The Annals of Probability
1974, Vol. 2, No. 4, 738–740

ON A CHARACTERIZATION OF THE FAMILY OF DISTRIBUTIONS WITH CONSTANT MULTIVARIATE FAILURE RATES

BY PREM S. PURI[1] AND HERMAN RUBIN[2]

Purdue University

Let $f(t_1, \cdots, t_k)$ be the probability density function of a vector $(Y_1, \cdots, Y_k)$ of nonnegative random variables. Let the multivariate failure rate (M.F.R.) $r(t_1, \cdots, t_k)$ be defined by the ratio $f(t_1, \cdots, t_k)/P(Y_i > t_i, i = 1, 2, \cdots, k)$, for $t_i \geqq 0$, $i = 1, \cdots, k$. It is shown that $r(t_1, \cdots, t_k)$ is constant if and only if the distribution of $(Y_1, \cdots, Y_k)$ is a mixture of exponential distributions. Analogous results hold for the nonnegative integer valued random vector with mixture being of geometric distributions.

Let $(Y_1, Y_2, \cdots, Y_k)$ be a vector of nonnegative random variables admitting a probability density function (pdf) with respect to Lebesgue measure, given by $f(t_1, \cdots, t_k)$, for $t_i \geqq 0$, $i = 1, 2, \cdots, k$. Let the multivariate failure rate (M.F.R.) $r(t_1, \cdots, t_k)$ be defined by

$$r(t_1, \cdots, t_k) = [f(t_1, \cdots, t_k)/P(Y_i > t_i, i = 1, 2, \cdots, k)], \tag{1}$$

for $t_i \geqq 0$, $i = 1, 2, \cdots, k$. In [2], the question was raised whether or not mixtures of exponential distributions are the only absolutely continuous distributions with constant M.F.R., or equivalently

$$\beta f(t_1, \cdots, t_k) = P(Y_i > t_i, i = 1, 2, \cdots, k), \tag{2}$$

for all $t_i \geqq 0$, $i = 1, 2, \cdots, k$, and for some positive constant β. The following Theorem gives the answer to this question in the affirmative.

THEOREM 1. *For a given $\beta > 0$, the only absolutely continuous distributions satisfying* (2) *are the ones which are mixtures of exponential distributions with* pdf *given by*

$$f(t_1, \cdots, t_k) = \beta^{-1} \int_0^\infty \cdots \int_0^\infty \exp[-\sum_{i=1}^k u_i t_i] G(du_1, \cdots, du_k), \tag{3}$$

for $t_i \geqq 0$, $i = 1, \cdots, k$, where the probability measure G is concentrated on the set $A = [\prod_{i=1}^k u_i = \beta^{-1}, u_i > 0, i = 1, 2, \cdots, k]$.

PROOF. It is easy to check that a pdf given by (3) satisfies (2). Conversely,

Received October 10, 1972; revised September 24, 1973.

[1] This investigation was supported in part by the Air Force Office of Scientific Research, AFSC, USAF, under Grant No: AFOSR71-2009 at Indiana University, Bloomington. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes not withstanding any copy right notation hereon.

[2] Research was supported in part by the Office of Naval Research Contract N0014-67-A-226-0008, project number NR042-216 at Purdue University.

AMS 1970 *subject classifications.* Primary 60E05.

Key words and phrases. Multivariate failure rate, characterization of distributions, mixture of distribution, exponential distribution, geometric distribution.

let a pdf f satisfy (2), or equivalently

$$\beta f(t_1, \cdots, t_k) = \int_{t_1}^{\infty} \cdots \int_{t_k}^{\infty} f(x_1, \cdots, x_k)\, dx_1 \cdots dx_k\,, \qquad t_i > 0,\ i = 1, \cdots, k\,. \tag{4}$$

It is clear from (4) that the pdf f must have derivatives of all orders with respect to $t_1, \cdots, t_k$, and that each of these is zero when evaluated at infinity for any of its arguments. In particular (4) is equivalent to

$$f(t_1, \cdots, t_k) = (-1)^k \beta \frac{\partial^k f(t_1, \cdots, t_k)}{\partial t_1 \partial t_2 \cdots \partial t_k}\,, \qquad t_i > 0,\ i = 1, 2, \cdots, k\,. \tag{5}$$

By a routine induction argument on $r_1, \cdots, r_k$, which involves a repeated use of (4), it can be easily shown that the functions

$$\psi_{r_1,\cdots,r_k}(t_1, \cdots, t_k) = (-1)^{\sum_{i=1}^k r_i} \cdot \beta \cdot \frac{\partial^{\sum_{i=1}^k r_i} f(t_1, \cdots, t_k)}{\partial t_1^{r_1} \cdots \partial t_k^{r_k}}\,, \tag{6}$$

are nonnegative for all $t_i > 0$, $r_i = 0, 1, 2, \cdots$, $i = 1, 2, \cdots, k$, and like the function f they also satisfy the relation (5). On the other hand the nonnegativity of $\psi_{r_1,\cdots,r_k}$ for all r_i's and for positive t_i's, imply that f must be given by (3) for some probability measure G (see for instance page 87, Bochner [1]). Finally, in order that (3) satisfies (2), it is easy to see that G must be concentrated on the set A. This completes the proof.

We now consider briefly the analogous problem for the discrete case (see Puri [2]). Let $(Y_1, \cdots, Y_k)$ be a vector of k nonnegative integer valued rv's satisfying for some $\beta > 0$, the relation

$$P(Y_i > n_i, i = 1, 2, \cdots, k) = \beta P(Y_i = n_i, i = 1, 2, \cdots, k)\,, \tag{7}$$

for $n_i = 0, 1, 2, \cdots$, $i = 1, 2, \cdots, k$. The solution to the problem of characterizing the distributions of such vectors $(Y_1, \cdots, Y_k)$ is given in the following theorem.

THEOREM 2. *For a given $\beta > 0$, the only distributions of $(Y_1, \cdots, Y_k)$ satisfying (7) are the ones which are mixtures of geometric distributions given by*

$$P(Y_i = n_i, i = 1, 2, \cdots, k) = \int_0^1 \cdots \int_0^1 [\prod_{i=1}^k p_i^{n_i}] H(dp_1, \cdots, dp_k)\,, \tag{8}$$

for $n_i = 0, 1, 2, \cdots$, $i = 1, 2, \cdots, k$, where the probability measure H is concentrated on the set $B = [\prod_{i=1}^k \{p_i/(1 - p_i)\} = \beta, 0 < p_i < 1, i = 1, 2, \cdots, k]$.

OUTLINE OF THE PROOF. Let $\mu(n_1, \cdots, n_k) = p(Y_i = n_i, i = 1, 2, \cdots, k)$. The proof is analogous to that of Theorem 1 and involves showing, by a similar induction argument, the differences Φ's defined by

$$\begin{aligned} \Phi_{r_1,\cdots,r_2}(n_1, \cdots, n_k) &\equiv \beta \Delta_1^{r_1} \cdots \Delta_k^{r_k} \mu(n_1, \cdots, n_k) \\ &\equiv \beta \sum_{m_1=0}^{r_1} \sum_{m_k=0}^{r_k} \{\prod_{i=1}^k \tbinom{r_i}{m_i}\} (-1)^{m_1+\cdots+m_k} \\ &\qquad \times \mu(n_1 + m_1, \cdots, n_k + m_k)\,, \end{aligned} \tag{9}$$

are nonnegative for all $r_i = 0, 1, 2, \cdots$; $n_i = 0, 1, 2, \cdots$; $i = 1, 2, \cdots, k$.

On the other hand, using the known results concerning the multi-dimensional Hausdorff moment problem (see for instance Shohat and Tamarkin [3]), the nonnegativity of these differences implies (8) for some probability measure H. Finally, it is easy to see that the measure H must be concentrated on the set B in order that the distribution given by (8) satisfies (7).

REFERENCES

[1] BOCHNER, S. (1960). *Harmonic Analysis and the Theory of Probability*. University of California Press, Berkeley.

[2] PURI, P. S. (1973). On a property of exponential and geometric distributions and its relevance to multivariate failure rates. *Sankhyā, Ser. A* **35** 61–68.

[3] SHOHAT, J. A. and TAMARKIN, J. D. (1943). *The problem of moments*. Mathematical Surveys No. 1, American Mathematical Society.

DEPARTMENT OF STATISTICS
PURDUE UNIVERSITY
MATHEMATICAL SCIENCES BUILDING
WEST LAFAYETTE, INDIANA 47907