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THE MULTIPLICITY OF AN INCREASING FAMILY OF σ-FIELDS[1]

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Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathscr{F}_t$, $t \in R_+$, be an increasing family of sub-σ-fields of $\mathscr{F}$ such that $\mathscr{F}_0$ is trivial and $\mathscr{F} = V_t \mathscr{F}_t$. Let $\mathscr{M}^2$ be the family of all square-integrable martingales m_t with $m_0 = 0$. Suppose that $L^2(\Omega, \mathscr{F}, P)$ is separable. Then there exists a finite or countable sequence in $\mathscr{M}^2$, $m_t^1, m_t^2, \cdots$, such that (i) the stable subspaces generated by m_t^i, m_t^j are orthogonal for $i \neq j$; (ii) $\langle m^1\rangle \succ \langle m^2\rangle \succ \cdots$ where $\langle m^i\rangle$ is the nonnegative measure on the predictable σ-field on $\Omega \times R_+$ induced by the quadratic variation process $\langle m^i\rangle$ of m^i, and (iii) every m in $\mathscr{M}^2$ has a representation $m_t = \sum_i \int_0^t \phi_i(s)\, dm_s^i$ a.s. for some predictable integrands ϕ_i. Furthermore, if $n_t^1, n_t^2, \cdots$ is another such sequence, then $\langle n^i\rangle \sim \langle m^i\rangle$ for all i.

1. Introduction. This paper extends the results of Cramér (1964), (1967) and Motoo and Watanabe (1965) to the case of all processes defined on a separable probability space. The main results are in Section 3.

2. Preliminaries. Throughout $(\Omega, \mathscr{F}, P)$ is a fixed probability space and $\mathscr{F}_t$, $t \in R_+$, is a fixed, increasing family of sub-σ-fields of $\mathscr{F}$ with $\mathscr{F}_0$ trivial and $\mathscr{F} = V_t \mathscr{F}_t$. $\mathscr{M}^2$ denotes the family of all martingales m_t (with respect to $(\Omega, \mathscr{F}_t, P)$, $t \in R_+$) such that $m_0 = 0$ and $\sup_{t \in R_+} Em_t^2 < \infty$. $\mathscr{M}^2$ is a Hilbert space under the inner product $(m, n) = Em_\infty n_\infty$ (see [4] Theorem 1). The *predictable σ-field* $\mathscr{P}$ is the σ-field of subsets of $\Omega \times R_+$ generated by all the adapted process y_t on $(\Omega, \mathscr{F}_t, P)$ which have left-continuous sample paths. A process x_t on $(\Omega, \mathscr{F}_t, P)$ is said to be *predictable* if the function $(\omega, t) \to x_t(\omega)$ is $\mathscr{P}$-measurable. The set of all predictable processes is also denoted by $\mathscr{P}$.

To each $m \in \mathscr{M}^2$ is associated a unique predictable process denoted $\langle m\rangle$ with non-decreasing sample paths and with $\langle m\rangle_0 = 0$, such that $m_t^2 - \langle m\rangle_t$ is a martingale. If m, n are in $\mathscr{M}^2$, then $m_t n_t - \langle m, n\rangle_t$ is a martingale where $2\langle m, n\rangle \equiv \langle m + n\rangle - \langle m\rangle - \langle n\rangle$. Let $L^p(\langle m, n\rangle) = \{\phi \in \mathscr{P} \mid E \int_0^\infty |\phi_s|^p |d\langle m, n\rangle_s| < \infty\}$, $p = 1, 2$. For each $\phi \in L^2(\langle m\rangle)$ there is a unique martingale $\phi \circ m \in \mathscr{M}^2$ such that

$$\langle \phi \circ m, n\rangle_t = \int_0^t \phi_s\, d\langle m, n\rangle_s\,, \qquad \text{for all } n \in \mathscr{M}^2\,.$$

A closed subspace $\mathscr{L}$ of the Hilbert space $\mathscr{M}^2$ is *stable* if $m \in \mathscr{L}$, $\phi \in L^2(\langle m\rangle)$ implies $\phi \circ m \in \mathscr{L}$. If $\mathscr{N} \subset \mathscr{M}^2$ then $\mathscr{L}(\mathscr{N})$ is the smallest stable subspace

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containing $\mathscr{N}$. $\mathscr{L}(m) = \{\phi \circ m \mid \phi \in L^2(\langle m\rangle)\}$. If $\mathscr{L}$ is a stable subspace $\mathscr{L}^\perp = \{m \in \mathscr{M}^2 \mid (m, n) = Em_\infty n_\infty = 0$ for all $n \in \mathscr{L}\}$. If $\mathscr{L}$ is stable so is $\mathscr{L}^\perp$ and each $m \in \mathscr{M}^2$ can be uniquely decomposed as $m = n + n'$ such that $n \in \mathscr{L}$ and $\langle n', y\rangle \equiv 0$ for all $y \in \mathscr{L}$. If $\mathscr{L} = \mathscr{L}(y)$ then in the decomposition above $n = (d\langle m, y\rangle / d\langle y\rangle) \circ y$.

3. The main result. For m, n in $\mathscr{M}^2$, $\langle m\rangle \succ \langle n\rangle$ ($\langle m\rangle \sim \langle n\rangle$, $\langle m\rangle \perp \langle n\rangle$) means that the measure $\langle n\rangle$ on $\mathscr{P}$ is absolutely continuous (mutually absolutely continuous, singular) with respect to $\langle m\rangle$. It may be worthwhile noting that by measure $\langle n\rangle$ on $\mathscr{P}$ we mean the measure

$$\langle n\rangle(\Lambda) = E\{\textstyle\int_0^\infty I_\Lambda(\omega, t)\, d\langle n\rangle_t(\omega)\}\,, \qquad \Lambda \in \mathscr{P}.$$

The next result is elementary and is included in ([5] Theorem 4.1).

PROPOSITION 1. *Let $\mathscr{L} = \mathscr{L}(m^1, \cdots, m^K)$, $K \leqq \infty$[2]. Then there exists a sequence $n^1, \cdots, n^K$ in $\mathscr{L}$ such that $n^1 = m^1$ and*

(i) $\mathscr{L}(n^1, \cdots, n^K) = \mathscr{L}$,
(ii) $\mathscr{L}(n^i) \perp \mathscr{L}(n^j)$, $i \neq j$.

THEOREM 1. *Let $\mathscr{L} = \mathscr{L}(m^1, \cdots, m^K)$, $K \leqq \infty$. Then there exists a sequence $n^1, \cdots, n^R$, in $\mathscr{L}$ with $R \leqq K$ and $n^i \not\equiv 0$ for all i such that*

(i) $\mathscr{L}(n^1, \cdots, n^R) = \mathscr{L}$,
(ii) $\mathscr{L}(n^i) \perp \mathscr{L}(n^j)$, $i \neq j$
(iii) $\langle n^1\rangle \succ \langle n^2\rangle \succ \langle n^3\rangle \succ \cdots$.

PROOF. By Proposition 1 it can be assumed that $\mathscr{L}(m^i) \perp \mathscr{L}(m^j)$, $i \neq j$. Applying the Lebesgue decomposition theorem to the measures $\langle m^i\rangle$ for $i > 1$, there exist measures $\mu_1{}^i$, $\mu_2{}^i$ on $\mathscr{P}$ such that

$$\langle m_i\rangle = \mu_1{}^i + \mu_2{}^i$$
$$\mu_1{}^i \prec \textstyle\sum_{j=1}^{i-1} \langle m^j\rangle$$
$$\mu_2{}^i \perp \textstyle\sum_{j=1}^{i-1} \langle m^j\rangle\,,$$

and sets $A_i \in \mathscr{P}$ such that

$$\mu_2{}^i(A_i{}^c) = 0\,, \qquad \langle m^j\rangle(A_i) = 0 \qquad j = 1, \cdots, i-1\,.$$

Now set $A_1 = \Omega \times R_+$ and define

$$n^1 = \sum_{i=1}^K \frac{1}{2^i E\langle m^i\rangle_\infty} I_{A_i} \circ m^i\,. \tag{1}$$

Clearly $n^1 \in \mathscr{L}$. It will be shown that it has the following properties:

(a) $m^1 \in \mathscr{L}(n^1)$
(b) $\langle n^1\rangle \succ \langle m^i\rangle$ for all i.

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[2] $K = \infty$ simply means that $m^1, m^2, \cdots$ is a countable sequence.

To prove (a) let $B = (\bigcup_{i=2}^{K} A_i)^c$. Then

$$2E\langle m_\infty{}^1\rangle I_B \circ n^1 = \sum_{i=1}^{K} \frac{2E\langle m^1\rangle_\infty}{2^i E\langle m^i\rangle_\infty} I_{B\cap A_i} \circ m^i = I_B \circ m^1 .$$

Now $\langle m^1\rangle(A_i) = 0$ for $i \geqq 2$ so $\langle m^1\rangle(B^c) = 0$. Thus $I_{B^c} \circ m^1 = 0$ and hence

$$m_1 = 2E\langle m_\infty{}^1\rangle I_B \circ n^1 \in \mathscr{L}(n^1) .$$

To prove (b) take $E \in \mathscr{P}$ such that $\langle n^1\rangle(E) = 0$. Since the m^j are orthogonal, this implies, from (1), that

$$\langle m^i\rangle(E \cap A_i) = 0 \qquad i = 1, 2, \cdots . \tag{2}$$

For $i = 1$ this says $\langle m^1\rangle(E) = 0$ so that $\langle n^1\rangle \succ \langle m^1\rangle$ and the result is established by induction, if

$$\langle m^j\rangle(E) = 0 \qquad \text{for } j = 1, \cdots, i-1 \tag{3}$$

implies $\langle m^i\rangle(E) = 0$. Suppose (3) is true. Now

$$\begin{aligned} \langle m^i\rangle(E) &= \langle m^i\rangle(E \cap A_i) + \langle m^i\rangle(E \cap A_i{}^c) \\ &= \langle m^i\rangle(E \cap A_i{}^c) && \text{from (2)} \\ &= \mu_1{}^i(E \cap A_i{}^c) \qquad \text{since} \quad \mu_2{}^i(A_i{}^c) = 0 . \end{aligned}$$

But $\mu_1{}^i \prec \sum_{j=1}^{i-1} \langle m^j\rangle$ so that, using (3), $\langle m^i\rangle(E) = 0$ as required.

Thus n^1 satisfies (a) and (b) above. Now let q^i, $i \geqq 2$, be the projection of m^i on $(\mathscr{L}(n^1))^\perp$ i.e.,

$$q^i = m^i - \frac{d\langle m^i, n^1\rangle}{d\langle n^1\rangle} \circ n^1 , \quad = m^i - \alpha_i \circ n^1 \quad \text{say.} \tag{4}$$

Apply Proposition 1 to $\mathscr{L}(q^2, q^3, \cdots)$ to obtain a sequence $p^2, p^3, \cdots$ with $p^2 = q^2$ such that $\mathscr{L}(q^2, q^3, \cdots) = \mathscr{L}(p^2, p^3, \cdots)$ and $\mathscr{L}(p^i) \perp \mathscr{L}(p^j)$, $i \neq j$. Then, using (4), it follows that

$$\mathscr{L} = \mathscr{L}(n^1, m^2, m^3, \cdots) = \mathscr{L}(n^1, p^2, p^3, \cdots) .$$

To obtain n^2, start with the sequence $p^2, p^3, \cdots$ and construct n^2 in the same way that n^1 was constructed from $m^1, m^2, \cdots$. Then n^2 will have the properties

(a′) $p^2 \in \mathscr{L}(n^2)$
(b′) $\langle n^2\rangle \succ \langle p^i\rangle$ for all $i \geqq 2$.

Now $m^2 \in \mathscr{L}(n^1, n^2)$ since $m^2 - p^2 = m^2 - q^2 \in \mathscr{L}(n^1)$ by (4) and $p^2 \in \mathscr{L}(n^2)$ by (a′). Also from (4)

$$\langle q^i\rangle = \langle m^i\rangle + \langle \alpha_i \circ n^1\rangle - 2\langle m^i, \alpha_i \circ n^1\rangle .$$

From (b) above, $\langle n^1\rangle \succ \langle m^i\rangle$ for all i so that this equation implies $\langle n^1\rangle \succ \langle q^i\rangle$ and consequently $\langle n^1\rangle \succ \langle n^2\rangle$.

Having constructed n^2 one projects $p^3, p^4, \cdots$ on $(\mathscr{L}(n^1, n^2))^\perp$ and then n^3 is constructed. This procedure continues for $n^4, n^5, \cdots$ and it is clear that the sequence thus obtained satisfies the assertion. □

THEOREM 2. *Let* $\mathscr{L} = \mathscr{L}(m^1, \cdots, m^K) = \mathscr{L}(n^1, \cdots, n^R)$ *for some* $K, R \leqq \infty$ *and suppose that*

(i) $\mathscr{L}(m^i) \perp \mathscr{L}(m^j)$; $\mathscr{L}(n^i) \perp \mathscr{L}(n^j)$, $i \neq j$

(ii) $\langle m^1 \rangle \succ \langle m^2 \rangle \succ \cdots$; $\langle n^1 \rangle \succ \langle n^2 \rangle \succ \cdots$.

Then $\langle m^i \rangle \sim \langle n^i \rangle$ *for all* i, *in particular* $K = R$.

PROOF. Because of (ii) there exist predictable processes $\phi_i \in L^2(\langle n^1 \rangle)$,

$$\phi_i \equiv \frac{d\langle n^i \rangle}{d\langle n^1 \rangle},$$

such that for $A \in \mathscr{P}$

$$\langle n^i \rangle(A) = E \textstyle\int_0^\infty I_A \phi_i \, d\langle n^1 \rangle \qquad i = 1, 2, \cdots. \tag{5}$$

Also for each i $m^i \in \mathscr{L}(n^1, \cdots, n^R)$ so that there exist predictable processes $f_{ij} \in L^2(\langle n^j \rangle)$ such that

$$m^i \equiv \textstyle\sum_j f_{ij} \circ n^j .$$

Because of (i) it follows from this representation that

$$\begin{aligned} \langle m^i, m^j \rangle_t &= \textstyle\sum_k \int_0^t f_{ik} f_{jk} \, d\langle n^k \rangle \\ &= \textstyle\sum_k \int_0^t f_{ik} f_{jk} \phi_k \, d\langle n^1 \rangle \qquad \text{from (5).} \end{aligned}$$

In particular, putting $i = j$ gives

$$\frac{d\langle m^i \rangle}{d\langle n^1 \rangle} = \textstyle\sum_k f_{ik}^2 \phi_k , \tag{6}$$

and since $\mathscr{L}(m^i) \perp \mathscr{L}(m^j)$ for $i \neq j$

$$\textstyle\sum_k f_{ik} f_{jk} \phi_k = 0 \quad \text{a.s.} \quad \langle n^1 \rangle \qquad \text{for } i \neq j . \tag{7}$$

It is immediate from (6) that $\langle m^1 \rangle \prec \langle n^1 \rangle$ and by symmetry $\langle n^1 \rangle \prec \langle m^1 \rangle$, thus $\langle m^1 \rangle \sim \langle n^1 \rangle$.

Now assume that $\langle m^i \rangle \sim \langle n^i \rangle$ for $i = 1, \cdots, r$. It will be shown that $\langle m^{r+1} \rangle \sim \langle n^{r+1} \rangle$. By the Lebesgue decomposition theorem there are measures μ^1, μ^2 on $\mathscr{P}$ with $\langle m^{r+1} \rangle = \mu^1 + \mu^2$, such that

$$\mu^1 \prec \langle n^{r+1} \rangle , \qquad \mu^2 \perp \langle n^{r+1} \rangle$$

i.e., there exists $B \in \mathscr{P}$ such that $\mu^2(B^c) = 0$, $\langle n^{r+1} \rangle(B) = 0$. Suppose $\mu^2(B) > 0$. Then $\langle m^{r+1} \rangle(B) > 0$ and there exists $B_0 \subseteq B$, $B_0 \in \mathscr{P}$ such that

$$\textstyle\int_0^\infty I_{B_0} \, d\langle n^{r+1} \rangle = 0 \quad \text{a.s.} \tag{8}$$

$$E \textstyle\int_0^\infty I_{B_0} \, d\langle m^{r+1} \rangle > 0 \qquad \text{and} \qquad \dfrac{d\langle m^{r+1} \rangle}{d\langle n^1 \rangle} > 0 \quad \text{a.s.} \quad \langle n^1 \rangle \quad \text{on} \quad B_0 . \tag{9}$$

Now $\langle n^1 \rangle \succ \langle n^{r+1} \rangle \succ \langle n^k \rangle$ for $k \geqq r + 1$, so, using (5), (8) implies

$$I_{B_0} \phi_k = 0 \qquad \text{a.s.} \quad \langle n^1 \rangle \qquad \text{for } k \geqq r + 1 . \tag{10}$$

Also $\langle n^1\rangle \sim \langle m^1\rangle \succ \langle m^k\rangle \succ \langle m^{r+1}\rangle$ for $1 \leqq k \leqq r+1$ so that from (9)

$$\frac{d\langle m^k\rangle}{d\langle n^1\rangle} > 0 \quad \text{a.s. } \langle n^1\rangle \text{ on } B_0 \quad \text{for } 1 \leqq k \leqq r+1\,. \tag{11}$$

Combining (7) with (10) it follows that

$$\textstyle\sum_{k=1}^{r} f_{ij} f_{jk} \phi_k = 0 \quad \text{a.s. } \langle n^1\rangle \text{ on } B_0 \qquad \text{for } i \neq j \tag{12}$$

whereas from (6), (10) and (11)

$$\textstyle\sum_{k=1}^{r} f_{ik}^2 \phi_k > 0 \quad \text{a.s. } \langle n^1\rangle \text{ on } B_0 \quad \text{for } 1 \leqq i \leqq r+1\,. \tag{13}$$

Now fix $(\omega, t) \in B_0$ such that (12), (13) hold, and let $x^i \in R^r$ be the vector with jth component equal to $(\phi_j(\omega, t))^{\frac{1}{2}} f_{ij}(\omega, t)$. Then (12) implies that x^i, x^j are orthogonal in R^r for $1 \leqq i \neq j \leqq r+1$ whereas (13) implies that these vectors are nonzero which is a contradiction so it must be the case that $\langle m^{r+1}\rangle \prec \langle n^{r+1}\rangle$ and by symmetry $\langle m^{r+1}\rangle \succ \langle n^{r+1}\rangle$. The result follows by induction. □

DEFINITION. *The multiplicity* of the family $(\Omega, \mathscr{F}_t, P)_{t\in R_+}$ is the length K of (any) sequence $m^1, m^2, \cdots, m^K$ of nonzero martingales which satisfied (i), (ii), (iii) of Theorem 2 for $\mathscr{L} = \mathscr{M}^2$. Denote the multiplicity by $M(\mathscr{M}^2)$.

From Theorems 2 and 3 follows immediately the following result.

COROLLARY. *If* $\mathscr{L}(n^1, \cdots, n^R) = \mathscr{M}^2$ *then* $R \geqq M(\mathscr{M}^2)$.

REMARK 1. Kunita and Watanabe ([6] page 227) have shown that every martingale which is measurable with respect to the family of σ-fields generated by a vector Brownian motion can be represented as a stochastic integral with respect to the Brownian motion. This result implies that if $(\Omega, \mathscr{F}_t, P)_{t\in R_+}$ is generated by a Gaussian process then the definition of multiplicity given here coincides with the one due to Cramér.

REMARK 2. The definition of multiplicity and the main results similarly extend the corresponding ones given in Motoo and Watanabe (1965) for the case of martingales of a Hunt process.

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