The Annals of Probability
1974, Vol. 2, No. 6, 1179–1184

ON SAWYER'S RATES OF CONVERGENCE FOR SOME FUNCTIONALS IN PROBABILITY

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An improvement in the rate of convergence for some functionals in probability given by Sawyer when a moment of specified order exists is obtained.

1. Introduction. Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables with $E(X_k) = 0$ and $E|X_k|^p < \infty$ for some $p \geqq 4$. Set $S_k = X_1 + X_2 + \cdots + X_k$ and let $(W(t), 0 \leqq t < \infty)$ denote a standard Wiener process. Then the following result is obtained in this paper.

THEOREM. *Let $f(t, x) \in C^1(R^2)$ be such that*

$$|Df(t, x)| \leqq \Omega(1 + |x|^a),$$

where D denotes either the identity operator or a first partial derivative, and the distribution function $F(\lambda) = P(\int_0^1 f(t, W(t))\,dt \leqq \lambda)$ satisfies a first order Lipshitz condition. Then if $F_n(\lambda) = P(n^{-1} \sum_1^n f(k/n, S_k n^{-\frac{1}{2}}) \leqq \lambda)$ one has that

$$\sup |F_n(\lambda) - F(\lambda)| = O(n^{-\gamma}\{\log (n)\}^{a/2}) \tag{1.1}$$

for all $\gamma < \frac{1}{2}$ if $p = 6$ and $\gamma = (p-2)/(p+2)$ if $4 \leqq p < 6$.

Under the same conditions Sawyer [5] obtained a bound of order $n^{-\gamma'}\{\log (n)\}^{ap/8}$ where $\gamma' = p/(2p+8)$. Since $p \geqq 4$, the result given here represents an improvement in Sawyer's bound both in terms of the exponent of n and $\log (n)$. For example if $E(X_k^4) < \infty$ and $f(t, x) = x^2$ then the bound here is of order $n^{-\frac{1}{3}} \log (n)$ compared with Sawyer's $n^{-\frac{1}{4}} \log (n)$. If $E(X_k^6) < \infty$ the comparison is $n^{-\gamma} \log (n)$ for all $\gamma < \frac{1}{2}$ against $n^{-6/20}\{\log (n)\}^{\frac{3}{2}}$. The improvement results from sharpening the estimates

$$P(|\Phi_i| > \delta), \qquad i = 1, 2, \cdots, 6 \tag{1.2}$$

of (2.8) in [5], where $\delta = n^{-\gamma}\{\log (n)^{2/a}\}$. The tool for this is provided by the lemma of Section 2. The proof of the theorem is completed in Section 3.

2. Lemma. *Let $Y_1, Y_2, \cdots, Y_n$ be independent and identically distributed random variables with $E(Y_1) = 0$ and $E|Y_1|^r < \infty$ for some $2 \leqq r < \infty$ such that Y_k is $\mathscr{F}_k$ measurable and $E(Y_{k+1} | \mathscr{F}_k) = 0$ where $\{\mathscr{F}_k\}$ is a sequence of increasing σ-fields. Also let $d_1, d_2, \cdots, d_n$ be random variables with d_k measurable with respect to $\mathscr{F}_{k-1}$ and* $\max(|d_k(w)|, k = 1, 2, \cdots, n) = O(\delta_n)$ *uniformly in w for a given sequence of*

Received September 13, 1973; revised January 25, 1974.
AMS 1970 *subject classifications.* Primary F15, G50, G99, E20.
Key words and phrases. Rate of convergence, functionals in probability.

real constants δ_n. *Then for any* $\varepsilon > 0$

$$P(|\sum d_i Y_i| > \varepsilon\delta_n n^{s/r}) = o(n^{-s+1}), \qquad \tfrac{1}{2} < s/r \leq 1. \tag{2.1}$$

PROOF. Let

$$\begin{aligned} Y_i' &= Y_i, && \text{if } |Y_i| \leq n^{s/r} \\ &= 0, && \text{if } |Y_i| > n^{s/r} \end{aligned}$$

and $X_i = Y_i' - E(Y_i')$ for $1 \leq i \leq n$. Then

$$\begin{aligned} &P(|\sum d_i Y_i| > 2\varepsilon\delta_n n^{s/r}) \\ &\quad\leq P(|\sum d_i Y_i'| > 2\varepsilon\delta_n n^{s/r}) + nP(|Y_1| > n^{s/r}) \\ &\quad\leq P(|\sum d_i X_i| > 2\varepsilon\delta_n n^{s/r} - \delta_n n|E(Y_i')| + nP(|Y_1| > n^{s/r}). \end{aligned} \tag{2.2}$$

Since $E(Y_1) = 0$,

$$|E(Y_1')| = |E(Y_1 I_{[|Y_1| > n^{s/r}]})| \leq n^{-s(r-1)/r} E(|Y_1|^r)$$

and so

$$n|E(Y_1')| \leq n^{-s+1} n^{s/r} = o(n^{s/r}), \qquad \text{since } s > 1.$$

Hence

$$\begin{aligned} &n^{s-1}P(|\sum_1^n d_i Y_i| > 2\varepsilon\delta_n n^{s/r}) \\ &\quad\leq n^{s-1}P(|\sum_1^n d_i X_i| > \varepsilon\delta_n n^{s/r}) + n^s P(|Y_1| > n^{s/r}). \end{aligned} \tag{2.3}$$

If G denotes the distribution function of Y_1,

$$n^s P(|Y_1| > n^{s/r}) \leq \int_{(|y| > n^{s/r})} |y|^r G(dy) \to 0, \qquad \text{as } n \to \infty. \tag{2.4}$$

For some β with $\beta > (s-1)/(s-r/2)$ and βr a positive even integer, use the Markov inequality on the first term on the right side of (2.3) to get

$$\begin{aligned} &n^{s-1}P(|\sum_1^n d_i X_i| > \varepsilon\delta_n n^{s/r}) \\ &\quad\leq c(\varepsilon)\delta_n^{-\beta r} n^{-s(\beta-1)-1} E((\sum_1^n d_i X_i)^{\beta r}) \\ &\quad\leq c(\varepsilon, r) n^{-s(\beta-1)-1} E((\sum_1^n X_i^2)^{\beta r/2}) \\ &\quad\leq c(\varepsilon, r) n^{-s(\beta-1)-1}(nE(X_1^{\beta r}) + n^2 E(X_1^{\beta r-2})E(X_1^2) + \cdots), \end{aligned} \tag{2.5}$$

where the second inequality is an application of Burkholder's martingale inequality. Each term on the right of (2.5) has the form

$$cn^{-s(\beta-1)-1} n^q E(X_1^{2i_1}) E(X_1^{2i_2}) \cdots E(X_1^{2i_q}) \tag{2.6}$$

where $2i_1 + 2i_2 + \cdots + 2i_q = \beta r$. If $2i_j \leq r$ then $E(X_1^{2i_j})$ is bounded. If $2i_j > r$ then integration by parts shows that $E(X_1^{2i_j}) = o(n^{s(2i_j - r)/r})$. In the case that all the $2i_j$ are less than or equal to r, (2.5) is bounded by $cn^{-s(\beta-1)-1} n^{\beta r/2}$ which goes to zero as $n \to \infty$ since $\beta > (s-1)/(s-r/2)$. If, say m, of the $2i_j$, $1 \leq m \leq q$, are larger than r then $\sum 2i_j \leq \beta r - 2(q-m)$ where the summation extends over $2i_j$ with $2i_j > r$. Thus (2.6) will go to zero provided

$$s(\beta - 1) - 1 + 2\frac{s}{r}(q - m) + ms \geq q + \beta s. \tag{2.7}$$

Since $(m-1)(s-1) \geq 0$, (2.7) holds and the proof of the lemma is complete.

3. Proof of Theorem. Let $\tau_1^{(n)}, \tau_2^{(n)}, \cdots, \tau_n^{(n)}$ be independent random Skorokhod stopping times such that $S_k n^{-\frac{1}{2}}$ is distributed as $W(\sum_1^k \tau_j^{(n)})$. Then as in the proof of Theorem 2 of [5] we must shows that

$$(3.1)\qquad P(|n^{-1} \textstyle\sum_1^n f'(k/n, W(\sum_1^k \tau_j^{(n)})) - \int_0^1 f'(t, W(t))\,dt| > n^{-\gamma}(\log (n))^{\beta}) = o(n^{-\gamma}(\log (n))^{\beta})$$

where

$$f'(t, x) = f(t, x)\,, \qquad |x| \leqq (\log (n))^{\frac{1}{2}}\,,$$

$|Df'(t, x)| \leqq c(\log (n))^{\alpha/2}$ uniformly in t and x. The difference on the left side of (3.1) is bounded as in [5] page 278; that is, suppressing the prime in $f'(t, x)$,

$$(3.2)\qquad P(|\textstyle\int_0^1 f(t, W(t))\,dt - n^{-1} \sum_1^n f(k/n, W(\sum_1^k \tau_j^{(n)}))| > 6\delta) \leqq \sum_1^6 P(|\Phi_i| > \delta)$$

where (see [3] page 278), if $V_k = \sum_1^k \tau_j^{(n)}$,

$$\Phi_1 = n^{-1} \textstyle\sum_1^n f(k/n), W(\sum_1^k \tau_j^{(n)})(n\tau_{k+1} - 1)$$
$$\Phi_2 = \textstyle\int_1^{\sum_1^n \tau_j^{(n)}} f(t, W(t))\,dt$$
$$\Phi_3 = f(0, 0)\tau_1^{(n)}$$
$$\Phi_4 = n^{-1} f(1, W(\textstyle\sum_1^n \tau_j^{(n)}))$$
$$\Phi_5 = \textstyle\sum_0^{n-1} \int_{V_k}^{V_{k+1}} \frac{\partial f}{\partial t}(k/n + \theta_{kn}(t), W(\sum_1^k \tau_j^{(n)}) + \bar\theta_{kn}(t))(t - k/n)\,dt$$
$$\Phi_6 = \textstyle\sum_0^{n-1} \int_{V_k}^{V_{k+1}} \frac{\partial f}{\partial x}(k/n + \bar\theta_{kn}(t), W(\sum_1^k \tau_j^{(n)}) + \theta_{kn}(t)) \times [W(t) - W(\sum_1^k \tau_j^{(n)})]\,dt\,.$$

The terms Φ_i in (3.2) are now estimated individually with the use of the lemma of Section 2. Let $r = p/2$; then, dropping the superscript n in $\tau_j^{(n)}$,

$$E(n\tau_j) = 1$$

and

$$E|n\tau_j|^r < \infty\,.$$

To estimate the term involving Φ_1, choose γ such that $0 \leqq \gamma < \frac{1}{2}$ and let $s = r(1 - \gamma)$. For any $\varepsilon > 0$, by the lemma of Section 2

$$(3.3)\qquad P(|\Phi_1| > \varepsilon n^{-\gamma}(\log n)^{\alpha/2}) = o(n^{-s+1})\,.$$

If $r \geqq 3$, $s - 1 = r(1 - \gamma) - 1 \geqq r/2 - 1 \geqq \frac{1}{2}$. In this case then for any $\gamma < \frac{1}{2}$ one has that

$$(3.4)\qquad P(|\Phi_1| > \varepsilon n^{-\gamma}(\log n)^{\alpha/2}) = o(n^{-\frac{1}{2}})\,.$$

If $2 \leqq r < 3$, the best choice of γ satisfies

$$\gamma = r(1 - \gamma) - 1$$

so that

$$\gamma = (r - 1)/(r + 1)$$

and one has

$$P(|\Phi_1| > \varepsilon n^{-\gamma}(\log n)^{\alpha/2}) = o(n^{-\gamma}) \tag{3.5}$$

for $\gamma = (r-1)/(r+1)$.

For the second term in (3.2)

$$|\Phi_2| \leqq c(\log n)^{\alpha/2} n^{-1} |\sum_1^n (n\tau_j - 1)| .$$

Hence for $0 \leqq \gamma < \frac{1}{2}$,

$$P(|\Phi_2| > \varepsilon n^{-\gamma}(\log n)^{\alpha/2}) \leqq P(n^{-1}|\sum_1^n (n\tau_j - 1)| > \varepsilon' n^{-\gamma}) . \tag{3.6}$$

Again applying the lemma with $s = r(1-\gamma)$ yields

$$P(|\Phi_2| > \varepsilon n^{-\gamma}(\log n)^{\alpha/2}) = o(n^{-\frac{1}{2}}) \tag{3.7}$$

for any $\gamma < \frac{1}{2}$ if $r \geqq 3$; while for $r < 3$

$$P(|\Phi_2| > \varepsilon n^{-\gamma}(\log n)^{\alpha/2}) = o(n^{-\gamma}) \tag{3.8}$$

where now $\gamma = (r-1)/(r+1)$.

As in [5] the terms Φ_3 and Φ_4 give no trouble. Now

$$\begin{aligned} |\Phi_5| &\leqq cn^{-2}(\log n)^{\alpha/2} \textstyle\sum_1^n (n\tau_k)^2 + cn^{-2}(\log n)^{\alpha/2} \sum_1^n (n\tau_k)|\sum_1^{k-1} (n\tau_j - 1)| \\ &= \Phi_5^{(1)} + \Phi_5^{(2)} , \quad \text{say.} \end{aligned} \tag{3.9}$$

For $\Phi_5^{(1)}$ and any $\gamma \geqq 0$ one has that

$$\begin{aligned} P(|\Phi_5^{(1)}| &> \varepsilon n^{-\gamma}(\log n)^{\alpha/2}) \\ &\leqq P(n^{-1} \textstyle\sum_1^n (n\tau_k)^2 \geqq \varepsilon' n^{-\gamma+1}) \\ &\leqq n^{r(\gamma-1)/2} E(n^{-1} \textstyle\sum_1^n (n\tau_k)^2)^{r/2} \\ &\leqq n^{r(\gamma-1)/2} E(n\tau_1)^r . \end{aligned} \tag{3.10}$$

The best choice of γ here satisfies

$$\gamma = r(1-\gamma)/2 , \qquad \gamma = r/(r+2) \geqq \tfrac{1}{2} \qquad \text{since} \quad r \geqq 2 .$$

Hence, certainly

$$P(|\Phi_5^{(1)}| > \varepsilon n^{-\frac{1}{2}}(\log n)^{\alpha/2}) = O(n^{-\frac{1}{2}}) . \tag{3.11}$$

Let $M > 0$ and $0 \leqq \gamma < \frac{1}{2}$. Then application of the lemma given in [3] page 12 along with Chebyshev's inequality shows that

$$\begin{aligned} P(\max_{1 \leqq k \leqq n} &|\textstyle\sum_1^n (n\tau_k - 1)| > n^{1-\gamma} \\ &\leqq \tfrac{4}{3} P\{|\textstyle\sum_1^n (n\tau_k - 1)| > n^{1-\gamma} - 2(nE(n\tau_1)^2)^{\frac{1}{2}}\} \\ &\leqq \tfrac{4}{3} P(|\textstyle\sum_1^n (n\tau_k - 1)| > \tfrac{1}{2} n^{1-\gamma}) . \end{aligned} \tag{3.12}$$

Now

$$\begin{aligned} P(|\Phi_5^{(2)}| &> Mn^{-\gamma}(\log n)^{\alpha/2}) \\ &\leqq P\left(cn^{-2} \textstyle\sum_1^n (n\tau_k - 1)|\sum_1^{k-1} (n\tau_j - 1)| > \dfrac{M}{2} n^{-\gamma}\right) \\ &\quad + P\left(cn^{-2} \textstyle\sum_1^n |\sum_1^{k-1} (n\tau_j - 1)| \geqq \dfrac{M}{2} n^{-\gamma}\right) . \end{aligned} \tag{3.13}$$

By (3.12) applied to the right side of (3.13) one has that

$$\begin{aligned}(3.14)\qquad & P(|\Phi_5^{(2)}| > Mn^{-\gamma}(\log n)^{a/2}) \\ & \leqq P\left(c|\textstyle\sum_1^n (n\tau_k - 1)| > \frac{M}{2} n\right) \\ & \quad + \tfrac{4}{3}P(|\textstyle\sum_1^n (n\tau_k - 1)| > \frac{1}{2}n^{1-\gamma}), \qquad n \text{ large.}\end{aligned}$$

By the lemma the first term on the right of (3.4) is $o(n^{-1})$. For $r \geqq 3$ the lemma shows the second term to be $o(n^{-\frac{1}{2}})$ for any $\gamma < \frac{1}{2}$. If $r < 3$ then the second term is $o(n^{-\gamma})$ for $\gamma = (r-1)/(r+1)$.

It remains now to estimate the term involving Φ_6. As in [5],

$$(3.15)\qquad |\Phi_6| \leqq c(\log n)^{a/2}n^{-\frac{3}{2}} \textstyle\sum_0^{n-1} Z_k$$

where the Z_k are independent and identically distributed and

$$Z_k \cong \textstyle\int_0^\tau |W(s)|\, ds$$

where $\tau = \tau_1^{(1)}$. Let $\mu = E(Z_k)$. Then for $0 < \gamma \leqq \frac{1}{2}$,

$$\begin{aligned}(3.16)\qquad & P(|\Phi_6| > M(\log n)^{a/2}n^{-\gamma}) \\ & \leqq P(\textstyle\sum_0^{n-1} Z_k > Mn^{\frac{3}{2}-\gamma}) \\ & \leqq P(\textstyle\sum_0^{n-1}(Z_k - \mu) > (M-\mu)n^{\frac{3}{2}-\gamma}), \qquad M \text{ large enough.}\end{aligned}$$

For $2 \leqq r \leqq 3$, apply the Markov inequality to bound the right side of (3.16) by

$$\begin{aligned}(3.17)\qquad & n^{-r(3-2\gamma)/3}E|\textstyle\sum_0^{n-1}(Z_k - \mu)|^{2r/3} \\ & \leqq 2n^{-r(3-2\gamma)/3+1}E|Z_1 - \mu|^{2r/3} \\ & = O(n^{-r(3-2\gamma)/3+1}), \qquad \text{provided } EZ_1^{2r/3} < \infty .\end{aligned}$$

The inequality in (3.17) is a result of von-Bahr and Esseen [6]. Since $Z_k \leqq \tau \max_{t<\tau} |W(t)|$, application of Hölder's inequality holds

$$(3.18)\qquad EZ_k^{2r/3} \leqq [E(\tau^r)]^{\frac{2}{3}}[E(\max_{t<\tau} |W(t)|^{2r})]^{\frac{1}{3}} < \infty ,$$

since by Doob's inequality, [2] page 317,

$$E(\max_{t<\tau}|W(t)|^{2r}) \leqq \left(\frac{2r}{2r-1}\right)^{2r} E(|W(\tau)|^{2r}) < \infty .$$

Now choose

$$\begin{aligned}(3.19)\qquad \gamma &= (r-1)/(1+2r/3), \qquad \text{for } 2 \leqq r < 2.25 \\ &= \tfrac{1}{2}, \qquad \text{for } 2.25 \leqq r .\end{aligned}$$

Combining the estimates for the terms involving $\Phi_1, \Phi_2, \cdots, \Phi_6$ one sees that

$$(3.20)\qquad (\sup |F_n(\lambda) - F(\lambda)| : -\infty < \lambda < \infty) = O(n^{-\gamma}(\log n)^{a/2})$$

for all $\gamma < \frac{1}{2}$ if $6 \leqq p < \infty$ and $\gamma = (p-2)/(p+2)$ if $2 \leqq p < 6$.

Acknowledgment. The author wishes to thank one of the referees for correcting and improving the original estimate of the term involving Φ_6.

REFERENCES

[1] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.

[2] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[3] FREEDMAN, D. (1970). *Brownian Motion and Diffusion.* Holden-Day, San Francisco.

[4] LOÈVE, M. (1963). *Probability Theory.* Van Nostrand, Princeton.

[5] SAWYER, S. (1972). Rates of convergence for some functionals in probability. *Ann. Math. Statist.* **43** 273–284.

[6] VON BAHR, B. and ESSEEN, C. G. (1965). Inequalities for the rth absolute moment of a sum of random variables. *Ann. Math. Statist.* **36** 299–303.

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