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SYMMETRIC MARKOV CHAINS[1]

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W. Feller's decomposition into a road map and speed measure are adapted to symmetric Markov chains with instantaneous states. The road map is a family of matrices defined by time changing the process with measures concentrated on finite sets.

0. Introduction and notations. Let $\mathbf{P}_t(x, y)$ be a standard transition matrix satisfying a condition of symmetry

$$m(x)\mathbf{P}_t(x, y) = m(y)\mathbf{P}_t(y, x) . \tag{0.1}$$

The rate

$$q(x) = \lim (1/t)\{1 - \mathbf{P}_t(x, x)\} \qquad (t \downarrow 0) \tag{0.2}$$

exists with $0 \leqq q(x) \leqq +\infty$. If every state is stable, that is, if $q(x) < +\infty$ then also

$$\lim (1/t)\mathbf{P}_t(x, y) = q(x)P(x, y) \tag{0.3}$$

for $x \neq y$ with P a substochastic matrix which vanishes on the diagonal. (See [1].) If P is recurrent, then $\mathbf{P}_t(x, y)$ is completely determined by (0.2) and (0.3). This need not be the case if there are transient state for P. In [10] we completely classified solutions of (0.2) and (0.3) under the assumption that P is transient, irreducible and strictly stochastic. Our main tools were certain techniques introduced in [8] and some discrete time potential theory associated with the matrix P.

We begin here a study of symmetric standard transition matrices which *need not satisfy* the *restriction* $q(x) < +\infty$. Thus *we do not rule out instantaneous states*. The main result in this paper is that Feller's decomposition [2] into a road map and a speed measure still makes sense if we replace the matrix P by a family of matrices P_M indexed by finite subsets M of the state space. Roughly speaking P_M is Feller's road map determined by "looking at the process only when it is in the set M."

After some preliminaries in Section 1 the road map is introduced axiomatically in Section 2 and the connection with continuous time Markov chains is established in Section 5. These three sections form an independent unit which can

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be read independent of the rest of the paper. In the remaining two sections we develop some discrete time potential theory.

I am grateful to the referee for informing me that the general technique of approximating a Markov chain by time changes onto a finite set has also been exploited by D. Williams, D. Freedman and others. We refer the reader to the recent monograph [3], for a survey of results and for detailed references to the literature. In Section 5 we interpret Freedman's sufficient condition for existence in our set up.

Added in proof. The author is now preparing a monograph [11] which unifies and extends the results in [8] and [9]. This will be preferable to either [8] or [9] as a reference for background material. Also the material in Section 27 of [11] will supplement the results established here.

Our notations are consistent with [10]. Functions and measures on **I** are viewed as column vectors and row vectors and then matrices are viewed according to the usual conventions. The indicator of a set will be denoted both by 1_A and $I(A)$. The integral of a function ξ over the set determined by a particular condition, say $X_t = y$, will be denoted both by $\mathscr{E}[X_t = y; \xi]$ and $\mathscr{E}I(X_t = y)\xi$. Questions of measurability are taken for granted. All functions are real valued. In particular $L^2(m)$ or $L^2(\mathbf{X}, m)$ is the real Hilbert space of square integrable functions on the measure space $(\mathbf{X}, \mu)$.

Elementary results for martingales and Markov chains are taken for granted. We list [5] as a general reference.

1. Preliminaries. In this section M' is a finite set and $P(x, y)$ is a substochastic matrix on M' which vanishes on the diagonal. Thus

$$P(x, y) \geqq 0 ; \qquad P(x, x) = 0 ; \qquad \textstyle\sum_y P(x, y) \leqq 1 . \tag{1.1}$$

We assume that there exists an everywhere positive measure α on M' which symmetrizes P:

$$\alpha(x)P(x, y) = \alpha(y)P(y, x) . \tag{1.2}$$

We also make the harmless assumption that P is irreducible.

We augment M' by introducing a *dead point* ∂. Functions f on M' are extended to $M' \cup \{\partial\}$ by the convention $f(\partial) = 0$. The *preliminary* discrete time sample space $\Lambda_{M'}$ is the set of sequences $\omega = \{\omega(n)\}_{n=0}^{\infty}$ in $M' \cup \{\partial\}$ which satisfy the following condition.

CONDITION 1.1. There exists a life time $\zeta(\omega)$ with $0 \leqq \zeta(\omega) \leqq +\infty$ such that $\omega(n) = \partial$ if and only if $n \geqq \zeta(\omega)$.

Of course the dead point ∂ can be eliminated when P is recurrent. It will turn out however that our main concern is with the transient case.

The trajectory variables X_n are defined on $\Lambda_{M'}$ by

$$X_n(\omega) = \omega(n) .$$

For x in M' the probability $\mathfrak{P}_x$ is the unique probability on $\Lambda_{M'}$ such that

(1.3) $$\mathfrak{P}_x[X_0 = x, X_1 = x_1, \cdots, X_n = x_n] = P(x, x_1) \cdots P(x_{n-1}, x_n)$$

for $n \geqq 0$ and for $x_1, \cdots, x_n$ in M'. For E a subset of M' the hitting time $\sigma(E)$ and the delayed hitting time $\sigma^+(E)$ are defined by

$$\sigma(E) = \inf\{n \geqq 0 : X_n \text{ is in } E\}$$
$$\sigma^+(E) = \inf\{n > 0 : X_n \text{ is in } E\}$$

with the understanding that these times are $+\infty$ when not otherwise defined. The *hitting probability* $H^E(x, y)$ is defined by

$$H^E(x, y) = \mathfrak{P}_x[\sigma(E) < +\infty; X_{\sigma(E)} = y] .$$

The last exit time $\sigma^*(E)$ is defined by

$$\sigma^*(E) = \sup\{n \geqq 0 : X_n \text{ is in } E\}$$

with the understanding that $\sigma(E) = -\infty$ when not otherwise defined.

For M a subset of M' we define

$$P_M{}^0(y, z) = \mathfrak{P}_y[\sigma^+(M) < +\infty; X_{\sigma^+(M)} = z]$$
$$P_M(y, z) = \mathfrak{P}_y[\sigma(M - \{y\}) < +\infty; X_{\sigma(M-\{y\})} = z]$$

for y, z in M, and also

$$\alpha_M(y) = \{1 - P_M{}^0(y, y)\}\alpha(y)$$

for y in M and we are ready for

THEOREM 1.1. (i) *α_M symmetrizes P_M. That is,*

$$\alpha_M(y)P_M(y, z) = \alpha_M(z)P_M(z, y)$$

for y, z in M.

(ii) *If f defined on M' satisfies $f = H^M f$, then*

(1.4) $$\tfrac{1}{2} \sum_{x,y \text{ in } M'} \alpha(x)P(x, y)\{f(x) - f(y)\}^2 + \sum_{x \text{ in } M'} \alpha(x)\{1 - P1(x)\}f^2(x)$$
$$= \tfrac{1}{2} \sum_{x,y \text{ in } M} \alpha_M(x)P_M(x, y)\{f(x) - f(y)\}^2$$
$$+ \sum_{x \text{ in } M} \alpha_M(x)\{1 - P_M 1(x)\}f^2(x) .$$

This simple theorem is our starting point. Note first that

$$P_M{}^0 = \sum_{k=0}^{\infty} 1_M P\{1_D P\}^k 1_M$$

and therefore by (1.2)

$$\alpha(y)P_M{}^0(y, z) = \alpha(z)P_M{}^0(z, y) .$$

But it is easy to check that for $y \neq z$

(1.5) $$P_M{}^0(y, z) = \{1 - P_M{}^0(y, y)\}P_M(y, z)$$

and (i) is proved. Our main tool for the proof of (ii) is a technique introduced in Section 7 of [8].

We consider first the case when P is transient so that the potential operator

$$G(x, y) = \sum_{k=0}^{\infty} P^k(x, y)$$

is everywhere finite. It is easy to see that $P^n 1 \to 0$ pointwise and therefore $1 = G(1 - P1)$. The symmetry condition (1.2) implies an analogous symmetry condition for G and this leads in turn to the identity

$$\sum \alpha(x)\{1 - P1\}(x)\mathfrak{E}_x \sum_{n=0}^{\infty} \varphi(X_n) = \sum \alpha(x)\varphi(x)$$

which plays an important role below. Next we introduce entrance and return times for excursions into $D = M' - M$. These are defined by

$$\begin{aligned} e(1) &= \inf\{n > \sigma(M) : X_n \text{ is in } D\} \\ r(1) &= \inf\{n > e(1) : X_n \text{ is in } M\} \\ e(2) &= \inf\{n > r(1) : X_n \text{ is in } D\} \quad \text{etc.} \end{aligned}$$

with the understanding that $e(i)$ or $r(i) = +\infty$ and therefore $X_{e(i)}$ or $X_{r(i)} = \partial$ when not otherwise defined. By the martingale property of $f(X_n)$ along excursions into D,

$$\begin{aligned} &\tfrac{1}{2} \textstyle\sum_x \alpha(x)\{1 - P1\}(x)\mathfrak{E}_x \sum_i \{f(X_{r(i)}) - f(X_{e(i)-1})\}^2 \\ &\quad = \tfrac{1}{2} \textstyle\sum_{y \text{ in } M} \alpha(y) \sum_{x \text{ in } D} P(y, x)\{f(y) - f(x)\}^2 \\ &\qquad + \tfrac{1}{2} \textstyle\sum_{y \text{ in } M} \alpha(y)\{1 - P1\}(y)f^2(y) \\ &\qquad + \tfrac{1}{2} \textstyle\sum_x \alpha(x)\{1 - P1\}(x)\mathfrak{E}_x \sum_{n > \sigma(M)} 1_D(X_n)\{f(X_{n+1}) - f(X_n)\}^2 . \end{aligned}$$

On the other hand

$$\begin{aligned} &\tfrac{1}{2} \textstyle\sum_x \alpha(x)\{1 - P1\}(x)\mathfrak{E}_x[\sigma(M) < \infty; f^2(X_{\sigma(M)})] \\ &\quad = \tfrac{1}{2} \textstyle\sum_x \alpha(x)\{1 - P1\}(x)f^2(x) \\ &\qquad + \tfrac{1}{2} \textstyle\sum_x \alpha(x)\{1 - P1\}(x)\mathfrak{E}_x \sum_{n < \sigma(M)} 1_D(X_n)\{f(X_{n+1}) - f(X_n)\}^2 \end{aligned}$$

and putting all this together we get

$$\begin{aligned} &\tfrac{1}{2} \textstyle\sum_x \alpha(x)\{1 - P1\}(x)(\mathfrak{E}_x \sum_i \{f(X_{r(i)}) - f(X_{e(i)-1})\}^2 \\ &\quad + I[\sigma(M) < +\infty]f^2(X_{\sigma(M)})) \\ &\qquad = \tfrac{1}{2} \textstyle\sum_{x,y} \alpha(x)P(x, y)\{f(x) - f(y)\}^2 \\ &\qquad\quad + \textstyle\sum_x \alpha(x)\{1 - P1\}(x)f^2(x) \\ &\qquad\quad - \tfrac{1}{2} \textstyle\sum_{y,z \text{ in } M} \alpha(y)P(y, z)\{f(y) - f(z)\}^2 \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\tfrac{1}{2} \textstyle\sum_{x,y} \alpha(x)P(x, y)\{f(x) - f(y)\}^2 + \sum \alpha(x)\{1 - P1\}(x)f^2(x) \\ &\quad = \tfrac{1}{2} \textstyle\sum_{y,z \text{ in } M} \alpha(y)P_M{}^0(y, z)\{f(y) - f(z)\}^2 + \sum_{y \text{ in } M} \alpha(y)\{1 - P_M{}^0 1(y)\}f^2(y) . \end{aligned}$$

Finally we apply (1.5) and its corollary

$$1 - P_M{}^0 1(y) = \{1 - P_M{}^0(y, y)\}\{1 - P_M 1(y)\} \tag{1.5$'$}$$

and (1.4) is proved for P transient.

To handle P recurrent, fix a reference point 0 in M and define

$$\begin{aligned} Q(x, y) &= P(x, y) \qquad \text{if} \quad x, y \neq 0 \\ &= 0 \qquad\qquad\ \ \text{if} \quad x = 0 \quad \text{or} \quad y = 0 . \end{aligned}$$

Then Q is transient and the theorem can be applied for Q. Since P and P_M must be strictly stochastic there is no loss of generality in assuming that $f(0) = 0$ in which case

$$\begin{aligned}\tfrac{1}{2} \textstyle\sum_{x,y} \alpha(x)P(x, y)\{f(x) - f(y)\}^2 \\ = \tfrac{1}{2} \textstyle\sum_{x,y} \alpha(x)Q(x, y)\{f(x) - f(y)\}^2 + \sum_x \alpha(x)\{1 - Q1(x)\}f^2(x) .\end{aligned}$$

On the other hand

$$\begin{aligned} Q_M(y, z) &= P_M(y, z) && \text{if } \ y, z \neq 0 \\ &= 0 && \text{if } \ y = 0 \ \text{ or } \ z = 0 \\ Q_M{}^0(y, y) &= P_M{}^0(y, y) && y \neq 0 \end{aligned}$$

and the theorem is proved for P recurrent.

2. Road maps. The *state space* $\mathbf{I}$ is a denumerably infinite set. We assume given for each finite subset M of $\mathbf{I}$ an irreducible substochastic matrix P_M on M and a measure α_M on M such that α_M symmetrizes P_M. We assume further that for M a subset of M'

$$P_M(y, z) = \mathfrak{P}_{(M')y}[\sigma(M - \{y\}) < +\infty; X_{\sigma(M-\{y\})} = z] . \tag{2.1}$$

$$\alpha_M(y) = \{1 - \mathfrak{P}_{(M')y}[\sigma^+(M) < +\infty; X_{\sigma^+(M)} = y]\}\alpha_{M'}(y) . \tag{2.2}$$

Of course $\mathfrak{P}_{(M')y}$ is defined in the same way as $\mathfrak{P}_y$ in Section 1 with $P_{M'}$ playing the role of P. Such a collection will be referred to as a *road map* on $\mathbf{I}$. It is easy to check that the P_M are either all transient or all recurrent and so it makes sense to speak of the road map as being transient or recurrent.

For f defined on $\mathbf{I}$ and for M a finite subset of $\mathbf{I}$ we define the Dirichlet norm

$$\begin{aligned}\mathfrak{E}^M(f, f) = \tfrac{1}{2} \textstyle\sum_{y,z \text{ in } M} \alpha_M(y)P_M(y, z)\{f(y) - f(z)\}^2 \\ + \textstyle\sum_{y \text{ in } M} \alpha_M(y)\{1 - P_M1(y)\}f^2(y) .\end{aligned} \tag{2.3}$$

The right side of

$$H^Mf(x) = \mathfrak{E}_{(M')x}[\sigma(M) < +\infty; f(X_{\sigma(M)})] \tag{2.4}$$

is independent of the choice of M' containing $M \cup \{x\}$ and so H^Mf is well defined by (2.4). By Theorem 1.1

$$\mathfrak{E}^M(f, f) = \mathfrak{E}^{M'}(H^Mf, H^Mf)$$

for M' containing M. Now consider g vanishing on M and h satisfying $H^Mh = h$ and as in Section 1 let $D = M' - M$. Then

$$\begin{aligned}\mathfrak{E}^{M'}(h, g) &= \tfrac{1}{2} \textstyle\sum_{x,w \text{ in } D} \alpha_{M'}(x)P_{M'}(x, w)\{h(x) - h(w)\}\{g(x) - g(w)\} \\ &\quad + \textstyle\sum_{x \text{ in } D} \sum_{y \text{ in } M} \alpha_{M'}(x)P_{M'}(x, y)\{h(x) - h(y)\}g(x) \\ &\quad + \textstyle\sum_{x \text{ in } D} \alpha_{M'}(x)\{1 - P_{M'}1\}(x)f(x)g(x) \\ &= \textstyle\sum_{x \text{ in } D} \alpha_{M'}(x)\{h(x)g(x)P_{M'}1_D(x) - g(x)P_{M'}1_Dh(x) \\ &\quad + h(x)g(x)P_{M'}1_M(x) - g(x)P_{M'}1_Mh(x) + h(x)g(x)(1 - P_{M'}1)(x)\} \\ &= \textstyle\sum_{x \text{ in } D} \alpha_{M'}(x)\{h(x)g(x) - g(x)P_{M'}h(x)\} = 0\end{aligned}$$

since $P_{M'}h = h$ on D by the "optional stopping time theorem" for martingales. Thus

$$\mathfrak{E}^{M'}(h, g) = 0 \tag{2.5}$$

and it follows that for arbitrary f

$$\mathfrak{E}^{M'}(f, f) = \mathfrak{E}^{M}(f, f) + \mathfrak{E}^{M'}(f - H^M f, f - H^M f) . \tag{2.6}$$

In particular $\mathfrak{E}^{M}(f, f)$ increases with M and we can define the *associated Dirichlet norm*

$$\mathfrak{E}(f, f) = \sup \mathfrak{E}^{M}(f, f) . \tag{2.7}$$

The *associated Dirichlet space* is the collection $\mathfrak{F}$ of functions f on $\mathbf{I}$ for which $\mathfrak{E}(f, f) < +\infty$.

It follows from (2.2) that $\alpha_M(y)$ increases with M and therefore

$$\alpha(y) = \sup \alpha_M(y) \qquad M \text{ in } I$$

is well defined.

DEFINITION 2.1. y in $\mathbf{I}$ is *stable* if $\alpha(y) < +\infty$ and *instantaneous* if $\alpha(y) = +\infty$.

For y in $\mathbf{I}$ let e_y be the indicator of the set $\{y\}$. Thus

$$\begin{aligned} e_y(x) &= 1 \qquad & x = y \\ &= 0 \qquad & x \neq y . \end{aligned}$$

Clearly

$$\begin{aligned} \mathfrak{E}^{M}(e_y, e_y) &= \alpha_M(y)\{1 - P_M 1(y)\} + \textstyle\sum_{x \neq y} \alpha_M(y) P_M(y, x) \\ &= \alpha_M(y) . \end{aligned}$$

and so e_y belongs to $\mathfrak{F}$ if and only if y is stable in which case

$$\mathfrak{E}(e_y, e_y) = \alpha(y) .$$

3. Discrete time potential theory. In this section we consider a *transient road map* $\{P_M\}$ with corresponding symmetrizing measures α_M. Sets M, M' below are understood to be finite subsets of $\mathbf{I}$.

If x, y belong to M and if M' contains M then clearly

$$\textstyle\sum_{n=0}^{\infty} (P_M)^n(x, y) = \sum_{n=0}^{\infty} (P_M)^n(x, y)\{1 - \mathfrak{P}_{(M')y}[\sigma^+(M) < +\infty;\, X_{\sigma+(M)} = y]\}$$

and from this it follows that the *potential kernel*

$$N(x, y) = \textstyle\sum_{n=0}^{\infty} (P_M)^n(x, y)/\alpha_M(y) \tag{3.1}$$

is independent of M containing x, y. Clearly every entry $N(x, y) > 0$.

A function h on $\mathbf{I}$ is *excessive* if it is finite and

$$h \geqq 0 ; \qquad h \geqq P_M h \quad \text{on} \quad M \tag{3.2}$$

for every M. If h is excessive and nontrivial then $h(x) > 0$ for every x in $\mathbf{I}$. Also

$$P_{M'} h \geqq P_M h \qquad \text{on} \quad M \tag{3.3}$$

whenever $M' \supset M$. This follows upon observing that $h(X_n)$ is a supermartingale relative to $P_{(M')x}$. The conditioned matrix $P_M{}^h$ is defined by

$$P_M{}^h(x, y) = (1/h(x))P_M(x, y)h(y) .$$

The conditioned probability $\mathfrak{P}^h_{(M)x}$ is defined in the same way as $\mathfrak{P}_{(M)x}$ except that P_M is replaced by $P_M{}^h$.

For each M let the preliminary sample space Λ_M be as in Section 1. The dead trajectory δ_M is the unique member of Λ_M such that $X_0(\delta_M) = \partial$. If $M' \supset M$ there exists a natural mapping $J_{M'M}$ from $\Lambda_{M'}$ to Λ_M. This is defined by setting

$$J_{M'M}\,\omega(n) = \omega(\tau_n) \qquad n \geqq 0$$

where

$$\tau_0 = \sigma(M)$$
$$\tau_1 = \inf\{n > \sigma(M) \colon X_n \text{ is in } M, X_n \neq x_{\sigma(M)}\} \quad \text{etc.}$$

with the understanding that $\tau_n = +\infty$ and therefore $J_{M'M}\,\omega(n) = \partial$ when not otherwise defined. The inverse limit $\Lambda_*{}^0$ is the collection of families $\{\omega_M\}$ with each ω_M in Λ_M and such that $J_{M'M}\,\omega_{M'} = \omega_M$ whenever $M \subset M'$. The *discrete time sample space* is the reduced inverse limit $\Lambda_* = \Lambda_*{}^0 - \{\delta\}$ where δ is the point in Λ^0 whose components are the dead trajectories δ_M. We denote by J_M the natural projection of Λ_* onto Λ_M.

For x in M and for $M \subset M'$ clearly

$$\mathfrak{E}_{(M')x}\xi \circ J_{M'M} = \mathfrak{E}_{(M)x}\xi$$

for $\xi \geqq 0$ on Λ_M. Since $\mathfrak{P}_{(M)x}$ does not charge the dead trajectory δ_M, it follows by a trivial modification of the argument on page 138 of [7] that there exists a unique probability $\mathfrak{P}_x$ on Λ_* such that

$$\mathfrak{E}_x\xi \circ J_M = \mathfrak{E}_{(M)x}\xi$$

whenever M contains x and whenever $\xi \geqq 0$ on Λ_M. More generally for $h > 0$ excessive there exists a unique probability $\mathfrak{P}_x{}^h$ on Λ_* such that

$$\mathfrak{E}_x{}^h\xi \circ J_M = \mathfrak{E}^h_{(M)x}\xi .$$

Trajectory variables, hitting times and exit times are not well defined on Λ_*. However first hitting positions $X_{\sigma(E)}$ and last exit positions $X_{\sigma^*(E)}$ are well defined at least for E finite. Once and for all *we adopt the convention that* $X_{\sigma(E)} = \partial$ *and* $X_{\sigma^*(E)} = \partial$ *when not otherwise defined.*

For $E \subset M \subset M'$ for $h > 0$ excessive and for x in E

$$\mathfrak{P}^h_{(M')x}[\sigma^+(E) = +\infty] = \{1 - \mathfrak{P}_{(M')x}[X_{\sigma^+(M)} = x]\}\mathfrak{P}^h_{(M)x}[\sigma^+(E) = +\infty]$$

and therefore

$$L_E{}^h(x) = 1_E(x)\alpha_M(x)\mathfrak{P}^h_{(M)x}[\rho^+(E) = +\infty] \tag{3.4}$$

is independent of M containing E. The following relations are easily checked

$$h(x)\mathfrak{P}_x{}^h[X_{\sigma^*(E)} = y] = N(x, y)h(y)L_E{}^h(y) \tag{3.5}$$

$$H^E h(x) = \textstyle\sum_y N(x, y)h(y)L_E{}^h(y) . \tag{3.6}$$

(See Section 1 in [10] where analogous relations are discussed in the stable case.)

For E a finite subset of $\mathbf{I}$ the local time reversal operator ρ_E and the truncation operator τ_E are defined on $\Lambda_M \cap [\sigma(E) < +\infty, \sigma^*(E) < +\infty]$ for M containing E by

$$\begin{aligned} \rho_E\omega(n) &= \omega[\sigma^*(E) - n] && 0 \leqq n \leqq \sigma^*(E) \\ &= \partial && n > \sigma^*(E) \\ \tau_E\omega(n) &= \omega(n) && 0 \leqq n \leqq \sigma^*(E) \\ &= \partial && n > \sigma^*(E) . \end{aligned}$$

These induce operators ρ_E and τ_E on $\Lambda_* \cap [X_{\sigma^*(E)} \neq \partial]$ by

$$\begin{aligned} J_M\rho_E\omega &= \rho_E J_M\omega \\ J_M\tau_E\omega &= \tau_E J_M\omega . \end{aligned}$$

The behavior of the probabilities $\mathfrak{P}_x^h$ under these operators is described in

THEOREM 3.1. *Let h, $h' > 0$ be excessive, let E, E' be finite subsets of I and let $\xi \geqq 0$ be defined on Λ. Then*

$$(3.7) \qquad \begin{aligned} &\textstyle\sum_x h'(x)L_{E'}^{h'}(x)h(x)\mathfrak{E}_x^h(X_{\sigma^*(E)} \neq \partial; \xi \circ \rho_E] \\ &\quad = \textstyle\sum_y h(y)L_E^h(y)h'(y)\mathfrak{E}_y^{h'}[X_{\sigma^*(E')} \neq \partial; \xi \circ \tau_{E'}] . \end{aligned}$$

This theorem extends results first obtained by G. A. Hunt in [4]. For the proof it suffices to combine the technique of [4] with some "diagram chasing." (See for example the proof of Theorem 1.2 in [10].)

From now on I_k, $k \geqq 1$ *is an increasing sequence of finite subsets of* $\mathbf{I}$ *such that* $I_k \uparrow \mathbf{I}$. Consider h, $h' > 0$ and excessive and for each k define $\mathfrak{P}^{(k)}$ on Λ_k by

$$\mathfrak{E}^{(k)}\xi = \textstyle\sum_x h'(x)L_k^{h'}(x)h(x)\mathfrak{E}_{(k)x}^h\xi .$$

Here and below we replace a subscript or superscript I_k by k for typographical convenience. With the help of Theorem 3.1 it is easy to check that

$$\mathfrak{E}^{(k+1)}\xi \circ J_{k+1,k} = \mathfrak{E}^{(k)}\xi$$

for $\xi \geqq 0$ on Λ_k and vanishing on δ_k. (See the proof of Theorem 1.3 in [10].) Again by a modification of the argument on page 138 of [7] there exists a unique countably, additive measure $\mathfrak{P}_{h'}^h$ on Λ_* such that

$$(3.8) \qquad \mathfrak{E}_{h'}^h\xi \circ J_k = \textstyle\sum h'(x)L_k^{h'}(x)h(x)\mathfrak{E}_x^h\xi$$

for $\xi \geqq 0$ on Λ_k and vanishing on δ_k. Clearly $N(\cdot, x)$ is excessive and $L_k^{N(\cdot,x)} = \varepsilon_x$ when I_k contains x and so

$$(3.9) \qquad \mathfrak{E}_{N(\cdot,x)}^h\xi = h(x)\mathfrak{E}_x^h\xi .$$

It is easy to check that the local time reversal operators ρ_k are related by

$$J_{k+1,k}\rho_{k+1} = \rho_k J_{k+1,k}$$

and so there is a unique global time reversal operator ρ such that

$$J_k\rho = \rho_k J_k$$

for all k. Routine arguments extend Theorem 3.1 to

THEOREM 3.2. *Let h, $h' > 0$ be excessive and let $\xi \geqq 0$ on Λ_*. Then*

$$\mathfrak{E}^h_{h'}\xi \circ \rho = \mathfrak{E}_h^{h'}\xi \,. \tag{3.10}$$

If $h' = N\nu$ with ν a measure on $\mathbf{I}$, then $\sum_x \nu(x)h(x)\mathfrak{E}_x^h$ agrees with $\mathfrak{E}^h_{h'}$ and so a special case of (3.10) is

$$\sum \nu(x)h(x)\mathfrak{E}_x^h\xi \circ \rho = \mathfrak{E}_h^{N\nu}\xi \tag{3.11}$$

which is valid if $N\nu$ converges.

We return now to the Dirichlet space defined in Section 2. Let ν be a measure on M and let $f = N\nu$. Then clearly

$$f = \textstyle\sum_{k=0}^{\infty} (P_M)^k(\nu/\alpha_M)$$
$$f - P_M f = \nu/\alpha_M$$

and so for any function g

$$\begin{aligned} \mathfrak{E}^M(f, g) &= \textstyle\sum_x \alpha_M(x)(f - P_M f)(x)g(x) \\ &= \textstyle\sum_x \nu(x)g(x) \,. \end{aligned} \tag{3.12}$$

It is easy to check that $H^M f = f$ and it follows from (2.5) that (3.12) is valid with M replaced by M' containing M, which proves

THEOREM 3.3. *Let $f = N\nu$ where ν is a measure on $\mathbf{I}$ with finite support. Then f belongs to $\mathfrak{F}$ and*

$$\mathfrak{E}(f, g) = \textstyle\sum_x \nu(x)g(x) \tag{3.13}$$

for arbitrary g in $\mathfrak{F}$.

The capacity of a set E is defined for E finite by

$$\operatorname{Cap}(E) = \textstyle\sum_x L_E(x) \tag{3.14}$$

and for E arbitrary by

$$\operatorname{Cap}(E) = \sup\{\operatorname{Cap}(K) : K \subset E,\ K \text{ finite}\} \,. \tag{3.14'}$$

The basic connection between capacity and the Dirichlet norm E is established in

THEOREM 3.4. *If f belongs to $\mathfrak{F}$ and if $|f| \geqq 1$ on E then*

$$\mathfrak{E}(f, f) \geqq \operatorname{Cap}(E) \,. \tag{3.15}$$

PROOF. Clearly $\mathfrak{E}(f, f) \geqq \mathfrak{E}(|f|, |f|)$ and so we can assume $f \geqq 0$. Also it suffices to consider E finite. But then

$$\begin{aligned} E(f, f) &= \mathfrak{E}(H^E1, H^E1) + \mathfrak{E}(f - H^E1, f - H^E1) + 2\mathfrak{E}(H^E1, f - H^E1) \\ &= \mathfrak{E}(H^E1, H^E1 + 2(f - H^E1)) \,. \end{aligned}$$

Since $H^E 1 = N(L_E)$ by (3.6), we can apply Theorem 3.3 to conclude that

$$\mathfrak{E}(f, f) \geqq \sum_x L_E(x)\{H^E 1(x) + 2(f(x) - H^E 1(x))\}$$
$$\geqq \operatorname{Cap}(E)$$

and the theorem is proved.

Replacing f by $(1/\varepsilon)f$ in (3.15) we get the Tchebychev type estimate

$$\operatorname{Cap}\{x: |f(x)| \geqq \varepsilon\} \leqq (1/\varepsilon^2)\mathfrak{E}(f, f) \tag{3.16}$$

for $\varepsilon > 0$ and for f in $\mathfrak{F}$ and taking $E = \{x\}$ and $\varepsilon = |f(x)|$ we get

$$L_{\{x\}}(x)f^2(x) \leqq \mathfrak{E}(f, f) . \tag{3.17}$$

This immediately implies

THEOREM 3.5. *$\mathfrak{F}$ is a Hilbert space relative to the inner product $\mathfrak{E}$. Morever if $f_n \to f$ strongly in $\mathfrak{F}$, then $f_n \to f$ pointwise on* **I**.

The *energy* of a measure ν on **I** is defined by

$$\mathfrak{E}(\nu) = \{\sum_{x,y} \nu(x)\nu(y)N(x, y)\}^{\frac{1}{2}} . \tag{3.18}$$

Simple limiting arguments establish the following two theorems.

THEOREM 3.6. *The following are equivalent for ν a measure on* **I**.

(i) *ν has finite energy.*
(ii) *The potential $N\nu$ belongs to $\mathfrak{F}$ and*

$$\mathfrak{E}(N\nu, g) = \sum_x \nu(x)g(x) \tag{3.19}$$

for g in $\mathfrak{F}$.

(iii) *There exists a constant $c > 0$ such that*

$$\sum_x \nu(x)|g(x)| \leqq c\{\mathfrak{E}(g, g)\}^{\frac{1}{2}}$$

for g in $\mathfrak{F}$.

THEOREM 3.7. *The following are equivalent for E a subset of* **I**.

(i) *E has finite capacity.*
(ii) *L_E has finite energy.*
(iii) *$H_E 1$ belongs to $\mathfrak{F}$.*

If these conditions are satisfied, then $H_E 1 = NL_E$ and

$$\mathfrak{E}(H_E 1, g) = \sum_x L_E(x)g(x)$$

for g in $\mathfrak{F}$.

REMARK. If E does not have finite capacity, then it may or may not be true that $H_E 1 = N_E L_E$.

If g in $\mathfrak{F}$ is orthogonal to all functions $H^M f$ as M runs over finite subsets of **I** and f runs over $\mathfrak{F}$, then in particular

$$\mathfrak{E}^M(g, g) = \mathfrak{E}(g, H^M g) = 0$$

for all M and therefore $\mathfrak{E}(g, g) = 0$ and $g = 0$. Thus the functions $H^M f$ are dense in $\mathfrak{F}$ and in particular $\mathfrak{F}$ is separable. Therefore we can argue as in the appendix of [8] to get a convenient compactification of **I**.

We say that g is a *normalized contraction* of f if

$$(3.20) \qquad |g(x)| \leqq |f(x)|; \qquad |g(x) - g(y)| \leqq |f(x) - f(y)| .$$

It is easy to check that if f is in $\mathfrak{F}$ and if g is a normalized contraction of f, then also g is in $\mathfrak{F}$ and $\mathfrak{E}(g, g) \leqq \mathfrak{E}(f, f)$. From this it follows that the subcollection of bounded functions in $\mathfrak{F}$ is an algebra. Let B_0 be the algebra in $\mathfrak{F}$ generated by the potential kernels $N(\cdot, y)$ and by the indicators e_x for x stable. It is easy to check that B_0 is dense in $\mathfrak{F}$ and that the uniform closure B is a separable Banach algebra. The desired compactification **X** is the maximal ideal space of B. It is well known and easy to check directly that **X** is a separable locally compact Hausdorff space. The algebra B_0 separates points on **I** and therefore **I** is naturally imbedded as a dense subset of **X**. The singleton $\{x\}$ is open in the induced topology if and only if x is stable.

All functions in B are identified with their continuous extensions to **X**. In particular the potential kernel $N(x, y)$ is well defined for y in **I** and x in **X** and for fixed y in **I** is continuous in x. If $h > 0$ on **I** is excessive then $H^M h \uparrow h$ as finite $M \uparrow$ **I** and since each $H^M h$ is a finite sum of the potential kernels $N(\cdot, y)$ it follows that h has a unique lower semicontinuous extension to **X** such that $H^M h \uparrow h$ on **X**. We identify h with this extension to **X**. This is true in particular for $h = N(x, \cdot)$ with x in **X** and therefore $N(x, y)$ is well defined on **X** × **X** and lower semicontinuous in each variable separately. Notice also that N separates points on **X**.

Consider E *finite*, let M in I contain x, y and E, and let $F = M - E$. Then

$$N(x, y)\alpha_M(y) = 1_F \textstyle\sum_{k=0}^{\infty} (1_F P_M 1_F)^k 1_F(x, y) + \sum_{z \text{ in } E} H^E(x, z) N(z, y) \alpha_M(y)$$

and therefore

$$(3.21) \qquad \textstyle\sum_{z \text{ in } E} H^E(x, z) N(z, y) = \sum_{z \text{ in } E} H^E(y, z) N(z, x) .$$

It follows in particular that for fixed y in **I** the sum

$$\textstyle\sum_{z \text{ in } E} H^E(\cdot, z) N(z, y)$$

has a continuous extension to **X**. But as y runs over E and *a fortiori* as y runs over **I** the functions $N(\cdot, y)$ restricted to E span the finite dimensional vector space of functions on E and therefore for each z in E the function $H^E(\cdot, z)$ has a continuous extension to **X**. From now on we regard $H^E(x, z)$ as being defined by this extension for x in **X**. With this understanding (3.21) makes sense and is valid for x, y in **X**. From this it follows directly that also

$$(3.22) \qquad N(x, y) = N(y, x)$$

for x, y in **X**.

From the very definition of the maximal ideal space, B is precisely the Banach space of continuous functions on $\mathbf{X}$ "vanishing at infinity." (Of course the last phrase is superfluous when $\mathbf{X}$ is compact.) It follows in particular that every measure ν on $\mathbf{I}$ with finite energy is a Radon measure on $\mathbf{X}$. More generally we say that a Radon measure ν on $\mathbf{X}$ has finite energy if there exists a constant $c > 0$ such that

$$\int \nu(dx)f(x) \leqq c\{E(f, f)\}^{\frac{1}{2}}$$

for f in B_0. In this case there exists a unique function $N\nu$ in $\mathfrak{F}$ such that

$$\mathfrak{E}(N\nu, f) = \int \nu(dx)f(x)$$

for f in B_0. We call $N\nu$ the potential of ν and we define the energy of ν by

$$\mathfrak{E}(\nu) = \{\mathfrak{E}(N\nu, N\nu\}^{\frac{1}{2}} .$$

Of course this is consistent with previous notations when ν is concentrated on $\mathbf{I}$. The proof of Proposition 1.2 in [8] shows that f in $\mathfrak{F}$ is a potential if and only if $\mathfrak{E}(f, g) \geqq 0$ whenever $g \geqq 0$ on $\mathbf{I}$. (This is not true if we consider only measures ν concentrated on $\mathbf{I}$.) Moreover it is easy to check that this is the case if and only if f is excessive.

For G open in $\mathbf{X}$ we define the capacity of G by

$$\text{Cap}\,(G) = \inf \mathfrak{E}(f, f) \tag{3.23}$$

as f runs over the functions in $\mathfrak{F}$ such that $f \geqq 1$ on $G \cap \mathbf{I}$. If no such f exists we put $\text{Cap}\,(G) = +\infty$. For general Borel subsets A of $\mathbf{X}$ we define

$$\text{Cap}\,(A) = \inf \text{Cap}\,(G) \tag{3.23$'$}$$

as G runs over the open supersets of A. With the help of (3.15) it is easy to check that this is consistent with (3.14) when A is a finite subset of $\mathbf{I}$. It follows from the proof of Proposition 1.5 in [8] that if open G in $\mathbf{X}$ has finite capacity, then there exists a unique function p^G in $\mathfrak{F}$ such that $p^G \geqq 1$ on $G \cap \mathbf{I}$ and

$$\text{Cap}\,(G) = \mathfrak{E}(p^G, p^G) .$$

Also $0 \leqq p^G \leqq 1$ on $\mathbf{I}$ and $p^G = 1$ on $G \cap \mathbf{I}$ and $p^G = N_\mu$ with μ concentrated on the closure $cl(G)$ in $\mathbf{X}$. Let ν be an arbitrary measure on $\mathbf{X}$ with finite energy. Then $N\nu$ in $\mathfrak{F}$ is excessive, $H^M N\nu \to N\nu$ strongly in $\mathfrak{F}$ as $M \uparrow \mathbf{I}$ and therefore $L_M{}^{N\nu}$ converges vaguely to ν. Thus

$$\begin{aligned}\nu(G) &\leqq \liminf \textstyle\sum_{x \text{ in } G} L_M{}^{N\nu}(x)\\ &= \liminf \textstyle\sum_x L_M{}^{N\nu}(x)p^G(x)\\ &= \liminf \mathfrak{E}(p^G, H^M N\nu)\\ &= \mathfrak{E}(p^G, N\nu)\end{aligned}$$

and therefore

$$\nu(G) \leqq \mathfrak{E}(\nu)\{\text{Cap}\,(G)\}^{\frac{1}{2}} . \tag{3.24}$$

By the proof of Proposition 1.8 in [8]

$$\text{Cap}\,(G_n) \uparrow \text{Cap}\,(G) \tag{3.25}$$

whenever open $G_n \uparrow G$ and

$$\text{Cap}(G_1 \cup G_2) + \text{Cap}(G_1 \cap G_2) \leqq \text{Cap}(G_1) + \text{Cap}(G_2). \tag{3.26}$$

Thus the Choquet extension theorem applies and for every Bore subset A of $\mathbf{X}$

$$\text{Cap}(A) = \sup \text{Cap}(K) \tag{3.27}$$

as K runs over the compact subsets of A. From this it follows that (3.23′) is consistent with (3.14′) when A is an arbitrary subset of $\mathbf{I}$.

A Borel subset A of $\mathbf{X}$ is said to be polar if $\text{Cap}(A) = 0$. A property is said to be valid *quasi-everywhere* on X if the exceptional set is polar. Two functions are said to be *quasi-equivalent* if they differ on a polar set, that is, if they are equal quasi-everywhere. A function f is *quasi-continuous* if there exists a decreasing sequence of open sets G_n in $\mathbf{X}$ with $\text{Cap}(G_n) \downarrow 0$ such that f is continuous on $\mathbf{X} - G_n$ for every n.

It follows from (3.24) that if a measure ν has finite energy then it charges no polar set. The proof of Lemma 1.17 in [8] establishes a partial converse. Every Radon measure which charges no polar set can be represented as a countable sum of measures with finite energy.

The proof of Theorem 1.11 in [8] shows that every f in $\mathfrak{F}$ has a quasi-continuous extension to $\mathbf{X}$ which is unique up to quasi-equivalence such that whenever $f_n \to f$, then for a subsequence the extensions f_n converge quasi-everywhere to the extension f. From now on we identify f in $\mathfrak{F}$ with this quasi-continuous extension to $\mathbf{X}$. Then the proof of Theorem 1.12 in [8] shows that

$$\mathfrak{E}(f, N\nu) = \int \nu(dx) f(x) \tag{3.28}$$

$$\text{Cap}\{x : |f(x)| \geqq \varepsilon\} \leqq (1/\varepsilon^2)\mathfrak{E}(f, f) \tag{3.29}$$

for f in $\mathfrak{F}$, for ν having finite energy and for $\varepsilon > 0$. This leads directly to the familiar

LEMMA 3.8. (*Maximum principle*) *if* $f = N\mu$ *and* $g = N\nu$ *are potentials in* $\mathfrak{F}$ *and if* $f \geqq g$ [a.e. ν], *then* $f \geqq g$ *quasi-everywhere on* $\mathbf{X}$.

We have already remarked that every f in $\mathfrak{F}$ has a unique quasi-continuous extension to $\mathbf{X}$. Fukushima's argument (see the proof of Lemma 1.15 in [8]) is easily adapted to improve the statement of uniqueness.

LEMMA 3.9. *Let* f, g *be quasi-continuous on an open subset* G *of* X *and let* $f \geqq g$ *on* $G \cap \mathbf{I}$. *Then* $f \geqq g$ *quasi-everywhere on* G.

Next consider h a potential in $\mathfrak{F}$ and suppose that g is excessive and $0 \leqq g \leqq h$. Then for every M

$$\begin{aligned} \mathfrak{E}^M(g, g) &= \sum \alpha_M(x) g(x)\{g - P_M g\}(x) \\ &\leqq \sum \alpha_M(x) h(x)\{g - P_M g\}(x) \\ &= \sum \alpha_M(x) g(x)\{h - P_M h\}(x) \\ &\leqq \mathfrak{E}^M(h, h) \end{aligned} \tag{3.30}$$

and it follows that also g is a potential in $\mathfrak{F}$. In particular if h is a potential in $\mathfrak{F}$ then so is $H^E h$ for E finite. Indeed if $h = N\nu$ then by (3.21)

$$H^E N\nu(x) = \sum_{z \text{ in } E} \int H^E(x, z)N(z, y)\nu(dy) = \sum_{z \text{ in } E} \int H^E(y, z)N(z, x)\nu(dy) = N(\pi^E\nu)(x) \tag{3.31}$$

where $\pi^E\nu$ is the balayaged measure

$$\pi^E\nu(z) = \int \nu(dy)H^E(y, z) . \tag{3.32}$$

We extend this now to E infinite.

Let $\mathscr{M}$ be the collection of Radon measures on $\mathbf{X}$ having finite energy. The proof of Proposition 1.4 in [8] shows that $\mathscr{M}$ is complete relative to the energy metric $\mathfrak{E}(\mu - \nu)$. For E an arbitrary subset of $\mathbf{I}$ let $\mathscr{M}(E)$ be the closure in $\mathscr{M}$ of measures concentrated on E and let $\mathfrak{F}(E)$ be the closed linear span in $\mathfrak{F}$ of $N\mathscr{M}(E)$. For ν in $\mathscr{M}$ the balayaged measure $\pi^E\nu$ is the unique measure in the closed convex set $\mathscr{M}(E)$ such that $\mathfrak{E}(\nu - \pi^E\nu)$ is minimal. The proof of Lemma 3.5 in [8] shows that $N\pi^E\nu$ is the orthogonal projection of $N\nu$ onto $\mathfrak{F}(E)$. Next consider finite subsets $E_n \uparrow E$. By the convergence theorem for reversed supermartingales $\lim N(X_{\sigma(E_n)}, y)$ exists [a.e. $\mathfrak{P}_x$] for x, y in $\mathbf{I}$. Also $\lim e_y(X_{\sigma(E_n)})$ exists for y in $\mathbf{I}$ and therefore $\lim f(X_{\sigma(E_n)})$ exists [a.e. $\mathfrak{P}_x$] for f in B_0. Therefore $X_{\sigma(E)} = \lim X_{\sigma(E_n)}$ is well defined [a.e. $\mathfrak{P}_x$] as a point in $X \cup \{\partial\}$. (We are identifying the dead point ∂ with the trivial homomorphism on B.) We denote the distribution of $X_{\sigma(E)}$ relative to $\mathfrak{P}_x$ by $H^E(x, dy)$ and we define the operator H^E by

$$H^E f(x) = \mathfrak{E}_x f(X_{\sigma(E)}) = \int H^E(x, dy)f(y) \tag{3.33}$$

when this converges. Similar arguments together with time reversal establish the existence of the last exit position $X_{\sigma^*(E)} = \lim X_{\sigma^*(E_n)}$.

THEOREM 3.10. *Let f belong to $\mathfrak{F}$ and let E be an arbitrary subset of* $\mathbf{I}$.

(i) *$H^E f$ converges everywhere on $\mathbf{I}$ and is the orthogonal projection of f onto $\mathfrak{F}(E)$. In particular*

$$H^E N\mu = N\pi^E\mu \tag{3.34}$$

for μ in $\mathscr{M}$.

(ii) *f belongs to the orthogonal complement of $\mathfrak{F}(E)$ if and only if $f = 0$ everywhere on E.*

PROOF. (ii) follows directly from (3.28). For (i) it suffices to consider $f = N\nu$ with ν supported by $\mathbf{I}$. It is easy to see that $H^E f$ is excessive and dominated by f and therefore is a potential in $\mathfrak{F}$. Since $f = H^E f$ on E, clearly $f - H^E f$ is orthogonal to $\mathfrak{F}(E)$. By the proof of Lemma 3.5 in [8], $f = N\pi^E\nu$ on E and therefore $H^E f = H^E N\pi^E\nu$. But $N\pi^E\nu$ can be approximated by potentials of measures concentrated on E and it follows easily that $H^E N\pi^E\nu$ and therefore $H^E f$ is dominated by $N\pi^E\nu$. Then by (3.30)

$$\mathfrak{E}(H^E f, H^E f) \leqq \mathfrak{E}(N\pi^E\nu, N\pi^E\nu)$$

and it follows that $H^E f = N\pi^E\nu$.

Finally we note that for E an arbitrary subset of $\mathbf{I}$ the function $H^E N(x, y)$ defined by

$$H^E N(x, y) = \int H^E(x, dz) N(z, y) \tag{3.35}$$

for x in $\mathbf{I}$ and y in $\mathbf{X}$ extends by our conventions to a function defined on $\mathbf{X} \times \mathbf{X}$ which is *lower semicontinuous* in each variable separately and *symmetric*. This is easily proved by approximating E from below by finite sets.

4. The Martin representation. We continue to work with the transient road map of Section 3. A function h on $\mathbf{I}$ is said to be harmonic if

$$h = H^E h \tag{4.1}$$

whenever $\mathbf{I} - E$ has compact closure in $\mathbf{X}$. (It is understood that the right side of (4.1) must converge everywhere on $\mathbf{I}$.)

NOTATION 4.1. $E \downarrow \varnothing$ canonically if each E is a subset of $\mathbf{I}$, if the complement $\mathbf{I} - E$ has compact closure in $\mathbf{X}$ and if the closure of E in $\mathbf{X}$ decreases to the empty set.

An excessive function $h > 0$ is said to be a *potential* if $H^E h \downarrow 0$ on $\mathbf{I}$ as $E \downarrow \varnothing$ canonically. It follows from Theorem 3.10 that every excessive function in $\mathfrak{F}$ is a potential and that $\mathfrak{F}$ contains no nontrivial harmonic functions. In particular $N(\cdot, y)$ is a potential for y in $\mathbf{I}$. Indeed $H^E N(\cdot, y) \downarrow 0$ quasi-everywhere on $\mathbf{X}$ as $E \downarrow \varnothing$ canonically and it follows by symmetry that $N(x, \cdot)$ is a potential for quasi-every x in $\mathbf{X}$. To pursue this further, we use the techniques of Hunt [4].

We begin by fixing a reference point 0 in I_1. For x, z in $\mathbf{I}$ clearly

$$N(0, z) \geqq \mathfrak{P}_0[X_{\sigma(\{x\})} = x] N(x, z) \tag{4.2}$$

and this extends by continuity to z in $\mathbf{X}$. It follows that if for a given z in $\mathbf{X}$ we have $N(z, 0) = 0$, then $N(z, x) = 0$ for all x in $\mathbf{I}$ and therefore $f(z) = 0$ for all f in B_0 which is impossible by the very definition of the maximal ideal space $\mathbf{X}$. Thus $N(\cdot, 0)$ is everywhere positive on $\mathbf{X}$ and therefore the *Martin kernel*

$$K(x, y) = N(x, y)/N(0, y) \tag{4.3}$$

defined for x, y in $\mathbf{X}$ is continuous in y for fixed x in $\mathbf{I}$. Also (4.2) and the corresponding estimate with the roles of x and 0 reversed imply

$$\mathfrak{P}_x[X_{\sigma(\{0\})} = 0] \leqq K(x, y) \leqq (\mathfrak{P}_0[X_{\sigma(\{x\})} = x])^{-1}. \tag{4.4}$$

We define a metric on $\mathbf{X}$ by

$$d(y, y') = \sum_x \beta(x) \mathfrak{P}_0[\sigma_{(\{x\})} = x] |K(x, y) - K(x, y')|$$

with $\beta(x) > 0$ chosen so that $\sum_x \beta(x) < +\infty$. The Martin closure $\mathbf{X}^*$ is the completion of $\mathbf{X}$ with respect to d. The Martin kernel extends by continuity to $\mathbf{I} \times \mathbf{X}^*$. But for fixed y in $\mathbf{X}^*$ the function $K(\cdot, y)$ is excessive on $\mathbf{I}$ and therefore $K(x, y)$ is actually well defined for x in $\mathbf{X}$ and y in $\mathbf{X}^*$.

For $h > 0$ excessive the last exit positions $X_{\sigma^*(I_k)}$ have the same distribution relative to $\mathfrak{P}_0^h$ as the first entrance positions $X_{\sigma(I_k)}$ have relative to $\mathfrak{P}_h^{N(\cdot,0)}$. Thus for fixed x in $\mathbf{I}$ the random variables $K(x, X_{\sigma^*(I_k)})$ form a reversed supermartingale and so $\lim K(x, X_{\sigma^*(I_k)})$ exists [a.e. $\mathfrak{P}_0^h$] as $k \uparrow \infty$. Therefore the terminal position

$$X_\infty = \lim X_{\sigma^*(I_k)}$$

is well defined [a.e. $\mathfrak{P}_0^h$] as a point in the Martin closure $\mathbf{X}^*$. By (3.5) and (3.6)

$$\begin{aligned} h(0)\mathfrak{E}_0^h K(x, X_{\sigma^*(I_k)}) &= \textstyle\sum_z K(x, z)N(0, z)L_k^h(z)h(z) \\ &= H^k h(x) \end{aligned}$$

and passing to the limit in k we conclude that

$$h(x) = \int K(x, y)\nu^h(dy) \tag{4.5}$$

with $\nu^h(dy)$ the distribution of X_∞ with respect to $h(0)\mathfrak{P}_0^h$. We denote by $\mathbf{X}_0$ the set of y in $\mathbf{X}^*$ such that $K(\cdot, y)$ is an extremal potential and by Δ_0 the set of x in $\mathbf{X}^*$ such that $K(\cdot, y)$ is extremal harmonic. The proof of Theorem 1.7 in [10] shows that ν^h is concentrated on $\mathbf{X}_0 \cup \Delta_0$ and is the unique measure concentrated on $\mathbf{X}_0 \cup \Delta_0$ such that (4.5) is valid. If h is a potential, then by Theorem 3.10

$$\textstyle\sum_{z \text{ in } E \cap I_k} N(0, z)L_k^h(z)h(z) \leqq H^E h(0)$$

which $\downarrow 0$ independent of k as $E \downarrow \varnothing$ canonically. Therefore ν^h is concentrated on $\mathbf{X}$ and it follows that $\mathbf{X}_0$ is contained in $\mathbf{X}$. If $h = N\nu$ belongs to $\mathfrak{F}$ then clearly $\pi^{I_k}\nu \to \nu$ vaguely. But also $\pi^{I_k}\nu = L_k^h h$ and it follows that $\nu^h = N(0, \cdot)\nu$. In particular ν does not charge $\mathbf{X} - \mathbf{X}_0$ and since every measure which charges no polar set is a countable sum of measures with finite energy, it follows that $\mathbf{X} - \mathbf{X}_0$ is a polar set. We summarize these results in

THEOREM 4.1. (i) *Every excessive function $h > 0$ has a unique representation* (4.5) *with ν^h concentrated on $\mathbf{X}_0 \cup \Delta_0$. Indeed ν^h is the distribution of the terminal positive X_∞ with respect to $h(0)\mathfrak{P}_0^h$ and $N(0, \cdot)\, L_k^h h \to \nu^h$ vaguely as $k \uparrow \infty$.*

(ii) *$\mathbf{X}_0$ is a subset of $\mathbf{X}$ and $\mathbf{X} - \mathbf{X}_0$ is polar.*

(iii) *If $h = N\nu$ with ν having finite energy, then $\nu^h = N(0, \cdot)\nu$.*

From now on we denote the restriction of ν^h to Δ_0 by l^h and we let κ^h be the unique measure on $\mathbf{X}_0$ such that

$$\nu^h(dy) = l^h(dy) + N(0, y)\kappa^h(dy)\,.$$

We also use the abbreviations

$$\begin{aligned} \nu^y &= \nu^{K(\cdot,y)} \\ l^y &= \nu^{K(\cdot,y)} \\ \nu_k^h &= N(0, \cdot)L_k^h h \quad \text{etc.} \end{aligned}$$

As in [10], Theorem 4.1 plus the martingale convergence theorem lead to

COROLLARY 4.2. (*Fatou*) *Let $h > 0$ be harmonic and let x in* **I**. *Then as $k \uparrow \infty$*

$$h(X_{\sigma^*(I_k)}) \to (dl^h/dl^1)(X_\infty) \quad [\text{a.e. } \mathfrak{P}_x] .$$

For x in **X** and y in **X*** the *Naim kernel* $\Theta(x, y)$ is defined by

$$\Theta(x, y) = K(x, y)/N(x, 0) .$$

Clearly

$$\Theta(x, y) = \Theta(y, x) \tag{4.6}$$

for x, y in **X**. The proof of Lemma 1.9 in [10] shows that

$$\textstyle\sum_z \nu_k^x(z)\Theta(z, y)$$

increases with k for x, y in **X*** and that we can extend Θ to $\mathbf{X}^* \times \mathbf{X}^*$ by setting

$$\Theta(x, y) = \lim \textstyle\sum_z \nu_k^x(z)\Theta(z, y) \qquad (k \uparrow \infty) .$$

This extension satisfies (4.6) and is lower semicontinuous in each variable separately. Finally, the proof of (1.24) in [10] establishes

$$\mathfrak{E}_1^1\varphi(X_{-\infty}, X_{+\infty}) = \int \nu^1(dy) \int \nu^1(dz)\Theta(y, z)\varphi(y, z) \tag{4.7}$$

for $\varphi \geqq 0$ on $\mathbf{X}^* \times \mathbf{X}^*$.

5. Speed measures. We return now to the general road map of Section 2.

DEFINITION 5.1. An everywhere positive measure m on **I** is a *speed measure* if $F \cap L^2(m)$ is dense in $L^2(m)$. It is an *active* speed measure if for each f in F there exists a sequence f_n, $n \geqq 1$ in $F \cap L^2(m)$ such that $f_n \to f$ pointwise on **I** and $\sup_n E(f_n, f_n) < +\infty$.

We fix a speed measure m and put

$$\mathbf{F} = \mathfrak{F} \cap L^2(m) .$$

It is easy to check that the pair $(\mathbf{F}, \mathfrak{E})$ is a Dirichlet space on $L^2(m)$ in the sense of Section 4 in [10]. Let $\mathbf{P}_t(x, y)$ be the unique standard transition matrix on **I** such that $(\mathbf{F}, \mathfrak{E})$ is the associated Dirichlet space. It is automatic that $\mathbf{P}_t$ satisfies the symmetry condition

$$m(x)\mathbf{P}_t(x, y) = m(y)\mathbf{P}_t(y, x) \tag{5.1}$$

for $t > 0$ and for x, y in **I**. It is well known [1] that

$$q(x) = \lim (1/t)\{1 - \mathbf{P}_t(x, x)\} \qquad t \downarrow 0 \tag{5.2}$$

exists for x in I with $0 \leqq q(x) \leqq +\infty$. (This and related results can also be proved using arguments based on Theorem 4.1 in [10].) In the literature, a state x is said to be *stable* if $q(x) < +\infty$ and *instantaneous* if $q(x) = +\infty$. It follows from Theorem 4.1 in [10] that the indicator e_x belongs to **F** if and only if x is stable and therefore this is consistent with our terminology in Section 2.

In order to obtain a decent process we compactify **I** as in Section 6 of [10]. Let $\mathbf{B}_0$ be a subset of $\mathbf{F} \cap L^1(m) \cap L^\infty$ satisfying

5.2.1. $\mathbf{B}_0$ is an algebra and contains the indicator e_x whenever x is stable.

5.2.2. $\mathbf{B}_0$ is dense in **F** relative to any of the inner products

$$\mathfrak{E}_u(f, g) = \mathfrak{E}(f, g) + u \sum m(x)f(x)g(x) \qquad u > 0 .$$

5.2.3. The uniform closure **B** of $\mathbf{B}_0$ is separable.

Then **B** is a commutative Banach algebra and its maximal ideal space **Y** is the desired compactification of **I**. This is a separable locally compact Hausdorff space which is compact if and only if 1 belongs to **B**. The original state space **I** is densely imbedded in **Y**. The singleton $\{x\}$ is open in **I** if and only if x is stable.

REMARK. The compactification **Y** is introduced only to get us started in our analysis and will not play an important explicit role. At least in special cases we can guarantee that **Y** is identical with **X** introduced in earlier sections.

Structures on **I** are carried over to **Y** in the obvious way. The point is that $(\mathbf{F}, \mathfrak{E})$ is a regular Dirichlet space on $L^2(\mathbf{Y}, m)$ and therefore the results of [8] are applicable. We adjoin a dead point ∂ to **Y** according to the usual conventions. The standard continuous time sample space Ω is the collection of mappings ω from $[0, \infty)$ into the augmented compactification $\mathbf{Y} \cup \{\partial\}$ which satisfy the following conditions.

5.3.1. $\omega(t)$ is right continuous and has left hand limits everywhere.

5.3.2. There exists a life time $\zeta(\omega)$ with $0 \leqq \zeta(\omega) \leqq +\infty$ such that $\omega(t) = \partial$ if and only if $t \geqq \zeta(\omega)$, and $\omega(t-0) \neq \partial$ for $0 < t < \zeta(\omega)$.

By the results of Section 2 in [8] there exists on exceptional set N of **Y** which is polar (and in particular does not intersect **I**) and a family of probabilities $\mathscr{P}_x$ indexed by x in the complement $\mathbf{Y} - N$ which form a strong Markov process taking values in $\mathbf{Y} - N$ and having the usual regularity properties. In particular each $\mathscr{P}_x$ is concentrated on Ω and therefore the trajectories are right continuous with left hand limits everywhere. Trajectory variables X_t and first hitting times $\sigma(E)$ and last exit times $\sigma^*(E)$ are defined in the usual manner. (See Section 5 of [10].)

For M a finite subset of **I** we introduce the additive functional

$$A^M(t) = \int_0^t ds\, 1_M(X_s) ,$$

the inverse functional

$$B^M(s) = \inf\{t > 0 : A^M(s) > t\}$$

and the time changed process

$$X_t^M = X_{B^M(t)} .$$

Roughly speaking, X_t^M is defined by "looking at X_t only when it is in M." The time changed resolvent R_a^M is defined by

$$R_a^M\varphi(x) = \mathscr{E}_x \int_0^\infty dt\, e^{-at}\varphi(X_t^M)$$

which after a change of variables can be written

$$R_a^M\varphi(x) = \mathscr{E}_x \int_0^\infty dt\, 1_M(X_t)e^{-aA^M(t)}\varphi(X_t)\,.$$

It is proved in [8] that the operators R_a^M, $a > 0$ form symmetric resolvents relative to $L^2(M, m)$. Suppose now that m is active. The associated Dirichlet norm is identified in Section 5 of [8] and also in Theorem 6.1 of [10] as

$$\mathfrak{E}(H^Mf, H^Mf)$$

where for the moment

$$H^Mf(x) = \mathscr{E}_x[\sigma(M) < \infty; f(X_{\sigma(M)})]\,. \tag{5.3}$$

From Theorem 8.4 in [8] it follows that (5.3) is consistent with (2.4) and therefore the associated Dirichlet norm is actually the norm $\mathfrak{E}^M$ defined by (2.3). From this it follows by an elementary case of Section 5 in [9] that

$$P_M(x, y) = \mathscr{P}_x[\sigma(M - \{x\}) < +\infty; X_{\sigma(M-\{x\})} = y]\,. \tag{5.4}$$

$$\alpha_M(x) = m(x)q_M(x)\,. \tag{5.5}$$

where q_M is defined by (5.2) except that $\mathbf{P}_t(x, x)$ is replaced by the transition matrix for the time changed process. Clearly $q_M \uparrow q$ as $M \uparrow \mathbf{I}$ and it follows again that x is stable in the sense of Section 2 if and only if $q(x) < +\infty$ and in this case

$$\alpha(x) = m(x)q(x)\,.$$

The proof of Theorem 4.2 in [10] is valid without the restriction that every state be stable and it follows that every standard transition matrix which satisfies a symmetry condition (5.1) and an obvious condition of irreducibility can be obtained by the above construction from a unique road map. We summarize in

THEOREM 5.1. *Let $\{P_M\}$ be a road map on* $\mathbf{I}$ *with corresponding symmetrizing measures α_M. Suppose that m is active. Then there is a unique irreducible standard transition matrix $\mathbf{P}_t(x, y)$ satisfying the symmetry condition* (5.1) *such that P_M and α_M can be recovered by* (5.4) *and* (5.5). *Conversely if $\mathbf{P}_t(x, y)$ is an irreducible standard transition matrix satisfying* (5.1) *for some everywhere positive measure m on* $\mathbf{I}$, *then there is a unique road map such that m is an active measure and such that the above is true.*

REMARK. By irreducibility in Theorem 5.1 we mean that for each pair x, y in $\mathbf{I}$ there exists $t > 0$ such that $\mathbf{P}_t(x, y) > 0$. It then follows from results in [1] that actually $\mathbf{P}_t(x, y) > 0$ for all $t > 0$ and for all x, y in $\mathbf{I}$. Obviously this restriction is harmless.

If the road map is recurrent and if m is bounded, then 1 belongs to $\mathbf{F}$ and

$\mathfrak{E}(1, 1) = 0$. From this it follows that the corresponding transition matrix is conservative. This is easily extended to general speed measures using random time change.

If the road map is transient and if the speed measure m is bounded, we can put $\mathbf{B}_0 = B_0$ and then $\mathbf{Y} = \mathbf{X}$. In this case it is easy to establish connections between various quantities in Sections 3 and 4 and their counterparts in continuous time: For example (5.3) is valid for arbitrary subsets E of $\mathbf{I}$ and

$$\mathscr{E}_x \int_0^\zeta dt f(X_t) = \sum_y N(x, y) f(y) m(y) .$$

We finish by interpreting a criterion of Freedman's in our set up. Fix a reference point 0 in $\mathbf{I}$ and for x in $\mathbf{I}$ define

$$\begin{aligned} O(0, x) &= \alpha_M(0)\mathscr{P}_0[\sigma_{\{x\}} < \sigma_{\{0\}}^+] \\ O(x, 0) &= \alpha_M(x)\mathscr{P}_x[\sigma_{\{0\}} < \sigma_{\{x\}}^+] \\ \sigma(0, x) &= O(0, x)/O(x, 0) \end{aligned}$$

where M is any finite subset of $\mathbf{I}$ which contains 0 and x. It is easy to check that these quantities are independent of the choice of M. For a given choice of m and for each finite M containing 0 define

$$\gamma_M(t) = \inf\{\int_0^s du\, 1_M(X_u) : \int_0^s du\, 1_{\{0\}}(X_u) > t\} .$$

It is easy to check that

$$\mathscr{E}_0 \gamma_M(t) = t(1 + \sum_{x \in M} \sigma(0, x) m(x)/m(0)) .$$

Sense can be made of this whether or not m is a speed measure. Freedman's sufficient condition for existence is

$$\sup_M \mathscr{E}_0 \gamma_M(t) < +\infty$$

or, equivalently

$$\sum_x \sigma(0, x) m(x) < +\infty . \tag{5.6}$$

(See [3] page 110.) This together with our results guarantees that (5.6) is a sufficient condition for an everywhere positive m on $\mathbf{I}$ to be a speed measure.

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