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ON THE CENTRAL LIMIT THEOREM FOR SAMPLE CONTINUOUS PROCESSES[1]

BY EVARIST GINÉ M.

Instituto Venezolano de Investigaciones Científicas

Let $\{X_k\}_{k=1}^\infty$ be a sequence of independent centered random variables with values in $C(S)$ (i.e., sample continuous processes in S), (S, d) being a compact metric space. This sequence is said to satisfy the central limit theorem if there exists a sample continuous Gaussian process on S, Z, such that $\mathscr{L}(\sum_{k=1}^n X_k/n^{\frac{1}{2}}) \rightarrow_{w^*} \mathscr{L}(Z)$ in $C'(C(S))$. In this paper some sufficient conditions are given for the central limit theorem to hold for $\{X_k\}_{k=1}^\infty$; these conditions are on the modulus of continuity of the processes X_k and they are expressed in terms of the metric entropy of distances associated to $\{X_k\}$. Then, in order to give some insight on these theorems, several results on the central limit theorem for particular processes (random Fourier and Taylor series, as well as more general processes on [0, 1]) are deduced.

1. Introduction. Let (S, d) be a compact metric space and let $\{X_k\}_{k=1}^\infty$ be a sequence of independent, centered $C(S)$-valued random variables (i.e., sample continuous processes). We say, following [10], that the central limit theorem holds for this sequence if the laws of $\sum_{k=1}^n X_k/n^{\frac{1}{2}}$ converge in the weak-star topology of $C'(C(S))$ to the law of a sample continuous Gaussian process on S. (This type of convergence is defined in [3] as weak convergence of probability measures).

Strassen and Dudley give in [10] a sufficient condition for the central limit theorem to hold in terms of the ε-entropy of S with respect to the pseudo-distance $e(s, t) = ||X_1(s) - X_1(t)||_\infty$ in the case of identically distributed random variables; this condition implies in particular that the random variables have equicontinuous range. The main results in this paper consist in two central limit theorems, one for not necessarily identically distributed random variables (Theorem 2.2) and the other for identically distributed random variables with not necessarily equicontinuous range (Theorem 2.4); the proof of the first follows the line of the theorem in [10], the main difference being that we use a different exponential bound (Lemma 2.1) and that this allows us to let the norm $||X_k(s) - X_k(t)||_\infty$ grow with k; then from this theorem we obtain the second one just via a very efficient truncation argument more or less suggested by [7]. Before describing more explicitly the sufficient condition in Theorem 2.4, let us recall the definition of ε-entropy: if (T, e) is a metric space (or pseudo-metric), then for every $\varepsilon > 0$,

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$N(T, e, \varepsilon)$ is defined as the infimum of the number of sets in any covering of T by sets of diameter less than or equal to 2ε, i.e.

$$N(T, e, \varepsilon) = \inf \{n : T \subset \bigcup_{i=1}^{n} V_i, \sup_{i; x, y \in V_i} e(x, y) \leqq 2\varepsilon\} ;$$

and the ε-entropy of (T, e), $H(T, e, \varepsilon)$, as

$$H(T, e, \varepsilon) = \log N(T, e, \varepsilon) .$$

The sufficient condition in Theorem 2.4 is that there exist a pseudo-distance e on S and a real random variable M such that

$$|X_1(\omega, s) - X_1(\omega, t)| \leqq e(s, t) M(\omega)$$

with

$$M \in L_2(\Omega, P) \qquad \text{and} \qquad \int_0^1 H(S, e, u)\, du < \infty .$$

On the other hand the condition in the theorem of Strassen and Dudley may be expressed in the same way but with $M \in L_\infty(\Omega, P)$ and $\int_0^1 H(S, e, u)^{\frac{1}{2}}\, du < \infty$. They prove in [10] that their condition on H can not be improved; the same is true for our condition on M.

Starting from the results just described, R. Dudley [5] proved a sort of interpolation between the theorem in [10] and Theorem 2.4; his conditions are: $M \in L_p(\Omega, P)$ and $\limsup_{\varepsilon \downarrow 0} \varepsilon^\alpha H(S, e, \varepsilon) < \infty$ for some $p > 2$ and $\alpha < 2p/(p+2)$. The ε-entropy condition can be somewhat improved but we will not treat this question here. We will use Dudley's theorem in some of the examples S below.

In the last section we apply the above theorems to processes defined by Fourier and Taylor random series and also to more general ones but with stronger covariance conditions. We obtain sufficient conditions for the central limit theorem for random series in terms of the moments of the summands just by using ways of translating such conditions into conditions on the modulus of continuity (Kahane [8]; Lemmas 3.4 and 3.9 in this paper). And for general processes we try the classical conditions for sample continuity in terms of the moments of the increments (Loève, [9] page 519) combined with a lemma of Garsia *et al.* [6] as suggested by A. de Araujo in [1]. These results are not necessarily best possible; the reasons for including them are that they may give some insight on the power of the theorems in the previous section and that they seem to be the first results in this direction. However we will not give detailed proofs of them.

2. The theorems. As customary, we will assume without further mention that all random variables and processes are defined on a probability space $(\Omega, \mathscr{F}, P)$, and the elements of Ω will be denoted by ω. We will denote by $\mathscr{L}(X)$ the law of any random variable X. If X is a random variable which values are real (or complex) functions on a metric space S, the notation for the two possible supremum norms will be

$$||X(\omega)||_\infty = \sup_{s \in S} |X(s, \omega)| , \qquad ||X(s)||_\infty = \sup_{\omega \in \Omega} |X(s, \omega)|$$

except if confusion may arise.

In the first theorem below we follow as closely as possible the proof of the theorem in [10], even in notation.

The following lemma is basic; a proof of it may be found in [2] (it contains some inaccuracies, but it is essentially correct: (6) there is valid whenever $cW < 1$, then, for proving the lemma, take $C = t/(\sigma + tW)$).

LEMMA 2.1 (*Bernstein's inequality*). *Let $x_1, \cdots, x_n$ be a set of independent, bounded real random variables with zero mean and set $S = \sum_{i=1}^{n} x_i$, $\sigma^2 = \sigma^2(S)$, and $M = \max_i \|x_i\|_\infty$; then for every $t > 0$,*

$$P\{S \geqq t\} \leqq \exp[-t^2/(2\sigma^2 + 2Mt/3)] .$$

THEOREM 2.2. *Let (S, d) be a compact metric space and let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent, centered, $C(S)$-valued random variables satisfying:*

(a) *there exists a centered Gaussian process Z on S such that the finite dimensional distributions of $(X_1 + \cdots + X_n)/n^{1/2}$ converge in law to the ones of Z (i.e., for every natural number k and $s_1, \cdots, s_k$ in S, the random vector $\sum_{i=1}^{n} (X_i(s_1), \cdots, X_i(s_k))/n^{1/2}$ converges in law to $(Z(s_1), \cdots, Z(s_k))$);*

(b) *for every $s, t \in S$, the number*

$$e(s, t) = \sup_k \|X_k(s) - X_k(t)\|_{L_2} + \sup_k \|X_k(s) - X_k(t)\|_\infty / k^{1/2}$$

is finite (thus defining a pseudo-distance on S) and moreover:

(i) *the identity map I: $(S, d) \to (S, e)$ is continuous*

(ii) $\int_0^1 H(s, e, u)\, du < \infty$ *(or what is the same, $\sum H(S, e, 2^{-m})/2^m < \infty$).*

Then, Z is sample continuous and

$$w^* - \lim_n \mathscr{L}\{(X_1 + \cdots + X_n)/n^{1/2}\} = \mathscr{L}(Z)$$

in $C'(C(S))$.

PROOF. By condition (a) and Prokhorov's theorem [3] we only need to prove uniform tightness of the sequence $\{\mathscr{L}(\sum_{i=1}^{n} X_i/n^{1/2})\}_{n=1}^{\infty}$. For simplicity, set

$$Z_n = (X_1 + \cdots + X_n)/n^{1/2} .$$

The main step in the proof of tightness of $\{\mathscr{L}(Z_n)\}_{n=1}^{\infty}$ is the following lemma.

LEMMA 2.3. *Under the conditions of the theorem, for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$P\{\sup [|Z_n(s) - Z_n(t)| : e(s, t) \leqq \delta] > \varepsilon\} \leqq \varepsilon \tag{2.1}$$

for every $n \in \mathbb{N}$.

PROOF OF THE LEMMA. By Lemma 2.1, if $e(s, t) \leqq \gamma$,

$$\sup_{n \geqq 1} P\{|Z_n(s) - Z_n(t)| > \varepsilon\} \leqq 2 \exp\{-\varepsilon^2/(2\gamma^2 + 2\gamma\varepsilon/3)\} \tag{2.2}$$

for every $\varepsilon > 0$.

For every $m \in \mathbb{N}$, let us cover S by $N(S, e, 2^{-m-4})$ sets each of diameter (with respect to e) at most 2^{-m-3}, and take one point from each forming a set A_m dense in S within 2^{-m-3}. Now we define

$$a_m = H(S, e, 2^{-m-4})/2^{m-1}\,,$$
$$b_m = \max(8a_m, m^{-2})$$

so that $\sum b_m < \infty$, and

$$\Omega_{mn} = \{\omega : \max[|Z_n(s) - Z_n(t)| : s, t \in A_{m-1} \cup A_m, e(s, t) \leqq 2^{-m}] > b_m\}\,.$$

The first thing to prove is that there exists $M > 0$ such that, for every $m > M$,

$$P_{mn} = P\Omega_{mn} \leqq b_m\,. \tag{2.3}$$

In fact, by (2.2),

$$P_{mn} \leqq 8N(S, e, 2^{-m-4})^2 \exp\{-b_m^2/(2 \cdot 4^{-m} + 2 \cdot 2^{-m}b_m/3)\}$$

and so, for $m \geqq 4$ (since $b_m \geqq m^{-2} \geqq 2^{-m}$)

$$P_{mn} \leqq 8N(S, e, 2^{-m-4})^2 \exp\{-3 \cdot 2^m b_m/8\}\,;$$

then

$$\log P_{mn} \leqq \log 8 + 2H(S, e, 2^{-m-4}) - 3 \cdot 2^{m-3}b_m$$

and, by the definition of b_m and a_m,

$$\log P_{mn} \leqq \log 8 + 2^{m-3}b_m - 3 \cdot 2^{m-3}b_m$$
$$= \log 8 - 2^{m-2}b_m\,.$$

Therefore, since $(2^{m-2}b_m - \log 8)/2 \log m$ tends to ∞ with m and both numerator and denominator are positive from some m on, there exists $M > 0$ such that for $m > M$,

$$\log P_{mn} \leqq -2 \log m$$

which implies

$$P_{mn} \leqq b_m$$

for $m > M$.

By condition (bii), $\sum b_m < \infty$; hence we can choose a number $r > M$ such that $\sum_{m \geqq r} b_m < \varepsilon/2$. Let us set $\delta = 2^{-r-3}$ and define $B_{nr} = \bigcup_{m \geqq r} \Omega_{mn}$. By (2.3), $PB_{nr} < \varepsilon$, hence we only need to prove that except on B_{nr}, if $e(s, t) \leqq \delta$ then $|Z_n(s) - Z_n(t)| \leqq \varepsilon$ for all n. In order to prove this, choose for each $s \in S$ and each m, a point s_m in A_m such that $e(s, s_m) \leqq 2^{-m-3}$; then, except on B_{nr},

$$|Z_n(s_m) - Z_n(s_{m-1})| \leqq b_m$$

for every $m \geqq r$, and so, since if $e(s, t) \leqq \delta$ then $e(s_r, t_r) \leqq 3\delta \leqq 2^{-r}$, we obtain that, except on B_{nr},

$$\begin{aligned}|Z_n(s) - Z_n(t)| &= |Z_n(s_r) + \textstyle\sum_{m>r} (Z_n(s_m) - Z_n(s_{m-1})) - Z_n(t_r) \\ &\qquad - \textstyle\sum_{m>r} (Z_n(t_m) - Z_n(t_{m-1}))| \\ &\leqq |Z_n(s_r) - Z_n(t_r)| + \textstyle\sum_{m>r} |Z_n(s_m) - Z_n(s_{m-1})| \\ &\qquad + \textstyle\sum_{m>r} |Z_n(t_m) - Z_n(t_{m-1})| \\ &\leqq b_r + 2 \textstyle\sum_{m>r} b_m < \varepsilon\end{aligned}$$

where the first equality above is justified by the compactness of S: the metrics e and d give rise to the same topology. □

If the limit law of $\{\mathscr{L}(Z_n(\omega, s))\}_{n=1}^{\infty}$, $\mathscr{L}(Z(s))$, is non-degenerate Gaussian, then the sequence of distribution functions converges uniformly and therefore, given s, for every $\varepsilon > 0$ there exists a positive number N such that

$$P\{|Z_n(s)| > N\} < \varepsilon \tag{2.4}$$

for all n. If $Z(s) = 0$ a.s. then $\{Z_n(\omega, s)\}_{n=1}^{\infty}$ converges in probability to zero and we still can find, for every $\varepsilon > 0$, a number N such that (2.4) is true for all n.

Now, by a standard argument (see, e.g., [3], page 55) it is clear that (2.1) and (2.4) yield the tightness of $\{\mathscr{L}(Z_n)\}_{n=1}^{\infty}$. □

The above theorem is mainly intended to be used for sequences of i.i.d. rv's with values in $C(S)$ and possibly of non-equicontinuous range through some kind of truncation; when the random variables are obtained in this way, condition (a) should not be difficult to verify: see Theorem 2.4 below. Otherwise, a good condition ensuring (a) may be, for example, the following: for every s, t in S the sequence $\{n^{-1}\sum_{i=1}^{n} EX_i(s)X_i(t)\}_{n=1}^{\infty}$ converges and for every choice of k points $s_1, \cdots, s_k \in S$, $k = 1, 2, \cdots$ the sequence of random vectors $\{(X_i(s_1), \cdots, X_i(s_k))\}_{i=1}^{\infty}$ satisfies the multidimensional form of the Lindeberg condition for the central limit theorem in $\mathbb{R}^k$; for example, the Lindeberg condition is automatically satisfied if for every $s \in S$ the sequence $\{X_k^2(s)\}_{k=1}^{\infty}$ is uniformly integrable.

Still following [10], given a sample continuous process X on S with $EX(s) = 0$, $EX^2(s) < \infty$, we will say that $\mathscr{L}(X)$ satisfies the central limit theorem if:

(a) the Gaussian process Z on S defined by $EZ(s) = 0$ and $EZ(s)Z(t) = EX(s)X(t)$ for every $s, t \in S$, is sample continuous (thus, defining a probability on $C(S)$), and

(b) whenever $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. rv's with values in $C(S)$ and such that $\mathscr{L}(X_1) = \mathscr{L}(X)$, then $w^* - \lim_n \mathscr{L}\{(X_1 + \cdots + X_n)/n^{\frac{1}{2}}\} = \mathscr{L}(Z)$ (in $C'(C(S))$). With this definition, we have:

THEOREM 2.4. *Let (S, d) be a compact metric space and let X be a process on S with $EX(s) = 0$, $EX^2(s) < \infty$ for every $s \in S$. Suppose that there exist a pseudo-distance e on S and a real nonnegative random variable M such that for every $s, t \in S$ and $\omega \in \Omega$,*

$$|X(\omega, s) - X(\omega, t)| \leqq e(s, t)M(\omega) \tag{2.5}$$

and verifying

(a) *the identity* I: $(S, d) \to (S, e)$ *is continuous and*

$$\int_0^1 H(S, e, u)\, du < \infty\,, \tag{2.6}$$

(b)

$$M \in L_2(\Omega, P)\,. \tag{2.7}$$

Then, $\mathscr{L}(X)$ satisfies the central limit theorem.

PROOF. (S, e) being compact and $X(\omega)$ continuous (by (2.5) and hypothesis a)), for every $\omega \in \Omega$ there exists $s_\omega \in S$ such that $|X(\omega, s_\omega)| = \|X(\omega)\|_\infty$ and therefore, by (2.5) and (2.7),

$$\|X(\omega)\|_\infty \leqq |X(\omega, t)| + [\text{diameter } (S, e)]M(\omega) \in L_2(\Omega, P)$$

$(X(t) \in L_2(\Omega, P))$; thus, taking $\max\{\|X(\omega)\|_\infty, M(\omega)\}$ for $M(\omega)$ in (2.5) if necessary, we may suppose that, in addition to (2.5) and (2.7), M also satisfies

$$\|X(\omega)\|_\infty \leqq M(\omega) . \tag{2.5'}$$

Let $\{X_k\}_{k=1}^\infty$ be a sequence of i.i.d. rv's with values in $C(S)$ and such that $\mathscr{L}(X_1) = \mathscr{L}(X)$. We may suppose that there exists a sequence of nonnegative independent random variables $\{M_k\}_{k=1}^\infty$ with $\mathscr{L}(M_k) = \mathscr{L}(M)$, $k = 1, 2, \cdots$, and such that

$$|X_k(\omega, s) - X_k(\omega, t)| \leqq e(s, t)M_k(\omega) , \qquad \|X_k(\omega)\|_\infty \leqq M_k(\omega) . \tag{2.8}$$

We define

$$\begin{aligned} Y_k(\omega) &= X_k(\omega) \quad \text{for} \quad M_k(\omega) \leqq k^{\frac{1}{2}} \\ &= 0 \quad \text{for} \quad M_k(\omega) > k^{\frac{1}{2}} , \end{aligned}$$

$$R_k(\omega) = X_k(\omega) - Y_k(\omega) \quad \text{and} \quad \mu_k = \int_\Omega \|R_k(\omega)\|_\infty \, dP(\omega) .$$

First we prove

$$\textstyle\sum_{k=1}^\infty \mu_k/k^{\frac{1}{2}} < \infty . \tag{2.9}$$

We have

$$\begin{aligned} \infty > \int M^2 \, dP &\geqq \textstyle\sum_{k=1}^\infty k^{\frac{1}{2}} \int_{k^{\frac{1}{2}} \leqq M < (k+1)^{\frac{1}{2}}} M \, dP \\ &= \int_{M \geqq 1} M \, dP + \lim_{N\to\infty} [\textstyle\sum_{k=2}^N (k^{\frac{1}{2}} - (k-1)^{\frac{1}{2}}) \int_{M \geqq k^{\frac{1}{2}}} M \, dP - N^{\frac{1}{2}} \int_{M \geqq (N+1)^{\frac{1}{2}}} M \, dP] \end{aligned}$$

and since

$$\lim_{N\to\infty} N^{\frac{1}{2}} \int_{M \geqq N^{\frac{1}{2}}} M \, dP \leqq \lim_{N\to\infty} \int_{M \geqq N^{\frac{1}{2}}} M^2 \, dP = 0$$

we have

$$\textstyle\sum_{k=1}^\infty (k^{\frac{1}{2}} - (k-1)^{\frac{1}{2}}) \int_{M \geqq k^{\frac{1}{2}}} M \, dP < \infty$$

and so

$$\textstyle\sum_{k=1}^\infty k^{-\frac{1}{2}} \int_{M \geqq k^{\frac{1}{2}}} M \, dP < \infty , \tag{2.10}$$

but by (2.8),

$$\mu_k = \int \|R_k(\omega)\|_\infty \, dP(\omega) = \int_{M_k \geqq k^{\frac{1}{2}}} \|X_k(\omega)\|_\infty \, dP(\omega) \leqq \int_{M_k \geqq k^{\frac{1}{2}}} M_k \, dP = \int_{M \geqq k^{\frac{1}{2}}} M \, dP ,$$

hence, (2.10) implies (2.9).

By (2.9)

$$\textstyle\sum_{k=1}^\infty \|R_k(\omega)\|_\infty/k^{\frac{1}{2}} < \infty \quad \text{a.s.} \tag{2.11}$$

Applying Kronecker's lemma to (2.9) and (2.11), we obtain

$$\lim_{n\to\infty} \textstyle\sum_{i=1}^n E\|R_i(\omega)\|_\infty/n^{\frac{1}{2}} = 0 \tag{2.12}$$

$$\lim_{n\to\infty} \textstyle\sum_{i=1}^n \|R_i(\omega)\|_\infty/n^{\frac{1}{2}} = 0 \quad \text{a.s.} \tag{2.13}$$

By (2.12) and (2.13)

$$\lim_{n\to\infty} \|\textstyle\sum_{i=1}^n X_i/n^{\frac{1}{2}} - \sum_{i=1}^n (Y_i - EY_i)/n^{\frac{1}{2}}\|_\infty = 0 \quad \text{a.s.} \tag{2.14}$$

(here $\|\cdot\|_\infty$ means supremum over all $s \in S$ for each fixed $\omega \in \Omega$). In fact,

$$\begin{aligned}\|\textstyle\sum_{i=1}^n X_i/n^{\frac{1}{2}} - \sum_{i=1}^n (Y_i - EY_i)/n^{\frac{1}{2}}\|_\infty &= \|\textstyle\sum_{i=1}^n R_i/n^{\frac{1}{2}} - \sum_{i=1}^n ER_i/n^{\frac{1}{2}}\|_\infty \\ &\leqq \textstyle\sum_{i=1}^n \|R_i(\omega)\|_\infty/n^{\frac{1}{2}} + \sum_{i=1}^n E\|R_i(\omega)\|_\infty/n^{\frac{1}{2}}\end{aligned}$$

and the last term converges to zero a.s. by (2.12) and (2.13).

The limit (2.14) proves that:

(a) $\{\mathscr{L}(\sum_{i=1}^n (Y_i - EY_i)/n^{\frac{1}{2}}\}_{n=1}^\infty$ and $\{\mathscr{L}(\sum_{i=1}^n X_i/n^{\frac{1}{2}})\}_{n=1}^\infty$ are w^*-convergence equivalent in $C'(C(S))$; and

(b) *a fortiori*, the finite dimensional distributions of both sequences are also w^*-convergence equivalent, and this implies that the finite dimensional distributions of the first sequence converge in law to the corresponding ones of the Gaussian process Z (defined by $EZ(s) = 0$ and $EZ(s)Z(t) = EX(s)X(t)$) because the same is true for the second sequence.

Hence, we need only prove the central limit theorem for the sequence $\{Y_k - EY_k\}_{k=1}^\infty$ which we already know satisfies hypothesis (a) of Theorem 2.2; hypothesis (b) of that theorem for this sequence is consequence of the following two estimates: if we set $\|M(\omega)\|_{L_2} = C$, by (2.5) and (2.7) we have

$$\begin{aligned}&\sup_k \|Y_k(s) - EY_k(s) - Y_k(t) + EY_k(t)\|_{L_2} \\ &\leqq 2\sup_k \|Y_k(s) - Y_k(t)\|_{L_2} \leqq 2\sup_k \|X_k(s) - X_k(t)\|_{L_2} \\ &\leqq 2e(s, t)\sup_k \|M_k(\omega)\|_{L_2} = 2Ce(s, t)\,,\end{aligned} \tag{2.15}$$

and moreover, by (2.5), (2.15) and the definition of Y_k,

$$\begin{aligned}&\sup_k \|Y_k(s) - EY_k(s) - Y_k(t) + EY_k(t)\|_\infty/k^{\frac{1}{2}} \\ &\leqq \sup_k |EY_k(s) - EY_k(t)|/k^{\frac{1}{2}} + \sup_k \|Y_k(s) - Y_k(t)\|_\infty/k^{\frac{1}{2}} \\ &\leqq Ce(s, t) + \sup_k \|(X_k(\omega, s) - X_k(\omega, t))\chi_{\{M_k(\omega)\leqq k^{\frac{1}{2}}\}}(\omega)\|_\infty/k^{\frac{1}{2}} \\ &\leqq (C + 1)e(s, t)\,.\end{aligned} \tag{2.16}$$

(2.15) and (2.16), together with the hypotheses of this theorem prove condition (b) of Theorem 2.2 for $\{Y_k - EY_k\}_{k=1}^\infty$ and therefore this sequence satisfies the central limit theorem and the proof is completed. □

The method for estimating the difference between $Y_k - EY_k$ and X_k in this theorem is inspired in the method used by Hartman and Wintner in [7] for handling a truncation problem.

One question that may be asked is whether the kind of hypotheses we have been considering for the central limit theorem are too strong. R. Dudley ([5] and [10]) has counterexamples showing that hypotheses only on $E|X(s) - X(t)|^2$ may not be adequate: there are sample continuous processes on [0, 1] with $E(X(s) - X(t))^2 \leqq |s - t|^{\frac{1}{2}-\varepsilon}$ for every $\varepsilon > 0$ for which the central limit theorem does not hold. On the other hand, as a corollary to the above theorems one can prove that the Kolmogorov conditions for sample continuity are also sufficient for the central limit theorem ([1], [5] and next section in this paper).

3. Examples. In this section we present some applications of the theorems in Section 2.

A. *Random Fourier series.* We consider first subgaussian series [8]. A real random variable ξ is called subnormal if

$$E\exp(\lambda\xi) \leqq \exp(\lambda^2/2)\,, \qquad -\infty < \lambda < \infty\,;$$

if ξ is subnormal then $E\xi = 0$, $E\xi^2 \leqq 1$ and more generally, for every $p > 0$ there exists a constant a_p independent of ξ such that $\|\xi\|_{L_p} \leqq a_p$ (e.g., Exercises 8 and 9, Chapter VI, [8], and use of symmetrization). Let T be the circle of length 1 and t arc length from some origin, then the process F defined on T as

$$F(\omega, t) = \textstyle\sum_{n=0}^{\infty} \xi_n(\omega) x_n \cos(2\pi nt + \varphi_n) \tag{3.1}$$

where x_n and φ_n, $n = 1, 2, \cdots$, are real numbers and $\{\xi_n\}_{n=0}^{\infty}$ is a sequence of independent subnormal random variables, is called a subgaussian series ([8] page 63).

For continuous functions $f\colon T \to \mathbb{R}$ the modulus of continuity W_f is defined as

$$W_f(h) = \sup\nolimits_{d(s,t)\leq h} |f(s) - f(t)|\,.$$

The following lemma, due to Kahane, gives properties of $W_{F(\omega)}$ in terms of properties of the moments of the summands in (3.1) and will allow us to apply Theorem 2.4 to $\mathscr{L}(F)$.

LEMMA 3.1. *Let $F(\omega)$ be as defined in* (3.1); *if*

$$\left(\textstyle\sum_{2^j\leq n<2^{j+1}} x_n{}^2\right)^{\frac12} \leqq Cj^{\gamma} \tag{3.2}$$

for some $\gamma < -1$ and constant C, then, for every $p > 0$ there exists a constant C_p such that

$$P\{\omega\colon W_{F(\omega)}(2^{-k}) < C_p k^{1+\gamma}\} \geqq 1 - 2\cdot 2^{-kp}\,, \tag{3.3}$$

$k = 1, 2, \cdots$; moreover $C_p = C'_{p,\gamma}\cdot C$ where $C'_{p,\gamma}$ does not depend on C, F or k, but only on p and γ.

PROOF. We can use the proof of Theorem 2, Chapter VII of [8] if it is modified just by using a slightly different version of Theorem 2, Chapter VI: if $P(t) = \sum \xi_n f_n(t)$ where $\{f_n\}$ is a sequence of real or complex trigonometric polynomials of degree less than or equal to N, $\{\xi_n\}$ is a sequence of independent subnormal random variables and $\sum$ is a finite sum, then for every $p > 0$ there exists a constant $\bar{C}_p$ such that

$$P\{\|P\|_\infty \geqq \bar{C}_p(\textstyle\sum \|f_n\|_\infty^2 \log N)^{\frac12}\} \leqq N^{-p}\,.$$

$\bar{C}_p$ may be taken to be $3(7 + p)^{\frac12}$. Using this fact instead of Theorem 2, Chapter VI, in the proof of Theorem 2, Chapter VII, and carrying the constants through all the computations, we obtain Lemma 3.4 with any constant $C'_{p,\gamma}$ such that

$$C'_{p,\gamma} \geqq (\log 2)^{-\gamma-\frac{1}{2}}(1 - 2^{1+\gamma})^{-1}\bar{C}_p . \qquad \square$$

(3.3), by application of the Borel–Cantelli lemma, proves sample continuity of F for $\gamma < -1$ (and gives an estimate of the modulus of continuity: Theorem 2, Chapter VII, [8]). To give an idea of how good a sufficient condition for sample continuity of F is (3.2) for $\gamma < -1$, we just note that there exist non-sample continuous subgaussian series for which $\sum_j (\sum_{2^j<n\leqq 2^{j+1}} x_n^2)^{\frac{1}{2}} < \infty$ ([8] page 65). On the central limit theorem for $\mathscr{L}(F)$ we can prove the following:

PROPOSITION 3.2. *Let F be a subgaussian series as defined in* (3.1) *and suppose that F satisfies* (3.2) *for some $\gamma < -\frac{3}{2}$. Then the central limit theorem holds for $\mathscr{L}(F)$ (in $C'(C(T))$).*

PROOF. By (3.2) and the properties of subnormal random variables, $EF(s) = 0$ and $EF^2(s) < \infty$ for every $s \in T$. Then, by the theorem in [5] we only need to prove

$$\sup_{h\in(0,\frac{1}{2}]} W_{F(\omega)}(h)|\log h|^{(p+2)/2p+\delta} \in L_p(\Omega, P)$$

for some $\delta > 0$ and for some $p \geqq 2$ (if $e(s, t) = |\log|s - t||^{-(p+2)/2p-\delta}$ then the ε-entropy condition for (T, e) in that theorem is satisfied). Using Lemma 3.1 we obtain that for every $q \geqq 1$ and $\alpha = -1 - \gamma$ (hence, $\alpha > (p + 2)/2p$ for some large p),

$$P\{\omega : \sup_{h\in(0,\frac{1}{2}]} W_{F(\omega)}(h)|\log h|^{\alpha} < 2^k \cdot 2^{\alpha+1} \cdot C_q\} \geqq 1 - 4 \cdot 2^{-kq} .$$

It is easy to see that if g is a nonnegative function satisfying $P\{\omega : g(\omega) \geqq 2^k\} \leqq c \cdot 2^{-kq}$ for some positive constant c and every $k \in N$, then $g \in L_r(\Omega, P)$ for every $r < q$; in fact, $\int_\Omega g^r\, dP \leqq 1 + 2^r + c(2^r - 1) \sum_{k=2}^\infty 2^{k(r-q)} < \infty$. Hence,

$$\sup_{h\in(0,\frac{1}{2}]} W_{F(\omega)}(h)|\log h|^{\alpha} \in L_r(\Omega, P)$$

for every positive r (in fact, $E[\sup_{h\in(0,\frac{1}{2}]} W_{F(\omega)}(h)|\log h|^{\alpha}] \leqq 2^{r(\alpha+1)}C_q{}^r C_{r,q,\alpha}$, where the constants only depend on their subindices). $\square$

If in the above proof we use Theorem 2.4 instead of the theorem in [5], then we obtain the proposition only for $\gamma < -2$. The fact that the ε-entropy condition in the theorem in [10] can not be improved makes it difficult to think that we can obtain stronger results than Proposition 3.2 for subgaussian series with our methods.

Kahane in [8] also considers series of the form

$$H(\omega) = \sum_{n=0}^\infty X_n(\omega) \cos(2\pi nt + \phi_n(\omega)) \tag{3.4}$$

where X_n, ϕ_n are real valued and $\{X_n \exp(i\phi)\}_{n=0}^\infty$ is a sequence of complex valued independent symmetric random variables. The distribution of H is equal to the distribution of the series $\sum_{n=0}^\infty \varepsilon_n(\omega_2)X_n(\omega_1)\cos(2\pi nt + \phi_n(\omega_1))$ where $\{\varepsilon_n\}_{n=0}^\infty$ is a sequence of Rademacher functions independent of $\{X_n \exp(i\phi_n)\}$ (so that the series may be supposed to be defined in a product space); this series has the property of being subgaussian for every fixed ω_1. In [8] this fact is called "reduction principle" and is used for reducing the study of the series (3.4) to the subgaussian case. Using this reduction principle and Proposition 3.2, we can prove:

PROPOSITION 3.3. *Let H be the process defined on T by* (3.4) *and let*

$$S_j = (\sum_{2^j \leqq n < 2^{j+1}} X_n^2)^{\frac{1}{2}} .$$

Suppose

$$ES_j^{2+\delta} < Cj^{\gamma}$$

for some $\delta > 0$, $C > 0$, and $\gamma < -5 - 2\delta$. Then, H is sample continuous and $\mathscr{L}(H)$ satisfies the central limit theorem.

Proposition 3.3 imposes conditions on the moments of order $2 + \delta$ of the terms of the series (3.4). Now we obtain a central limit theorem for general Fourier series just imposing conditions on the second moments of the terms (rather strong conditions). We will use the following lemma which is not too far from the Sobolev lemma for the circle.

LEMMA 3.4. *Let $f(t) = \sum_{n=-\infty}^{\infty} a_n \exp(2\pi int)$ be defined on the circle T and suppose*

$$\sum_{n\neq 0} |a_n||\log n|^{\alpha} < \infty , \qquad \alpha > 0 ;$$

then, for every $s, t \in T$,

$$|f(s) - f(t)| \leqq C|\log|s - t||^{-\alpha}$$

where

$$C = 2\pi(\alpha/\varepsilon)^{\alpha} \sum_{|n|\leqq[\exp\alpha]} |n||a_n| + 2\pi \sum_{|n|>[\exp\alpha]} |a_n|(\log|n|)^{\alpha} .$$

PROOF. We use without further mention the bounds $|\exp(2\pi inh) - 1| \leqq 2$ and $|\exp(2\pi inh) - 1| \leqq 2\pi|nh|$ and the fact that the function $y = x^{-\alpha}\log x$ is decreasing on $(\exp\alpha^{-1}, \infty)$ and its maximum, attained at $x = \exp\alpha^{-1}$ is $y = (\alpha e)^{-1}$. Then, defining $N = \max([\exp\alpha] + 1, 1/h)$, and taking $h \in (0, \frac{1}{2}]$, we have

$$\begin{aligned} &|f(t + h) - f(t)||\log h|^{\alpha} \\ &\qquad \leqq \sum_{n\neq 0} |a_n||\exp(2\pi inh) - 1||\log h|^{\alpha} \\ &\qquad \leqq 2\pi(\alpha/e)^{\alpha} \sum_{|n|<[\exp\alpha]} |n||a_n| + \sum_{n\geqq N} |a_n||\exp(2inh) - 1||\log h|^{\alpha} \\ &\qquad\qquad + \sum_{|n|>[\exp\alpha],|n|h<1} |a_n||\exp(2inh) - 1||\log h|^{\alpha} ; \end{aligned}$$

the second summand is bounded by $2\sum_{|n|\geqq N} |a_n|(\log|n|)^{\alpha}$ because in this case $(\log|n|)^{\alpha} \geqq |\log h|^{\alpha}$; as for the third summand, we have

$$\begin{aligned} &\sum_{|n|>[\exp\alpha],|n|h<1} |a_n||\exp(2\pi inh) - 1||\log h|^{\alpha} \\ &\qquad \leqq 2\pi \sum_{|n|>[\exp\alpha],|n|h<1} |a_n|((h^{-1})^{-1/\alpha}|\log h|)^{\alpha}(|n|^{-1/\alpha}\log|n|)^{-\alpha}(\log|n|)^{\alpha} \\ &\qquad \leqq 2\pi \sum_{|n|>[\exp\alpha],|n|h<1} |a_n|(\log|n|)^{\alpha} \end{aligned}$$

by the properties of the function $y = x^{-1/\alpha}\log x$ (since in this case $\exp\alpha \leqq |n| < h^{-1}$). □

PROPOSITION 3.5. *Let $F(\omega, t)$ be the process defined on T by*

$$F(\omega, t) = \sum_n Y_n(\omega) \exp(int)$$

where $\{Y_n\}_{n=-\infty}^{\infty}$ is a sequence of independent, centered, square integrable random variables satisfying

$$\sum_{n \neq 0} \|Y_n\|_{L_2} (\log |n|)^{1+\delta} < \infty$$

for some $\delta > 0$. Then F is sample continuous and the central limit theorem holds for $\mathscr{L}(F)$ (in $C'(C(T))$).

PROOF. This is a direct consequence of Lemma 3.4 and Theorem 2.4. □

By considering moments of order greater than 2 in the above proposition, we are only able to reduce somewhat the power of $\log |n|$. The condition in Proposition 3.5 is much stronger than Hunt's condition for sample continuity of Fourier series (see [8], notes to Chapter VII).

B. *Random Taylor series.* As a close look at Proposition 5 of Chapter V, Theorems 1 and 2 of Chapter VI and Theorem 2 of Chapter VII in [8] proves, Lemma 3.1 above depends only on Bernstein's inequality for trigonometric polynomials and on the properties of subgaussian series; the crucial fact is that, as a consequence of this inequality, if $p(t) = \sum_{n=1}^{N} b_n \cos(nt + \phi_n)$, then there exists, on the circle, an interval of length N^{-2} where $|p(t)| > \frac{1}{2}\|p\|_\infty$. Moreover N^{-2} may be substituted by αN^{-r}, (α, r, positive numbers) and the circle by some other set of finite measure, say, an interval of finite length. These considerations, together with the next lemma, will give us results for random Taylor series analogous to the ones obtained for Fourier series. We give the lemma and present the results without proof in order to avoid repetition.

LEMMA 3.6. *Let $p(t) = \sum_{n=1}^{N} a_n t^n$, $t \in [-1, 1]$; then there exists a subinterval of $[-1, 1]$ of length at least $2^{-4}N^{-2}$ where $|p(t)| > \frac{1}{2}\|p\|_\infty$.*

PROOF. The change of variables $t = \cos\theta$ gives a bijection between $[-1, 1]$ and $[0, \pi]$; $p(t)$ is transformed into $q(\theta) = \sum_{n=1}^{N} b_n \cos n\theta$ and as a consequence of Bernstein's inequality for trigonometric polynomials (see Proposition 5, Chapter V [8]), there exists an interval of length $2^{-1}N^{-1}$ where $|q(\theta)| > \frac{1}{2}\|q\|_\infty$. Since $1 - \cos(2N)^{-1} \geqq 2^{-4}N^{-2}$, we obtain the lemma. □

PROPOSITION 3.7. *For $t \in [0, 1]$, define the process $Y(\omega, t) = \sum_{n=0}^{\infty} \xi_n(\omega) x_n t^n$, where $\{\xi_n\}_{n=0}^{\infty}$ is a sequence of independent subnormal random variables. Then, if there exist constants $C > 0$ and $\gamma < -\frac{3}{2}$ such that $(\sum_{2^j \leq n < 2^{j+1}} x_n^2)^{\frac{1}{2}} \leqq Cj^{\gamma}$, Y is sample continuous and the central limit theorem holds for $\mathscr{L}(Y)$ (in $C'(C[-1, 1])$).*

PROPOSITION 3.8. *If $Y(\omega, t) = \sum_{n=0}^{\infty} X_n(\omega) t^n$ where $\{X_n\}_{n=0}^{\infty}$ is a sequence of independent symmetric random variables such that if S_j is defined by $S_j = (\sum_{2^j \leq n < 2^{j+1}} X_n^2)^{\frac{1}{2}}$ then there exist $\delta > 0$, $\gamma < -5 - 2\delta$ and $C > 0$ such that $ES_j^{2+\delta} \leqq Cj^{\gamma}$, then Y is sample continuous and the central limit theorem holds for $\mathscr{L}(Y)$ (in $C'(C[-1, 1])$).*

Lemma 3.4 and hence Proposition 3.5 have also analogues for random Taylor series. For $\delta > 0$ define the following distance on $[-1, 1]$: set $A_1 = [-1, -\frac{1}{2}]$, $A_2 = [-\frac{1}{2}, \frac{1}{2}]$, and $A_3 = [\frac{1}{2}, 1]$, then, $e_\delta(s, t) = |t - s|$ on A_2, $e_\delta(s, t) = |\log|t - s||^{-1-\delta}$ on A_1 and on A_3 (not on $A_1 \cup A_3$), for $s \in A_1$ and $t \in A_2$ $e_\delta(s, t) = e_\delta(s, -\frac{1}{2}) + e_\delta(-\frac{1}{2}, t)$, for $s \in A_2$ and $t \in A_3$ $e_\delta(s, t) = e_\delta(s, \frac{1}{2}) + e_\delta(\frac{1}{2}, t)$ and for $s \in A_1$ and $t \in A_3$ $e_\delta(s, t) = 1 + e_\delta(s, -\frac{1}{2}) + e_\delta(\frac{1}{2}, t)$. Then:

LEMMA 3.9. *Let $f(t) = \sum_{n=0}^\infty a_n t^n$ and suppose*

$$\sum_{n=2}^\infty |a_n|(\log n)^{1+\delta} < \infty \qquad (\delta > 0)\,.$$

Then, for every $s, t \in [-1, 1]$,

$$|f(s) - f(t)| \leqq Ce_\delta(s, t)$$

where C may be taken to be

$$C = 2((1 + \delta)/e)^{1+\delta} \sum_{n=1}^{[e^{1+\delta}]} n|a_n| + 4 \sum_{n=[e^{1+\delta}]+1}^\infty |a_n|(\log n)^{1+\delta} + \sum_{n=1}^\infty n2^{-n+1}|a_n|\,.$$

PROOF. Similar to the proof of Lemma 3.4, using the bounds $|t^n - s^n| \leqq 2$ and $|t^n - s^n| \leqq n|t - s|$ for $s, t \in A_3$, and $|t^n - s^n| \leqq n2^{-n+1}|t - s|$ for $s, t \in A_2$. □

PROPOSITION 3.10. *Let Y be the process on $[-1, 1]$ defined by $Y(\omega, t) = \sum_{n=0}^\infty Y_n(\omega)t^n$ where $\{Y_n\}_{n=0}^\infty$ is a sequence of centered, square integrable, independent random variables. Then, if for some $\delta > 0$, $\sum_{n\neq 0} \|Y_n\|_{L_2}(\log n)^{1+\delta} < \infty$, the process Y is sample continuous and $\mathscr{L}(Y)$ satisfies the central limit theorem in $C'(C[-1, 1])$.*

Again, the condition in this proposition is stronger than what is needed for just sample continuity: using Kolmogorov's inequality and Abel's lemma on convergence of series, we may easily prove that $\sum \|Y_n\|_{L_2}^2 < \infty$ is sufficient for sample continuity of the process Y.

C. *Processes on* $[0, 1]$. The following proposition has been suggested by an example in [1].

PROPOSITION 3.11. *Let $X(\omega, t)$ be a process on $[0, 1]$ with $EX(t) = 0$ and $EX^2 < \infty$ for every $t \in [0, 1]$ and such that for some $p \geqq 2$, $\|X(s) - X(t)\|_{L_p} \leqq K|s - t|^{1/p}/|\log|s - t||^{(p+1)/p+\tau}$ with $K > 0$, $\tau > (p + 2)/2p$; then the process is sample continuous and the central limit theorem holds for $\mathscr{L}(X) \in C'(C[0, 1])$.*

PROOF. If only $\tau > 0$, then $X(t)$ is sample continuous by virtue of a classical result (Loève, [9] page 519). In Lemma 1.1 of Garsia, Rodemich and Rumsey, [6] page 566, take $\psi(u) = |u|^p$ and $P(u) = |u|^{2/p}/|\log|s - t||^{1+\tau-\rho}$ where $0 < \rho < \min(p^{-1}, \tau - (p + 2)/2p)$. Then, assuming $K = 1$,

$$\int_0^1 \int_0^1 E|(X(s) - X(t))/P(|s - t|)|^p\, ds\, dt \leqq \int_0^1 \int_0^1 |s - t|^{-1}|\log|s - t||^{-1-\rho p}\, ds\, dt < \infty\,.$$

Hence the random variable $B = \int_0^1 \int_0^1 |(X(s) - X(t))/P(|s - t|)|^p\, ds\, dt$ is in $L_1(\Omega, P)$ and in particular, is a.s. finite. By Lemma 1.1 in [6] we then have that, almost surely,

$$|X(s) - X(t)| \leqq 8 \cdot 4^{1/p} \cdot B^{1/p} \int_0^{|s-t|} u^{-2/p}\, d(u^{2/p}/|\log u|^{1+\tau-\rho})$$
$$\leqq C \cdot B^{1/p} |\log|s-t||^{\rho-\tau}$$

for some positive C and s, t in any interval of length less than, say, $\frac{1}{2}$. Then, if $p = 2$, we apply Theorem 2.4, and if $p > 2$, the theorem in [5].

The example in Proposition 7.1 of [4], for the case $k = 1$, satisfies $E|x_s - x_t|^2 \leqq |s - t|$ as it is easily verified, and x_s has all its versions with almost all their sample functions unbounded; hence, at least for $p = 2$, the best possible power of $|s - t|$ for sample continuity gives also the central limit theorem for the process X.

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