl ([10] page 159).

THEOREM 2. *Suppose for each $z \in J$ that $a(\cdot, z) > 0$, $b(\cdot, z)$, and $c(\cdot, z)$ are piecewise analytic functions on I°, Z is piecewise constant, and either $\lambda > 0$ or $\lambda = 0$ and the system is nonconservative. Then the differential equation*

$$\min_{\omega \in Z^p} \{(A_\omega - \lambda I)v + c_\omega\} = 0 \tag{8}$$

has a unique piecewise analytic solution $v = \hat{v} \in \mathscr{D}(\lambda \hat{v})$. Moreover $v = \min_{\omega \in Z^p} v_\omega$,

$$N \equiv \{\omega \in Z\colon (A_\omega - \lambda I)\hat{v} + c_\omega = 0\} \neq \varnothing \tag{9}$$

is piecewise constant, $N^p \neq \varnothing$, and $\omega \in Z^p$ minimizes v_ω over Z^p if and only if $\omega \in N^p$.

In the case where the problem is conservative, $v_\omega(x, 0)$ is infinite. In this setting, Mandl [10] chose to minimize the long run average cost. For a control ω, we define its long run average cost, Θ_ω, by

$$\Theta_\omega = \lim_{t\to\infty} t^{-1}C_\omega(t) ,$$

which exists almost everywhere and is independent of x. It is well known that

$$\Theta_\omega = \lim_{\lambda\downarrow 0} \lambda v_\omega(x, \lambda) .$$

Letting λ decrease to zero in (8) we can formally pass to the following result which appears in the case where $a(x, z)$, $b(x, z)$ and $c(x, z)$ are continuous and J is compact as Theorem 6, Mandl ([10] page 168). Theorem 3 follows from Pliska [12].

THEOREM 3. *Suppose the hypotheses of Theorem 2 are satisfied and in addition the problem is conservative. Then there exists a unique number $\hat{\Theta}$ such that*

$$\min_{\omega\in Z}\{A_\omega v + c_\omega - \hat{\Theta}\} = 0 \tag{10}$$

has a solution v in $\mathscr{D}(\hat{\Theta})$, and v is unique up to an additive constant.

Furthermore, $\hat{\Theta} = \min_{\omega\in Z^p}\Theta_\omega$,

$$N \equiv \{\omega\in Z\colon A_\omega v + c_\omega - \hat{\Theta} = 0\} \neq \emptyset$$

is piecewise constant, $N^p \neq \emptyset$, and $\omega\in Z^p$ minimizes $\hat{\Theta}_\omega$ over Z^p if and only if $\omega\in N^p$.

3. The Laurent series expansion of $v_\omega(x, \lambda)$. In this section we prove the existence of the Laurent series expansion of $v_\omega(x, \lambda)$ and present a recursive scheme from the computation of the coefficients of the expansion. The validity of the Laurent series expansion depends on the fact that v_ω has the representation

$$v_\omega = R(\lambda, A_\omega)c_\omega \tag{11}$$

where $R(\lambda, A_\omega)$ denotes the *resolvent of A_ω*. In the nonconservative case $R(\lambda, A_\omega)$ admits an ordinary Taylor series expansion around $\lambda = 0$, but in the conservative case $R(0; A_\omega)$ is unbounded and the expansion does not exist.

The recursive scheme for computation of the coefficients generalizes the results of Veinott [16] for the finite state case and enables us to actually compute these expansions. In the past this has proved to be a much more complicated task.

We introduce the following notation before stating Theorem 4. Let $c_n^\omega(x) = 0$ for all $x\in I$ and $n\neq 0$, and for $n = 0$ define $c_0^\omega(x)$ by

$$\begin{aligned} c_0^\omega(x) &= c_\omega(x) , \qquad x\in I^\circ \\ &= -\theta_i \textstyle\int_I \nu_i(y)\, d\mu_i(y) - \sigma_i c_i - \kappa_i\gamma_i , \qquad x = r_i,\ i = 0, 1 . \end{aligned} \tag{12}$$

Notice that $c_n^\omega(r_i) = c_n(r_i)$ is independent of ω for all i and n. Further define $\Phi_i^n(f, g)$ for $n = -1, 0, 1, \cdots$ by

$$\begin{aligned} \Phi_i^n(f, g) = {}& (\theta_i + \kappa_i)f(r_i) - \theta_i \textstyle\int_I f(x)\, d\mu_i(x) - (-1)^i\pi_i f'(r_i) \\ &+ \sigma_i\, g(r_i) + c_n(r_i) \qquad \text{for } i = 0, 1 . \end{aligned} \tag{13}$$

Let $y_n^\omega(x) = 0$ for all $n \leqq -2$, $\omega\in Z^p$, and $x\in I$, and define $\mathscr{D}_n(g)$ for all n by

$$\mathscr{D}_n(g) = \{f;\ f''\in C_I \text{ and } \Phi_i^n(f, g) = 0,\ i = 0, 1\} . \tag{14}$$

THEOREM 4. *Let $\omega \in Z^p$. Then for $0 < \lambda <$ constant*

$$v_\omega(x, \lambda) = \sum_{n=-1}^{\infty} \lambda^n y_n^\omega(x) \tag{15}$$

where

$$A_\omega y_n^\omega + c_n^\omega - y_{n-1}^\omega = 0 \qquad \text{and} \qquad y_n^\omega \in \mathscr{D}_n(y_{n-1}^\omega) . \tag{16}$$

Conversely if y_n, $-1 < n < m$ satisfies (16) *then $y_n = y_n^\omega$ for $-1 < n < m$ and $y_m = y_n^\omega + k$ where k is any constant in the conservative case and $k = 0$ in the nonconservative case.*

PROOF. The proof is in two parts. We first show that (15) holds. As noted above the result is valid in the nonconservative case. In the conservative case the result follows from noting that T_t^ω converges exponentially fast to P^*, the stationary measure of the diffusion process corresponding to ω. (Doob [3] page 256 and Friedlin [6] page 60.) For $\lambda > 0$ we have the representation

$$R(\lambda; A_\omega) - \lambda^{-1}P^* = \int_0^\infty e^{-\lambda t}(T_t^\omega - P^*)\, dt .$$

This together with the exponential convergence of T_t^ω is sufficient to ensure the boundedness of $R(\lambda; A_\omega) - \lambda^{-1}P^*$ at $\lambda = 0$. Hence this operator admits a Taylor series expansion. We conclude (15) from (11).

From Theorem 1 we have that v_ω is the unique piecewise analytic solution of

$$(\lambda I - A_\omega)v_\omega = c_\omega \qquad \text{in} \quad \mathscr{D}(\lambda v_\omega) . \tag{17}$$

Substituting (15) into (17) we have that

$$\sum_{n=-1}^{\infty} \lambda^n y_{n-1}^\omega - \sum_{n=-1}^{\infty} \lambda^n A_\omega y_n^\omega = c_\omega . \tag{18}$$

Further since $v_\omega \in \mathscr{D}(\lambda v_\omega)$, $\Phi_i(v_\omega, \lambda v_\omega) = 0$ for $i = 0, 1$. On substituting (15) into (1) we get

$$\begin{aligned}(\theta_i + \kappa_i) \sum_{n=-1}^{\infty} \lambda^n y_n^\omega(r_i) - \theta_i \int_I \Big[\sum_{n=-1}^{\infty} \lambda^n y_n^\omega(x) + \nu_i(x)\Big]\, d\mu_i(x) \\ - (-1)^i \pi_i \sum_{n=-1}^{\infty} \lambda^n y_n^{\omega\prime}(r_i) + \sigma_i\Big[\sum_{n=-1}^{\infty} \lambda^n y_{n-1}^\omega(r_i) - c_i\Big] - \kappa_i \gamma_i = 0\end{aligned} \tag{19}$$

for $i = 0, 1$ and all small enough $\lambda > 0$.

Equating the coefficients of like powers of λ to 0 in (18) and (19) we have that y_n^ω satisfies (16).

We will prove the converse by induction on m. We first consider the nonconservative case. For $m = -1$, (16) implies

$$A_\omega y_{-1} = 0 . \tag{20}$$

From Theorem 1 with $\lambda = 0$, (20) has the unique solution $y_{-1} = 0$ in $\mathscr{D}_{-1}(y_{-2}^\omega)$. The general step follows from Theorem 1 with c_ω replaced by $c_m^\omega - y_{m-1}^\omega$ therein.

Next consider the conservative case. For $m = -1$ we have

$$\Phi_i^{-1}(y_{-1}, y_{-2}) = \theta_i y_{-1}(r_i) - \theta_i \int_I y_{-1}(x)\, d\mu_i(x) - (-1)^i \pi_i y_{-1}'(r_i) = 0 . \tag{21}$$

Since A_ω is a second order linear differential operator, (20) has two linearly independent solutions, (Mandl [10] page 30). It is easy to see that the constant function satisfies (20) and there exists a strictly increasing solution of (20) u, such that $u' > 0$. (Feller [4] page 483 and [5].) Hence any solution of (20) has the form

$$y_{-1}(x) = k + ju(x)$$

where k and j are constants to be determined from (21).

Substituting $y_{-1}(x)$ into (21) we have

$$\theta_i(k + ju(r_i)) - \theta_i \int_I (k + ju(x))\, d\mu_i(x) - (-1)^i \pi_i j u'(r_i) = 0 . \tag{22}$$

Noting that $\int_I d\mu_i(x) = 1$ and regrouping terms (22) becomes

$$j[\theta_i u(r_i) - \theta_i \int_I u(x)\, d\mu_i(x) - (-1)^i \pi_i u'(r_i)] = 0 . \tag{23}$$

Recall both boundaries cannot be purely adhesive. Thus either $\pi_0 + \theta_0 > 0$ or $\pi_1 + \theta_1 > 0$. In the former event $u(r_0) = 0$ and $\theta_0 \int_I u(x)\, d\mu_0(x) + \pi_0 u'(r_0)$ is positive so by (23) for $i = 0$, $j = 0$.

In the latter event, $u(r_1) = 1 > \int_I u(x)\, d\mu_i(x)$ and hence $\theta_1 u(r_1) - \theta_1 \int_I u(x)\, d\mu_1(x) + \pi_1 u'(r_1)$ is positive. Thus, by (23) for $i = 1$, $j = 0$. Thus $j = 0$ always and k cannot be determined from (21). This implies that $y_{-1} = y_{-1}^{\omega} + k$ where k is an arbitrary constant. This completes the case $m = -1$.

Suppose the result is true for m, so $y_n = y_n{}^{\omega}$ for $n < m$ and $y_m = y_m{}^{\omega} + k_m$. Then the equation for y_{m+1} becomes

$$A_\omega y_{m+1} + c_{m+1}^{\omega} - y_m = 0 . \tag{24}$$

Substituting in the general form for y_m we have

$$A_\omega y_{m+1} = -c_{m+1}^{\omega} + y_m{}^{\omega} + k_m . \tag{25}$$

The general solution of (25) is of the form

$$y_{m+1} = k_{m+1} - j_{m+1} u + g - k_m h , \tag{26}$$

where k_{m+1} and j_{m+1} are arbitrary constants, $k_{m+1} - j_{m+1} u$ is the homogeneous solution of (25), g is a particular solution of

$$A_\omega g = -c_{m+1}^{\omega} + y_m{}^{\omega} ,$$

and h is a particular solution of

$$A_\omega h = -1 ,$$

subject to $h(r_0) = h(r_1) = 0$. Noting that $h(x) = E_x \tau$, where $\tau = \inf\{t > 0;\ X_t{}^{\omega} = r_0 \text{ or } r_1\}$ it follows that $h(x) > 0$ on I° and $h'(r_0) > 0$ and $h'(r_1) > 0$.

Define the operator G_i on the space of continuously differentiable functions f on I by

$$G_i f = \theta_i(f(r_i) - \int_I f(x)\, d\mu_i(x)) - (-1)^i \pi_i f'(r_i)$$

for $i = 0, 1$. In this notation, since $y_{m+1} \in \mathscr{D}_{m+1}(y_m)$, we have

$$\Phi_i{}^{m+1}(y_{m+1}, y_m) = G_i y_{m+1} + \sigma_i y_m(r_i) + c_{m+1}(r_i) = 0 \tag{27}$$

for $i = 0, 1$. Substituting (26) and $y_m = y_m{}^{\omega} + k_m$ into (27) gives

$$j_{m+1} G_i u + k_m(G_i h - \sigma_i) = \Phi_i{}^{m+1}(g, y_m{}^{\omega}) \tag{28}$$

for $i = 0, 1$. Note that (28) is independent of k_{m+1} which is therefore not restricted by the boundary conditions. Now we know that $k_m = 0$ satisfies (28) for some j_{m+1} because (16) is satisfied by $(y^{\omega}_{m+1}, y_m{}^{\omega})$. Thus, it remains to show that no other value of k_m satisfies (28) for some j_{m+1}. This will be so provided the determinant

$$\Delta \equiv \begin{vmatrix} G_0 u & G_0 h - \sigma_0 \\ G_1 u & G_1 h - \sigma_1 \end{vmatrix} = (G_0 u)(G_1 h - \sigma_1) + (-G_1 u)(G_0 h - \sigma_0) \tag{29}$$

is not zero. Now from the properties of h and u we have $G_0 u \leqq 0$, $-G_1 u \leqq 0$, $G_0 h \leqq 0$, and $G_1 h \leqq 0$, so $\Delta \geqq 0$. In fact, since $\theta_i + \pi_i + \sigma_i > 0$ for $i = 0, 1$, $G_i h - \sigma_i < 0$ for $i = 0, 1$. Also since $(\theta_0 + \pi_0) + (\theta_1 + \pi_1) > 0$, either $G_0 u < 0$ or $-G_1 u < 0$. Combining these facts we see that $\Delta > 0$, completing the proof.

4. Characterization of discount optimality. In this section we present a characterization of n-discount optimality that is analogous to that first presented by Blackwell [1] for $n = 0$ and $+\infty$ and Veinott [16] for all finite $n \geqq -1$. In the case of $n = +\infty$ we must modify Blackwell's definition to account for the nonfiniteness of the state space. A control is then shown to be n-discount optimal if and only if it lexicographically minimizes the first $n + 2$ terms in the Laurent series expansion of the expected discounted cost. We note that Veinott's [16] notion of a transient problem correspond to a nonconservative problem in our terminology.

We will say that a control $\hat{\omega} \in Z^p$ is *n-discount optimal* if for all $x \in I$ and $\omega \in Z^p$,

$$\limsup_{\lambda \downarrow 0} \lambda^{-n}[v_{\hat{\omega}}(x, \lambda) - v_{\omega}(x, \lambda)] \leqq 0 \,. \tag{30}$$

It is clear that if $\hat{\omega}$ is n-discount optimal, then it is m-discount optimal for $m = -1, \cdots, n$. In the nonconservative case all controls are -1-discount optimal since $v_{\omega}(x, 0)$ is finite for all $\hat{\omega} \in Z^p$ and hence $\limsup_{\lambda \downarrow 0} \lambda v_{\omega}(x, \lambda) = 0$. Equivalently this follows from noting that $y^{\omega}_{-1}(x) = 0$ in this case. In the conservative case we see that -1-discount optimality corresponds to minimizing the long run average cost Θ_{ω}. This follows from the fact that $\Theta_{\omega} = \lim_{\lambda \downarrow 0} \lambda v_{\omega}(x, \lambda)$. The results in this case appear in Theorem 3.

A control $\hat{\omega} \in Z^p$ will be called *∞-discount optimal* if for each $x \in I$, there exists a $\lambda^*(x) > 0$ such that

$$v_{\hat{\omega}}(x, \lambda) - v_{\omega}(x, \lambda) \leqq 0 \tag{31}$$

for all $0 < \lambda < \lambda^*(x)$, $x \in I$, and $\omega \in Z^p$.

We show that a control is n-discount optimal if and only if it lexicographically minimizes the first n-coefficients in the Laurent series expansion of $v(x, \lambda)$. Similarly a control will be shown to be ∞-discount optimal if and only if it is n-discount optimal for all n, or what is the same thing, lexicographically minimizes the entire vector of coefficients of the Laurent expansion of $v(x, \lambda)$. We follow the notation and development of Veinott [16].

Denote by D_n the set of $\omega \in Z^p$ for which ω is n-discount optimal, for

$n = -1, 0, 1, \cdots$. It is easy to see that $D_{-1} \supseteqq D_0 \supseteqq D_1 \supseteqq \cdots \supseteqq D_\infty$. Let $F_N = (f_{-1}, f_0, \cdots, f_N)$, $-1 \leqq N \leqq \infty$, be a sequence of real valued functions on I. We say that F_N is *lexicographically nonnegative*, written $F_N \succeq 0$, if for each $x \in I$, the first nonzero element is positive. We say that F_N is *lexicographically greater than* G_N, denoted by $F_N \succeq G_N$, if $F_N - G_N \succeq 0$.

For $\omega \in Z^p$, we let $Y_N^\omega = (y_{-1}^\omega(x), y_0^\omega(x), \cdots, y_N^\omega(x))$ for $N \geqq -1$, $Y_N^\omega = 0$ for $N < -1$ and $Y^\omega = (y_{-1}^\omega(x), y_0^\omega(x), \cdots)$. Suppose we have a Laurent series $A_\lambda = \sum_{i=-1}^\infty \lambda^i a_i$. Then $\limsup_{\lambda \downarrow 0} \lambda^{-n} A_\lambda \geqq 0$ if and only if the vector of the coefficients of A_λ up to the nth power of λ is lexicographically nonnegative. This is sufficient to give us the following proposition.

PROPOSITION 1. *Let D_n and D_∞ be defined as above. Then*

$$D_n = \{\omega \in Z^p : Y_n^\alpha \succeq Y_n^\omega \ \textit{for all} \ \alpha \in Z^p\} \quad n = -1, 0, \cdots$$

and

$$D = \{\omega \in Z^p : Y^\alpha \succeq Y^\omega \ \textit{for all} \ \alpha \in Z^p\}.$$

5. Existence of n-discount optimal controls. In this section we show for all finite $n \geqq -1$, that the set of piecewise constant n-discount optimal controls is nonempty. Further we present second order quasilinear differential equations that characterize the nth term in the Laurent series expansion of the expected discounted cost of an n-discount optimal control. The idea of the proof is similar to that used by Veinott [15] in studying the finite state and action Markov decision problem.

Let $y_{-2}^\omega = 0$ for all $\omega \in Z^p$, $E_{-2} \equiv Z$, and $D_{-2} \equiv Z^p$. If $D_n \neq \emptyset$, let $\hat{y}_n = y_n^\omega$ for all $\omega \in D_n$. We now generalize Theorem 4 to obtain the main result of this paper.

THEOREM 5. *For each $m \geqq -1$, $D_m \neq \emptyset$. Also $y_n = \hat{y}_n$ for $-1 \leqq n < m$ and $y_m = \hat{y}_m + k$ where k is any constant in the conservative case and $k = 0$ in the nonconservative case if and only if*

$$\min_{\omega \in E_{n-1}} (A_\omega y_n + c_n^\omega - y_{n-1}) = 0 \qquad \textit{and} \qquad y_n \in \mathscr{D}_n(y_{n-1}) \tag{32}$$

where

$$E_n \equiv \{\omega \in E_{n-1} : A_\omega y_n + c_n^\omega - y_{n-1} = 0\}. \tag{33}$$

for all $-1 \leqq n \leqq m$. Furthermore for each $m \geqq -1$, E_m is piecewise constant, $E_m^p = D_{m-1}$ in the conservative case, and $E_m^p = D_m$ in the nonconservative case.

PROOF. The proof is constructive and is by induction on m. We discuss the nonconservative and conservative cases separately, beginning with the former.

Let $m = -1$. Then for all $\omega \in E_{-1}^p$, $y_{-1} = 0$ is the unique solution of

$$A_\omega y_{-1} = 0 \qquad \text{and} \qquad y_{-1} \in \mathscr{D}_{-1}(0). \tag{34}$$

Hence $\hat{y}_{-1} = 0$ is the unique solution of (32) and $E_{-1} = Z$ is piecewise constant, $E_{-1}^p \neq \emptyset$, $D_{-1} = E_{-1}^p$. Suppose now that the theorem holds for m and consider

$m+1$. Then by Theorem 2, there is a unique y_{m+1} satisfying (32) for $n = m+1$, $y_{m+1} = \min_{\omega \in D_m} y^{\omega}_{m+1} = \hat{y}_{m+1}$, E_{m+1} is piecewise constant, and $D_{m+1} = E^p_{m+1}$.

We now consider the conservative case. It suffices to show by induction on m that (32) has a solution, E_n is piecewise constant and $E_n{}^p = D_{n-1} \neq \emptyset$ for all $n \leqq m$, and each solution of (32) has the form $y_n = \hat{y}_n$ for $n < m$ and y_m is unique up to an additive constant. This is trivially so for $m = -1$ since for all $\omega \in E_1{}^p$, y_{-1} satisfies (34) if and only if y_{-1} is a constant. Thus $E_{-1} = Z$ and $E^p_{-1} = D_{-2}$. Suppose the assertion holds for m and consider $m+1$. By the induction hypothesis we have

$$(35)\qquad \min_{\omega \in E_{m-1}} (A_\omega(y_m + k_m) + c_m{}^\omega - \hat{y}_{m-1}) = 0 \quad\text{and}\quad y_m + k_m \in \mathscr{D}_m(\hat{y}_{m-1})$$

for constant k_m. Notice that E_m is independent of k_m. Now for each $\omega \in E_m{}^p = D_{m-1}$, it follows from Theorem 3 with $Z = \{\omega\}$ that there is a unique constant $k_m{}^\omega$ for which there is a y_{m+1} satisfying

$$(36)\qquad A_\omega y_{m+1} + (c^\omega_{m+1} - y_m) - k_m{}^\omega = 0 \qquad\text{and}\qquad y_{m+1} \in \mathscr{D}_{m+1}(y_m + k_m{}^\omega)\,.$$

From (35) and the definitions involved, for each such ω, $\hat{y}_{m-1} = y^\omega_{m-1}$ and

$$(37)\qquad A_\omega(y_m + k_m{}^\omega) + c_m{}^\omega - y^\omega_{m-1} = 0 \qquad\text{and}\qquad (y_m + k_m{}^\omega) \in \mathscr{D}_m(y^\omega_{m-1})\,.$$

Thus from (37), (36), and Theorem 4, we have $y_m{}^\omega = y_m + k_m$. Now by Theorem 3 again, but with $Z = E_m$ this time, there is a unique constant $\hat{k}_m$ for which there is a y_{m+1}, unique up to an additive constant, satisfying

$$(38)\quad \min_{\omega \in E_m} (A_\omega y_{m+1} + (c^\omega_{m+1} - y_m) - \hat{k}_m) = 0 \quad\text{and}\quad y_{m+1} \in \mathscr{D}_{m+1}(y_m + \hat{k}_m)\,.$$

Also $\hat{k}_m = \min_{\omega \in D_{m-1}} k_m{}^\omega$, E_{m+1} is piecewise constant, and $\omega \in D_{m-1}$ and $k_m{}^\omega = \hat{k}_m$ if and only if $\omega \in E^p_{m+1}$. Hence

$$y_m + \hat{k}_m = y_m + \min_{\omega \in D_{m-1}} k_m{}^\omega = \min_{\omega \in D_{m-1}} y_m{}^\omega = \hat{y}_m\,,$$

and $\omega \in D_{m-1}$ and $y_m{}^\omega = \hat{y}_m$ if and only if $\omega \in E^p_{m+1}$. Thus $E^p_{m+1} = D_m$, completing the proof.

Implicit in the proof of Theorem 5 is an algorithm for finding the set of n-discount optimal controls, and the first $n+2$ terms in the Laurent series expansion of the expected discounted cost of an n-discount optimal control. We state the algorithm in the conservative case.

I. Let $m = 0$, $E_{-1} = Z$, $D_{-2} = E^{p'}_{-1}$ and $y_{-1} = 0$.

II. Find a constant, $\hat{k}_{m-1}$, such that

$$\min_{\omega \in E_m} (A_\omega y_m - \hat{k}_{m-1} + c_m{}^\omega - y_{m-1}) = 0$$

has a solution, y_m, in $\mathscr{D}_n(y_{m-1})$. Note that $\hat{y}_{m-1} = y_{m-1} + \hat{k}_{m-1}$.

III. Let $E_m = \{\omega \in E_{m-1} \colon A_\omega y_m - c_m{}^\omega - \hat{y}_{m-1} = 0\}$. Then $D_{m-1} = E_m{}^p$.

IV. If $m - 1 = n$, stop; otherwise let $m = m+1$ and go back to Step II.

6. Existence of ∞-discount optimal controls. In this section we prove that there exists a measurable ∞-discount optimal control in both the conservative and nonconservative cases. The author is not sure if there exists a diffusion process on $[r_0, r_1]$ corresponding to a measurable control. Partial affirmative results in the n-dimensional case appear in Stroock and Varadhan [13] and [14]. In the special case where there exists a unique n-discount optimal control for some finite n, that control is ∞-discount optimal.

THEOREM 6. *There exists a measurable $\omega \in Z$ that is ∞-discount optimal.*

PROOF. On supplying I with the usual topology for the real line and J with the discrete topology, we see that $I \times J$ is compact in the product topology. Let $\bar{E}_n$ be the closure of E_n, $\bar{E}_\infty \equiv \bigcap_{n=-1}^\infty \bar{E}_n$, and $E_\infty = \bigcap_{n=-1}^\infty E_n$. Let $E_{nx} = \{z \in J: (x, z) \in E_n\}$, and define $E_{\infty x}$, $\bar{E}_{nx}$, and $\bar{E}_{\infty x}$ similarly. For each $x \in I$, $\bar{E}_{-1x} \supset \bar{E}_{0x} \supset \bar{E}_{1x} \supset \cdots$ are nonempty compact subsets of J so $\bar{E}_{\infty x} = \bigcap_{n=-1}^\infty \bar{E}_{nx}$ is nonempty and compact for each $x \in I$. Similarly, $\bar{E}_\infty$ is nonempty and compact. Also $E_{\infty x}$ is nonempty for each $x \in I^\circ$. Now by Theorem 5, E_n is piecewise constant so E_{nx} differs from $\bar{E}_{nx}$ for at most finitely many $x \in I$ depending on n. Thus $E_{\infty x}$ differs from $\bar{E}_{\infty x}$ for at most a countable set Q_0 of $x \in I$.

Now denote by $z_1, \cdots, z_k$ the elements of J. Define the $Q_1, \cdots, Q_k$ inductively by $Q_i = \{x \in I - \bigcup_{j=0}^{i-1} Q_j: (x, z_i) \in \bar{E}_\infty\}$, $i = 1, \cdots, k$. Clearly, $Q_0, Q_1, \cdots, Q_k$ is a partition of I into measurable sets. Now let $\omega(x) = z_i$ for $x \in Q_i$ and $i = 1, \cdots, k$, and choose $\omega(x) \in E_{\infty x}$ arbitrarily for $x \in Q_0$. Then $\omega \in E_\infty$ and ω is measurable, so ω is ∞-discount optimal.

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