The Annals of Probability
1974, Vol. 2, No. 4, 702–713

ASYMPTOTIC MAXIMA OF CONTINUOUS GAUSSIAN PROCESSES[1]

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Let $X(t)$ be a stationary Gaussian process with continuous sample paths. The behavior of $|X(t)|$ as $t \to \infty$ is considered. In particular, conditions on the spectrum of the process are given which determine whether $\limsup_{t\to\infty} |X(t)|/(\log t)^{\frac{1}{2}} = \text{Const.} > 0$. These conditions are complete except when the spectrum of the process is continuous-singular. The main concern of this paper is to study the asymptotic behavior of some specific examples of $X(t)$ with continuous-singular spectra.

Many examples are given showing the asymptotic behavior of stationary Gaussian processes with discrete spectra and their indefinite integrals.

1. Introduction. This paper continues the program of Marcus (1972a) to study the asymptotic behavior of continuous Gaussian processes with minimal conditions imposed on the covariance or spectrum of the process. Let $X(t)$ be a separable, stationary Gaussian process, $EX(t+h)X(t) = \hat{F}(h) = \int_0^\infty \cos \lambda h \, dF(\lambda)$. Clearly, $EX^2(t) = \hat{F}(0) = F(\infty)$.

If $\limsup_{t\to\infty} |X(t)|/g(t) = C$, a.s., $0 < C < \infty$, then $X(t)$ will be called relatively stable with respect to $g(t)$. We will consider the question of when $X(t)$ is relatively stable with respect to $(\log t)^{\frac{1}{2}}$ and state our results in terms of the spectrum F of the process.

Belyaev (1958) has shown that when the spectrum has a continuous component $\limsup_{t\to\infty} |X(t)| = \infty$ a.s. If F is discrete denote the points of increase of F by $\{\lambda_k\}$ and the corresponding jumps by $\{a_k^2\}$. Belyaev (1958) also points out that if $\sum a_k < \infty$ the corresponding process is continuous and if $\sum a_k = \infty$ and the λ_k are incommensurable, then the corresponding process is unbounded with probability 1. The remaining case is obvious. If $\sum a_k = \infty$ and the λ_k are not incommensurable (i.e. there exist integers n_k and a $\theta > 0$ such that $\lambda_k = n_k \theta$) then, since the corresponding process is periodic, if it is continuous it must be bounded.

In Marcus (1972a) it was shown that for all continuous stationary Gaussian processes

$$\limsup_{t\to\infty} \frac{|X(t)|}{(2 \log t)^{\frac{1}{2}}} \leqq F^{\frac{1}{2}}(\infty) \quad \text{a.s.} \tag{1.1}$$

It also follows from Berman (1964) and Pickands (1967) that when

Received June 22, 1973; revised November 29, 1973.

[1] This research was supported in part by a National Science Foundation grant.

AMS 1970 *subject classifications.* Primary 60G15, 60G17; Secondary 60E05.

Key words and phrases. Maxima of Gaussian processes, asymptotic rates, processes with stationary increments.

$\limsup_{h\to\infty} \hat{F}(h) \leqq \delta$ ($\delta > 0$) then

$$\limsup_{t\to\infty} \frac{|X(t)|}{(2\log t)^{\frac{1}{2}}} \geqq (F(\infty) - \delta)^{\frac{1}{2}} . \tag{1.2}$$

It is well known that $F = F_{ac} + F_{cs} + F_d$ has a decomposition into an absolutely continuous, continuous singular and discrete measure (with respect to Lebesgue measure) with $\lim_{h\to\infty} \hat{F}_{ac}(h) = 0$ and $\limsup_{h\to\infty} \hat{F}_d(h) = \hat{F}_d(0) = F_d(\infty)$. Furthermore, $F_{cs} = M + M_0$ has a decomposition into pairwise mutually singular measures with $\lim_{h\to\infty} \hat{M}_0(h) = 0$ and $\limsup_{h\to\infty} M(h) \neq 0$.

Using the above notation (1.1) and (1.2) together imply that if $F = F_{ac} + M_0$ then

$$\limsup_{t\to\infty} \frac{|X(t)|}{(2\log t)^{\frac{1}{2}}} = (F_{ac}(\infty) + M_0(\infty))^{\frac{1}{2}} \quad \text{a.s.} \tag{1.3}$$

In Theorem 2.1 we show that if $F = F_d$ then

$$\limsup_{t\to\infty} \frac{|X(t)|}{(2\log t)^{\frac{1}{2}}} = 0 \quad \text{a.s.} \tag{1.4}$$

so that if $F = F_{ac} + M_0 + F_d$ (with $F_d(\infty) > 0$) (1.3) still holds although in this case $F(\infty) > F_{ac}(\infty) + M_0(\infty)$.

In order to fully answer the question of when $X(t)$ is relatively stable with respect to $(\log t)^{\frac{1}{2}}$ it remains only to consider the case $F = M$. As long as $\limsup_{h\to\infty} \hat{M}(h) < M(\infty)$ (this can be realized) (1.2) can be used to show that $X(t)$ is relatively stable with respect to $(\log t)^{\frac{1}{2}}$ although we cannot determine the proper constant. However, it is possible that $\limsup_{h\to\infty} \hat{M}(h) = M(\infty)$. In this case (1.2) is of no help.

In Theorem 3.2 we give examples of stationary Gaussian processes with spectrum of the type M where $\limsup_{h\to\infty} \hat{M}(h) = M(\infty)$ and for which

$$\limsup_{t\to\infty} \frac{|X(t)|}{(\log t)^{\frac{1}{2}}} \geqq C \quad \text{a.s.}$$

for some $C > 0$. Therefore, the conditions used in (1.2) are not necessary in general.

It seems reasonable to conjecture that if $X(t)$ is a continuous stationary Gaussian process and if its spectrum F has a continuous component of mass $F_c(\infty) = F_{ac}(\infty) + F_{cs}(\infty)$ then $\limsup_{t\to\infty} |X(t)|/(2\log t)^{\frac{1}{2}} = F_c^{\frac{1}{2}}(\infty)$. If this is true it would be an elegant extension of Belyaev's result.

Considering (1.4) it is interesting to see what growth rate can be achieved by continuous stationary Gaussian processes with discrete spectrum. We examine examples of these processes given by

$$X(t) = \sum_{k=0}^{\infty} a_k[\eta_k \cos 2^{-k}t2\pi + \eta_k' \sin 2^{-k}t2\pi] \tag{1.5}$$

where η_k, η_k', $k = 0, 1, \cdots$ are independent standard normal random variables with $\sum |a_k| = \infty$ ($\{a_k\} \in l^2$ is necessary for $EX^2(t) < \infty$). In Theorem 3.1 and

Corollary 3.2, utilizing the lacunary nature of the processes (1.5), we show that various growth rates are possible. In particular examples of these processes are found that are relatively stable with respect to $(\log t)^{(1-\varepsilon)/2}$ for any $\varepsilon > 0$.

In Marcus (1972a) the following upper bounds for the asymptotic maxima of a continuous Gaussian process with stationary increments is given: Let $Y(t)$ be such a process with $EY^2(t) \leqq Q^2(t)$, $Q \uparrow$, then

$$\limsup_{t\to\infty} \frac{|Y(t)|}{h(t)} \leqq C \qquad \text{a.s.} \quad (C > 0) \tag{1.6}$$

where

$$h(t) = Q(t)\left[(\log\log t)^{\frac{1}{2}} + \left(\frac{1}{Q(t)} \int_0^t \frac{Q(u)}{u}\,du\right)^2\right].$$

We obtain examples of continuous Gaussian processes with stationary increments by considering

$$Y(t) = \int_0^t X(u)\,du \tag{1.7}$$

for processes $X(u)$ given by (1.5). In Theorem 4.1 we show that many of these processes are relatively stable with respect to $h(t)$, i.e. the upper bound is achieved up to a possible multiplicative constant. Other results relating to the examples in (1.5) and (1.7) are given in Sections 3 and 4.

Throughout this paper the symbols C and C' are used to designate finite constants that are greater than zero. The appearance of the same symbol in two different equations does not imply that the constants necessarily have the same value.

I acknowledge with thanks many helpful discussions with my colleagues Colin Graham and Don Saari.

2. Processes with discrete spectra. Let $F = F_d$. The spectrum is characterized by a sequence $\{a_k^2\}$, $k = 0, 1, \cdots$ such that $F(\lambda_k) - F(\lambda_k-) = a_k^2$, $\sum_{k=0}^\infty a_k^2 = F(\infty)$. A stationary Gaussian process with this spectrum is given by

$$X(t) = \sum_{k=0}^\infty a_k[\eta_k \cos\lambda_k t + \eta_k' \sin\lambda_k t] \tag{2.1}$$

where η_k, η_k' are independent standard normal random variables. The following theorem is a simple application of Theorem 1.4 in Marcus (1972a).

THEOREM 2.1. *Let $X(t)$ be a real valued continuous stationary Gaussian process with discrete spectrum; then*

$$\limsup_{t\to\infty} \frac{|X(t)|}{(2\log t)^{\frac{1}{2}}} = 0 \quad \text{a.s.} \tag{2.2}$$

PROOF. All such processes can be represented as in (2.1). For any $\varepsilon > 0$ choose N sufficiently large so that $\sum_{k=N+1}^\infty a_k^2 < \varepsilon^2$. Consider

$$\begin{aligned} Y(t) &= \textstyle\sum_{k=0}^N a_k[\eta_k \cos\lambda_k t + \eta_k' \sin\lambda_k t] \\ Z(t) &= \textstyle\sum_{k=N+1}^\infty a_k[\eta_k \cos\lambda_k t + \eta_k' \sin\lambda_k t] \end{aligned} \tag{2.3}$$

Clearly $X(t) = Y(t) + Z(t)$. Since $|Y(t)| \leqq \sum_{k=0}^{N} a_k(|\eta_k| + |\eta_k'|)$, $|Y(t)|$ is finite almost surely. By (1.1)

$$\limsup_{t\to\infty} \frac{|Z(t)|}{(2\log t)^{\frac{1}{2}}} \leqq \varepsilon \quad \text{a.s.}$$

Consequently

$$\limsup_{t\to\infty} \frac{|X(t)|}{(2\log t)^{\frac{1}{2}}} \leqq \varepsilon \quad \text{a.s.}$$

and since this result holds for all ε the result follows.

As a Corollary of Theorem 2.1 we obtain a useful result about cosine series. (This is known in the theory of almost periodic functions.)

COROLLARY 2.2. *Let $\{a_k\} \in l^2$ and $\{\lambda_k\}$ be a sequence of nonnegative numbers; then*

$$\limsup_{h\to\infty} \sum_{k=0}^{\infty} a_k^2 \cos \lambda_k h = \sum_{k=0}^{\infty} a_k^2 . \tag{2.4}$$

PROOF. For these values of $\{a_k\}$ and $\{\lambda_k\}$ consider the process $Y(t)$ given in (2.3). The covariance of $Y(t)$ is $\sum_{k=0}^{N} a_k^2 \cos \lambda_k h$ and

$$\limsup_{h\to\infty} \sum_{k=0}^{N} a_k^2 \cos \lambda_k h = \sum_{k=0}^{N} a_k^2 . \tag{2.5}$$

To see that (2.5) holds note that $Y(t)$ is a continuous stationary Gaussian process with discrete spectrum. Its covariance is equal to $\sum_{k=0}^{N} a_k^2$ at $h = 0$ and if (2.5) is not correct then (1.2) implies a contradiction of Theorem 2.1. Since (2.5) holds for all N (2.4) follows.

Given Theorem 2.1 the question arises how fast can continuous Gaussian processes with discrete spectrum grow. We will show by examples that diverse growth rates are possible including processes that are relatively stable with respect to $(\log t)^{(1-\varepsilon)/2}$ for any $\varepsilon > 0$. Four lemmas are needed to do this.

LEMMA. 2.3. *Let $\{a_n\} \in l^2$ and ξ_n be independent standard normal random variables. Then*

$$\limsup_{N\to\infty} \frac{|\sum_{n=N}^{\infty} a_n \xi_n|}{(2\log N \sum_{n=N}^{\infty} a_n^2)^{\frac{1}{2}}} \leqq 1 \quad \text{a.s.}$$

PROOF. $Y_N = \sum_{n=N}^{\infty} a_n \xi_n / (\sum_{n=N}^{\infty} a_n^2)^{\frac{1}{2}}$ is a standard normal random variable; $\text{Prob}\,[Y_N \geqq (1+\varepsilon)(2\log N)^{\frac{1}{2}}] \leqq 1/N^{1+\varepsilon}$. The result follows from the Borel–Cantelli lemma.

LEMMA 2.4. *Let $\{a_n\} \in l^1$, $a_n \geqq 0$, $\sum_{n \geqq N} a_n > 0$ for all $N \geqq 0$ and ξ_n be independent standard normal random variables. Then*

$$\limsup_{N\to\infty} \frac{\sum_{n=N}^{\infty} a_n |\xi_n|}{\max\,[\sum_{n=N}^{\infty} a_n, (\log N \sum_{n=N}^{\infty} a_n^2)^{\frac{1}{2}}]} \leqq C \quad \text{a.s.}$$

for some constant $C < \infty$.

PROOF. This is Lemma 2.1 of Marcus (1972b).

The next lemma is similar to Lemma 2.4; however, since we consider the sums $\sum_{n=0}^{N} a_n|\xi_n|$ it is not necessary to require that $\{a_n\} \in l^i$, $i = 1, 2$.

LEMMA 2.5. *Let* $a_n \geqq 0$ *and* ξ_n *be independent standard normal random variables. Then*

$$\limsup_{N\to\infty} \frac{\sum_{n=0}^{N} a_n|\xi_n|}{\max\left[\sum_{n=0}^{N} a_n, (\log N \sum_{n=0}^{N} a_n^2)^{\frac{1}{2}}\right]} \leqq C \quad \text{a.s.}$$

for some constant $C < \infty$.

PROOF. This proof follows from the proof of Lemma 2.1 of Marcus (1972b). In the next lemma the random variables are not required to be normal.

LEMMA 2.6. *Let* $\{a_n\} \in l^2$ *but not in* l^1, $a \geqq 0$ *and* ξ_n *be independent identically distributed random variables with* $E|\xi_n|^2 < \infty$. *Then*

$$\limsup_{N\to\infty} \frac{\sum_{n=0}^{N} a_n|\xi_n|}{\sum_{n=0}^{N} a_n E|\xi_n|} \geqq 1 \quad \text{a.s.}$$

PROOF. Inequality II in Kahane (1968) page 6, states that for a positive random variable X such that $EX^2 < \infty$ and for $0 < \lambda < 1$, by Schwarz's inequality,

$$\text{Prob}\,[X \geqq \lambda E(X)] \geqq (1 - \lambda)^2 \frac{E^2(X)}{E(X^2)}\,.$$

With $X = \sum_{n=0}^{N} a_n|\xi_n|$ we obtain

$$\text{Prob}\left[\frac{\sum_{n=0}^{N} a_n|\xi_n|}{\sum_{n=0}^{N} a_n E|\xi_n|} \geqq \lambda\right] \geqq (1 - \lambda)^2 C \tag{2.6}$$

where $C = E^2|\xi_n|/E|\xi_n|^2$. By taking $\lambda = 1 - \delta$, $0 < \delta < 1$ we have

$$\limsup_{N\to\infty} \frac{\sum_{n=0}^{N} a_n|\xi_n|}{\sum_{n=0}^{N} a_n E|\xi_n|} \geqq 1 - \delta \tag{2.7}$$

which probability at least $C\delta^2$. However, since $\sum a_n = \infty$ the event in (2.7) is a tail event and therefore it holds with probability 1. Since this is true for all $0 < \delta < 1$ the lemma follows.

THEOREM 2.7. *Consider*

$$X(t) = \sum_{k=0}^{\infty} a_k[\eta_k \cos 2^{-k}t2\pi + \eta_k' \sin 2^{-k}t2\pi] \tag{2.8}$$

where η_k, η_k' *are independent standard normal random variables*, $\{a_k\} \in l^2$ *but not in* l^1 *and* $a_k \geqq 0$. *Then for each sample path in a set of measure* 1 *there exists a* T_0 *such that for* $T \geqq T_0$

$$\sup_{t\in[0,T]} |X(t)| \leqq C'\{\textstyle\sum_{k=0}^{[\log_2 T]} a_k + (\log_2 T \sum_{k=0}^{[\log_2 T]} a_k^2)^{\frac{1}{2}}\} \tag{2.9}$$

for some constant $C' < \infty$. *Furthermore*

$$\limsup_{t\to\infty} \frac{|X(t)|}{|\sum_{k=0}^{[\log_2 t]} a_k|} \geqq C \quad \text{a.s.} \tag{2.10}$$

for some constant $C > 0$. (Note $[\cdot]$ denotes integral part).

PROOF. Since the points of increase of the spectrum of $X(t)$ are bounded above, $X(t)$ is continuous (see Belyaev (1959)). We shall obtain the upper bounds for $X(t)$ first. Let $0 \leqq t \leqq 2^N$ and set $X(t) = Y_N(t) + Z_N(t)$ where

$$Y_N(t) = \sum_{k=0}^{N} a_k[\eta_k \cos 2^{-k}t2\pi + \eta_k' \sin 2^{-k}t2\pi] \tag{2.11}$$

$$Z_N(t) = \sum_{k=N+1}^{\infty} a_k[\eta_k \cos 2^{-k}t2\pi + n_k' \sin 2^{-k}t2\pi]\,. \tag{2.12}$$

We have

$$\sup_{t\in[0,2^N]} |Y_N(t)| \leqq \sum_{k=0}^{N} a_k(|\eta_k| + |\eta_k'|)\,.$$

Next consider

$$\begin{aligned} Z_N(t) &= \sum_{k=N+1}^{\infty} a_k \eta_k[1 - (1 - \cos 2^{-k}t2\pi)] \\ &\quad + \sum_{k=N+1}^{\infty} a_k \eta_k' \sin 2^{-k}t2\pi\,. \end{aligned} \tag{2.13}$$

$$\begin{aligned} |Z_N(t)| &\leqq |\sum_{k=N+1}^{\infty} a_k \eta_k| + \sum_{k=N+1}^{\infty} a_k|\eta_k| \sin^2 2^{-k}t\pi \\ &\quad + \sum_{k=N+1}^{\infty} a_k|\eta_k'| \sin 2^{-k}t2\pi\,. \end{aligned}$$

The last term in (2.13)

$$\begin{aligned} \sum_{k=N+1}^{\infty} a_k|\eta_k'| \sin 2^{-k}t2\pi &\leqq 2\pi 2^N \sum_{k=N+1}^{\infty} a_k 2^{-k}|\eta_k'| \\ &\leqq \hat{a}_N \pi 2^{N+1} \sum_{k=N+1}^{\infty} |\eta_k'| 2^{-k} \end{aligned}$$

where $\hat{a}_N = \sup_{n>N} a_n$. Similarly

$$\sum_{k=N+1}^{\infty} a_k|\eta_k| \sin^2 2^{-k}t\pi \leqq \pi \hat{a}_N 2^{2N} \sum_{k=N+1}^{\infty} |\eta_k| 2^{-2k}\,.$$

Combining these equations we obtain

$$\begin{aligned} |Z_N(t)| &\leqq |\sum_{k=N+1}^{\infty} a_k \eta_k| + \hat{a}_N \pi 2^{N+1} \sum_{k=N+1}^{\infty} |\eta_k'| 2^{-k} \\ &\quad + \hat{a}_N \pi 2^{2N} \sum_{k=N+1}^{\infty} |\eta_k| 2^{-2k}\,. \end{aligned} \tag{2.14}$$

We apply Lemmas 2.3 and 2.4 to (2.14) and obtain that for each sample path in a set of measure 1 there exists an N_0 such that for $N \geqq N_0$

$$\begin{aligned} |Z_N(t)| &\leqq [2 \log (N + 1) \sum_{k=N+1}^{\infty} a_k^2]^{\frac{1}{2}} + C\hat{a}_N(\log (N + 1))^{\frac{1}{2}} \\ &\leqq C'[\log (N + 1) \sum_{k=N+1}^{\infty} a_k^2]^{\frac{1}{2}}\,. \end{aligned} \tag{2.15}$$

Similarly we apply Lemma 2.5 to $Y_N(t)$ to obtain that for each sample path in a set of measure 1 there exists an N_0' such that for $N \geqq N_0'$

$$|Y_n(t)| \leqq C \max [\sum_{n=0}^{N} a_n, (\log N \sum_{k=0}^{N} a_k^2)^{\frac{1}{2}}]\,. \tag{2.16}$$

For each sample path on a set of measure 1 choose $N \geqq \max (N_0, N_0')$ then (2.15) and (2.16) both hold. Also since $\{a_k\} \in l^2$, $|Z_N(t)| = o(\log N)^{\frac{1}{2}}$ as $N \to \infty$ on this set. Note that the base of the logarithm whether 2 or e only affects the values of the constant. Therefore (2.9) is satisfied for $T = 2^M$, $M \geqq N$. We can readily extrapolate for the intermediate values of T because $\{a_k\} \in l^2$ implies $a_k \to 0$ as $k \to \infty$. Therefore for T_0 sufficiently large (dependent on the sample paths in a set of measure 1) we obtain (2.9).

We now obtain (2.10). Let $0 \leqq t \leqq 2^N$, set $h = t/2^N$ and define $\tilde{Y}_N(h) = Y_N(2^N h)$ for Y_N given in (2.11), i.e.

$$\tilde{Y}_N(h) = \sum_{k=0}^N a_k[\eta_k \cos 2^{N-k}h2\pi + \eta_k' \sin 2^{N-k}h2\pi]\,.$$

The random function $\tilde{Y}_N(h)$ is periodic on [0, 1] and is a lacunary series; therefore

$$\sup_{h\in[0,1]} |\tilde{Y}_N(h)| \geqq C \sum_{k=0}^N a_k|\eta_k| \tag{2.17}$$

for some constant $C > 0$. This inequality exhibits a property of lacunary series, for further reference see Marcus (1972 b) Section 2. It follows that

$$\sup_{t\in[0,2^N]} |Y_N(t)| \geqq C \sum_{k=0}^N a_k|\eta_k|\,. \tag{2.18}$$

Applying Lemma 2.6 to (2.18) we obtain that for each sample path in a set of measure 1 there exists a sequence of integers $N_k \to \infty$ such that

$$\sup_{t\in[0,2^{N_k}]}|Y_{N_k}(t)| \geqq (1-\varepsilon)C \sum_{k=0}^{N_k} a_k\,.$$

Note that $Z_{N_k}(t)$ (see (2.12)) is symmetric and for each k it is independent of $Y_{N_k}(t)$. Consequently with probability at least $\frac{1}{2}$ (2.18) holds with $X_{N_k}(t)$ replacing $Y_{N_k}(t)$. It follows that (2.10) holds with probability at least $\frac{1}{2}$. However, since $\sum a_n = \infty$ (2.10) is a tail event so the probability must be one.

In the following Corollary we present some examples obtained by employing Theorem 2.7.

COROLLARY 2.8. *Let $X(t)$ be given by* (2.8) *then under the following conditions on $\{a_k\}$ it is relatively stable with respect to the corresponding $g(t)$*, i.e.

$$\limsup_{t\to\infty} \frac{|X(t)|}{g(t)} = C \quad \text{a.s.} \tag{2.19}$$

The conditions are (*define* $a_0 = 1$):

a) $a_k = k^{-\alpha}$; $\frac{1}{2} < \alpha < 1$; $g(t) = (\log t)^{1-\alpha}$

b) $a_k = k^{-\frac{1}{2}}(\log k)^{-(1+\varepsilon)/2}$; $\varepsilon > 0$; $g(t) = (\log t)^{\frac{1}{2}}(\log\log t)^{-(1+\varepsilon)/2}$

c) $a_k = (\log k)^{\beta}/k$; $\beta \geqq -\frac{1}{2}$; $g(t) = (\log\log t)^{1+\beta}$.

PROOF. The only point to comment upon is that the lim sup in (2.19) is a constant. From Theorem 2.7 we see that the lim sup is bounded above and below by positive nonzero constants; however, since it is a tail event it must be equal to some constant a.s.

We have now shown that the processes given in (2.8) can have diverse growth rates. It seems natural to question how important is the fact that the points of increase of the spectrum converge to zero. Can similarly rapid growth rates be achieved if the points of increase are bounded away from zero? Using Slepian's lemma we show that the convergence of the points of increase of the spectrum to zero is of no particular significance.

Let us slightly modify the series given in (2.8) and consider

$$Y(t) = \xi/2^{\frac{1}{2}} + \sum_{k=0}^\infty a_k[\eta_k \cos 2^{-k}t2\pi + \eta_k' \sin 2^{-k}t2\pi] \tag{2.20}$$

where ξ like η_k and η_k' are independent standard normal random variables. Also let $\sum_{k=0}^{\infty} a_k^2 = \frac{1}{2}$. The covariance of $Y(t)$ is

$$\Gamma(h) = \tfrac{1}{2} + \textstyle\sum_{k=0}^{\infty} a_k^2 \cos 2^{-k}h2\pi \,. \tag{2.21}$$

Note that $\Gamma(0) = 1$ and $\Gamma(h) \geqq 0$. $Y(t)$ is only a slight modification of $X(t)$ and it is easy to see that the results in Corollary 2.8 also apply to $Y(t)$ (except to change the constant C in (2.19)). Now let $\Gamma_1(h)$, $\Gamma_1(0) = 1$ be any covariance function and let $Z(t)$ be a stationary Gaussian process with covariance $R(h) = \Gamma(h)\Gamma_1(h)$. It follows from Slepians' lemma (see Marcus, Shepp (1972) Lemma 2.2) that if $Y(t)$ satisfies (2.19) then

$$\limsup_{t\to\infty} \frac{|Z(t)|}{g(t)} \geqq C \quad \text{a.s.} \tag{2.22}$$

Suppose we choose $\Gamma_1(h) = \cos \lambda h2\pi$; then

$$R(h) = \tfrac{1}{2}\{\cos \lambda h2\pi + \textstyle\sum_{k=0}^{\infty} a_k^2[\cos (\lambda + 2^{-k})h2\pi + \cos (\lambda - 2^{-k})h2\pi]\} \,.$$

In this example the points of increase of the spectrum converge to λ. Similarly $\Gamma_1(h)$ can be taken to be any countable cosine series and $R(h)$ will then be the covariance of a stationary Gaussian process $Z(t)$ with discrete spectrum and its growth rate will be greater than that of $Y(t)$.

3. Processes with continuous singular spectrum. As stated in the introduction $F_{cs} = M + M_0$ can be decomposed into pairwise mutually singular measures with $\lim_{h\to\infty} \hat{M}_0(h) = 0$ and $\limsup_{h\to\infty} M(h) \neq 0$. From (1.3) we see that with regard to relative stability processes with spectrum of the type M_0 behave the same as those with spectrum of the type F_{ac}. The gap in our knowledge concerning the relative stability of stationary Gaussian processes occurs for processes with spectrum of the type M. We make one contribution in this direction by exhibiting examples of continuous stationary Gaussian processes with spectrum of the type M for which $\lim_{h\to\infty} \hat{M}(h) = M(\infty)$ and which are relatively stable with respect to $(\log t)^{\frac{1}{2}}$. This shows that $\limsup_{h\to\infty} \hat{M}(h) < M(\infty)$, which implies relative stability with respect to $(\log t)^{\frac{1}{2}}$ by (1.2), is not a necessary condition. We begin with a lemma.

LEMMA 3.1. *Let* $\Gamma(h) = \prod_{j=1}^{\infty} \Gamma_{N_j}(h)$ *where*

$$\Gamma_{N_j}(h) = \frac{1}{2}\left(1 + \frac{1}{N_j} \textstyle\sum_{n=0}^{N_j-1} \cos 2^{-n}h2\pi\right) \tag{3.1}$$

and N_j, $j = 1, 2, \cdots$ *is an increasing sequence of positive integers with* $\sum_{j=1}^{\infty} N_j^{-1} < \infty$. *Then* $\Gamma(h)$ *is a characteristic function*, $\Gamma(0) = 1$ *and* $\limsup_{h\to\infty} \Gamma(h) = 1$.

PROOF. It is clear that $\Gamma_{N_j}(h)$ is a characteristic function since it is a cosine transform of a discrete measure with mass at zero and $2\pi/2^n$, $n = 0, 1, \cdots, N_j - 1$. The function $\Gamma(h) = \lim_{m\to\infty} \prod_{j=1}^{m} \Gamma_{N_j}(h)$ is a limit of characteristic functions. To show that $\Gamma(h)$ is a characteristic function it is only necessary to show that it is continuous at $h = 0$. Let $h \leqq 1$, then

$$\begin{aligned}\Gamma(h) &\geqq \prod_{j=1}^{\infty}\left(\tfrac{1}{2} + \frac{1}{2N_j}\sum_{N=0}^{N_j-1}\left(1 - \frac{2^{-2n}h^2}{2}\right)\right)\\ &= \prod_{j=1}^{\infty}\left(1 - \frac{h^2}{4N_j}\sum_{n=0}^{N_j-1}2^{-2n}\right)\\ &\geqq \prod_{j=1}^{\infty}\left(1 - \frac{h^2}{2N_j}\right) \geqq \exp\left[-h^2\sum_{j=1}^{\infty}\frac{1}{N_j}\right].\end{aligned} \tag{3.2}$$

Since $\sum N_j^{-1} < \infty$, $\Gamma(h)$ is continuous at $h = 0$. It is obvious that $\Gamma(0) = 1$; we now show that $\limsup_{h\to\infty}\Gamma(h) = 1$. To do this we calculate $\Gamma_{N_j}(2^{N_k})$. If $j \leqq k$, $\Gamma_{N_j}(2^{N_k}) = 1$. If $N_j > N_k + 3$

$$\begin{aligned}\Gamma_{N_j}(2^{N_k}) &\geqq \tfrac{1}{2} + \frac{1}{2N_j}\left\{\sum_{n=0}^{N_k}\cos\frac{2^{N_k}2\pi}{2^n} - 1 + \sum_{n=N_k+3}^{N_j-1}\left(1 - \frac{2^{2N_k}}{2^{2n}}\frac{\pi^2}{2}\right)\right\}\\ &\geqq 1 - \frac{2}{N_j}.\end{aligned}$$

This calculation also shows that $\Gamma_{N_j}(2^{N_k}) \geqq (1 - 2/N_j)$ if $N_k < N_j \leqq N_k + 3$. Therefore

$$\Gamma(2^{N_k}) \geqq \prod_{j>k}\left(1 - \frac{2}{N_j}\right) \geqq \exp\left[-4\sum_{j>k}^{\infty}\frac{1}{N_j}\right]$$

and since $\sum N_j^{-1} < \infty$, $\limsup_{h\to\infty}\Gamma(h) = 1$.

THEOREM 3.2. *Let $X(t)$ be a stationary Gaussian process with covariance function $\Gamma(h) = \prod_{j=1}^{\infty}\Gamma_{N_j}(h)$ where $\Gamma_{N_j}(h)$ is given in* (3.1) *and N_j is an increasing sequence of positive integers with $\sum_{j=1}^{\infty}N_j^{-1} < \infty$. Then $X(t)$ has continuous sample paths, $\Gamma(h) = \hat{M}$ for some continuous singular measure M, $\limsup_{h\to\infty}\Gamma(h) = \Gamma(0) = 1$ and*

$$\limsup_{t\to\infty}\frac{|X(t)|}{(2\log t)^{\frac{1}{2}}} \geqq C \quad \text{a.s.} \tag{3.3}$$

for some constant $C > 0$.

PROOF. The continuity of $X(t)$ follows from (3.2). To obtain (3.3) we compare $X(t)$ with processes

$$X_{N_j}(t) = \eta/2^{\frac{1}{2}} + \frac{1}{(2N_j)^{\frac{1}{2}}}\sum_{n=0}^{N_j-1}\eta_n\cos 2^{-n}t2\pi + \eta_n'\sin 2^{-n}t2\pi$$

where η, η_n, η_n', $n = 0, 1, \cdots$ are independent standard normal random variables. By precisely the same method used to obtain (2.15) we obtain

$$\sup_{t\in[0,2^{N_j}]}|X_{N_j}(t)| \geqq \frac{C}{N_j^{\frac{1}{2}}}\sum_{n=0}^{N_j-1}|\eta_n|\,.$$

It follows from (2.6) that for some $C' < 1$

$$\text{Prob}\,[\sup_{t\in[0,2^{N_j}]}|X_{N_j}(t)| \geqq C'N_j^{\frac{1}{2}}] \geqq \alpha$$

for some $\alpha > 0$ independent of N_j. Therefore

$$\text{Prob}\left[\sup_{t\in[2,2^{N_j}]} \frac{|X_{N_j}(t)|}{(\log t)^{\frac{1}{2}}} \geqq C''\right] \geqq \alpha .$$

The covariance of $X_{N_j}(t)$ is $\Gamma_{N_j}(h)$ given in (3.1). Note that $\Gamma_{N_j}(h) \geqq 0$ and so $\Gamma(h) \leqq \Gamma_{N_j}(h)$ whereas $\Gamma(0) = \Gamma_{N_j}(0)$. Therefore by Slepian's lemma (see Marcus, Shepp (1972), Lemma 2.2 for further reference)

$$\text{Prob}\left[\sup_{t\in[2,2^{N_j}]} \frac{|X(t)|}{(\log t)^{\frac{1}{2}}} \geqq C''\right] \geqq \alpha \tag{3.4}$$

and this holds for all N_j. Consequently, (3.3) holds on a set of measure greater than or equal to α. Since (3.3) is a tail event and $\alpha > 0$ the measure must be equal to one.

4. Processes with stationary increments. We will now examine integrals of continuous stationary Gaussian processes of the type given in (2.8) to obtain examples of continuous Gaussian processes with stationary increments that have various rates of growth. Consider

$$Y(t) = \int_0^t X(s)\,ds = \sum_{k=0}^\infty 2^k a_k[\eta_k \sin 2^{-k}t2\pi + \eta_k' \cos 2^{-k}t2\pi] . \tag{4.1}$$

Note that

$$E(Y(t+h) - Y(t))^2 = 4 \sum_{k=0}^\infty 2^{2k}a_k^2 \sin^2 2^{-k}h\pi \leqq 4\pi h^2 \sum_{k=0}^\infty a_k^2$$

for h sufficiently small; therefore the random series in (4.1) is a continuous Gaussian process with stationary increments. In fact it is almost surely the integral of $X(s)$ as claimed. This follows since a random series of the type given in (2.8) can be taken to be the Karhunen–Lóeve expansion of $X(s)$. If $X(s)$ is continuous then almost surely the series is uniformly convergent (see Garsia, Rodemich and Rumsey (1970) for further reference) and therefore it can be integrated termwise.

Following the discussion in the introduction we will find a monotone majorant $Q^2(t)$ for $EY^2(t)$.

$$EY^2(t) = 4 \sum_{k=0}^\infty 2^{2k}a_k^2 \sin^2 2^{-k}t\pi .$$

Let $2^{N-1} < t \leqq 2^N$; then

$$EY^2(t) \leqq 4[\sum_{k=0}^N 2^{2k}a_k^2 + \pi^2 2^{2N} \sum_{k=N+1}^\infty a_k^2] . \tag{4.2}$$

When $t = t_N = (\frac{2}{3})2^N$

$$EY^2(t) \geqq 4\left[3^{\frac{1}{2}}/2 \sum_{k=0}^N 2^{2k}a_k^2 + \frac{\pi^2}{2} 2^{2N} \sum_{k=N+1}^\infty a_k^2\right] . \tag{4.3}$$

It follows from (4.2) that we can choose

$$Q^2(t) = C\{\sum_{k=0}^{[\log_2 t]} 2^{2k}a_k^2 + t^2 \sum_{k=[\log_2 t]+1}^\infty a_k^2\} \tag{4.4}$$

and we see from (4.3) that this function cannot be made appreciably smaller.

Let $g(t)$ be an increasing function for which

$$\limsup_{t\to\infty} \frac{|Y(t)|}{Q(t)g(t)} = C \quad \text{a.s.;} \tag{4.5}$$

we can now state Theorem 4.1.

THEOREM 4.1. *Let $Y(t)$ be given by* (4.1), *$Q(t)$ by* (4.4) *and $g(t)$ by* (4.5). *The following examples can be realized*:

	a_k	$Q(t)$	$g(t)$
1a.	$2^{-k}k^{-1}(\log k)^{\beta}$, $\beta \geqq -\frac{1}{2}$	1	$(\log\log t)^{1+\beta}$
1b.	$2^{-k}k^{-\alpha}$, $\frac{1}{2} < \alpha < 1$	1	$(\log t)^{1-\alpha}$
1c.	$2^{-k}k^{-\frac{1}{2}}(\log k)^{-(1+\varepsilon)/2}$, $\varepsilon > 0$	1	$(\log t)^{\frac{1}{2}}(\log\log t)^{-(1+\varepsilon)/2}$
2.	$2^{-k}k^{-\frac{1}{2}}$	$(\log\log t)^{\frac{1}{2}}$	$(\log t)^{\frac{1}{2}}(\log\log t)^{-\frac{1}{2}}$
3.	$2^{-k}k^{-\alpha}$, $\alpha < \frac{1}{2}$	$(\log t)^{\frac{1}{2}-\alpha}$	$(\log t)^{\frac{1}{2}}$
4.	$2^{-k}(2^{k^r}k^{r-1})^{\frac{1}{2}}$, $0 < r < 1$	$\exp\{\frac{1}{2}(\log t)^r\}$	$(\log t)^{(1-r)/2}$
5.	$2^{-k\beta}k^{-\alpha}$, $0 < \beta < 1$, $-\infty < \alpha < \infty$	$t^{1-\beta}(\log t)^{-\alpha}$	$(\log\log t)^{\frac{1}{2}}$

The upper bounds given in (1.6) *are achieved in Examples* 3, 4 *and* 5.

PROOF. For $2^{N-1} < t \leqq 2^N$ let $Y(t) = X_N(t) + Z_N(t)$ where

$$X_N(t) = \sum_{k=0}^{N} 2^k a_k[\eta_k \sin 2^{-k}t2\pi + \eta_k'(1 - \cos 2^{-k}t2\pi)]\,.$$

Therefore,

$$\sup_{t\in[0,2^N]} |X_N(t)| \leqq 2 \sum_{k=0}^{N} 2^k a_k(|\eta_k| + |\eta_k'|)\,. \tag{4.6}$$

Also with probability at least $\frac{1}{2}$

$$\sup_{t\in[0,2^N]} |X_N(t)| \geqq C \sum_{k=0}^{N} 2^k a_k|\eta_k| \tag{4.7}$$

for some $C > 0$ independent of N. Inequality (4.7) is obtained in a similar manner as (2.15) by considering the sine series, under a change of scale, as a lacunary series. Since the series in $(1 - \cos 2^{-k}t2\pi)$ is independent of the sine series and symmetric it is equally likely positive or negative when the sine series achieves its maximum. This accounts for the probability $\frac{1}{2}$ however, since we are ultimately concerned with tail events any positive probability suffices. Also since $Z_N(t)$ is symmetric and independent of $X_N(t)$ it follows that with probability $\frac{1}{4}$

$$\sup_{t\in[0,2^N]} |Y(t)| \geqq C' \sum_{k=0}^{N} 2^k a_k|\eta_k|\,. \tag{4.8}$$

The lower bounds for the limit superior of $Y(t)$ are all computed from (4.8) by means of Lemma (2.6) except for Example 5. In this case we make use of the fact that $\limsup_{k\to\infty} |\eta_k|/(2\log k)^{\frac{1}{2}} = 1$ so (4.8) is greater than $2^N a_N(2\log N)^{\frac{1}{2}}$ infinitely often.

It we could replace $X_N(t)$ in (4.6) by $Y(t)$ we would be finished since the upper bounds could be obtained by Lemma 2.5 and in the cases considered they would agree with (4.8). In fact this is possible; we now show that $Z_N(t)$ does not contribute to the upper bound (except perhaps to increase the multiplicative constant).

$$Z_N(t) = \sum_{k=N+1}^{\infty} 2^k a_k[\eta_k \sin 2^{-k}t2\pi + \eta_k' \sin^2 2^{-k}t\pi] .$$

It is enough to consider the first term

$$\begin{aligned}\sum_{k=N+1}^{\infty} 2^k a_k \eta_k \sin 2^{-k}t2\pi & \\ &= \sum_{k=N+1}^{\infty} 2^k a_k \eta_k[2^{-k}t2\pi + (\sin 2^{-k}t2\pi - 2^{-k}t2\pi)] \\ &\leqq 2\pi t|\sum_{k=N+1}^{\infty} a_k \eta_k| + \sum_{k=N+1}^{\infty} 2^k a_k|\eta_k||\sin 2^{-k}t2\pi - 2^{-k}t2\pi| \\ &\leqq 2\pi t|\sum_{k=N+1}^{\infty} a_k \eta_k| + \tfrac{4}{3}\pi^3 t^3 \sum_{k=N+1}^{\infty} a_k 2^{-2k}|\eta_k| .\end{aligned}$$

Using Lemmas 2.3 and 2.4 we see that for each sample path in a set of measure 1 there exists a T so that for $t \geqq T$, $Z_N(t)$ is a.s. bounded by

$$C[t(\sum_{k=N+1}^{\infty} a_k^2)^{\frac{1}{2}}(\log\log t)^{\frac{1}{2}} + \hat{a}_{N+1}(\log\log t)^{\frac{1}{2}}]$$

where $N = [\log_2 t]$ and $\hat{a}_{N+1} = \sup_{n>N} a_n$. From (4.4) we see that $Q(t)$ is at least comparable to $t(\sum_{k=N+1}^{\infty} a_k^2)^{\frac{1}{2}}$ so $Z_N(t)$ is a.s. bounded by $C'Q(t)$ $(\log\log t)^{\frac{1}{2}}$ for $t \geqq T$. In the examples given $Q(t)g(t)$ is calculated from (4.6) and is at least this large.

It is a simple matter to check that the upper bounds given in (1.6) are achieved in Example 3, 4 and 5.

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