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CONTACT INTERACTIONS ON A LATTICE[1]

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Let $\{\xi_t\}$ be a Markov process whose values are subsets of Z_d, the d-dimensional integers. Put $\xi_t(x) = 1$ if $x \in \xi_t$ and 0 otherwise. The transition intensity for a change in $\xi_t(x)$ depends on $\{\xi_t(y), y$ a neighbor of $x\}$. The chief concern is with "contact processes," where $\xi_t(x)$ can change from 0 to 1 only if $\xi_t(y) = 1$ for some y neighboring x. Let $p_t(\xi) = \text{Prob}\{\xi_t \neq \emptyset \mid \xi_0 = \xi\}$. Under appropriate conditions, p_t is increasing, subadditive, or submodular in ξ. In the case of contact processes, conditions are giving implying that $p_\infty(\xi) = 0$ for all finite ξ, or that the contrary is true. In other cases conditions for ergodicity are given.

1. Introduction. Let Z_d be the d-dimensional lattice of points ("vertices") $x, y, z, \cdots, x = (x^1, \cdots, x^d)$, where each x^i is an integer. Call x and y *neighbors*, and write $x \sim y$, if $\sum_i |x^i - y^i| = 1$; x is a *neighbor* of $E \subset Z_d$ (written $x \sim E$) if $x \notin E$ but $x \sim y$ for some $y \in E$. Let E^+ denote the union of E and its neighbors. Let Ξ be the set of subsets of Z_d and Ξ_0 the set of finite subsets of Z_d. Each $\xi \in \Xi$ can be considered a map from Z_d into $\{0, 1\}$, with $\xi(x) = 1$ if $x \in \xi$ and $\xi(x) = 0$ if $x \notin \xi$. If $E \subset Z_d$, $\xi(E)$ denotes the number of points of ξ in E. Let $|\xi|$ be the number of points in ξ, $|\xi| \leqq \infty$. We often use notation such as $\xi \cup x$ instead of $\xi \cup \{x\}$. Let N_x be the set of the $2d$ neighbors of x.

We will consider a class of Markov processes $\{\xi_t\}$ with state space Ξ, whose precise definition is given in Section 2. Roughly speaking, if $\xi_t = \xi$, if $\xi(x) = 0$ and if $\xi(N_x) = k$, there is a probability $\lambda_k \Delta + o(\Delta)$ that $\xi_{t+\Delta}(x) = 1$, where $\lambda_0, \lambda_1, \cdots, \lambda_{2d}$ are given positive numbers. If $\xi(x) = 1$, the probability is $\mu\Delta + o(\Delta)$ that $\xi_{t+\Delta}(x) = 0$; the assumption that μ does not depend on N_x might suit some applications ($0 \to 1$ = infection and $1 \to 0$ = recovery), but in any case seems to lead to mathematically nice classes of processes. A process $\{\xi_t\}$ is a special case of the birth-death interactions treated by Spitzer (1971), Chapter 5, and of a broad class of interactions treated by Dobrushin (1971). It is also related to the "artificial neuron" networks discussed, e.g., by Vasil'ev (1969), (1970).

We will deal chiefly with *contact interactions*, where $\lambda_0 = 0$. If one thinks of a contact interaction as the spread of an infection or the growth of a population, then the more general case $\lambda_0 > 0$ corresponds to spontaneous infection, or immigration. The case $\lambda_0 = 0$, $\lambda_k = \lambda_1$ for $k \geqq 1$ seems like a continuous-time version of the neuron networks mentioned above, where however the process is $1 - \xi_t(x)$ rather than $\xi_t(x)$.

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In Section 3 we define "contact chains," which correspond roughly to paths of propagation and are useful for getting bounds on the rate of propagation. In Section 5 we take up, for a contact interaction $\{\xi_t\}$, the "survival" probability $p_t(\xi) = \text{Prob}\{\xi_t \neq \emptyset \mid \xi_0 = \xi\}$ and the expectation $m_t(\xi) = \mathscr{E}\{|\xi_t| \mid \xi_0 = \xi\}$, which are respectively 1 and ∞ if $|\xi| = \infty$.

In a branching process with one initial object let π_t and n_t be respectively the probability of survival until t and the expected size at t. Then with M initial objects the probability and expectation are $1 - (1 - \pi_t)^M$ and Mn_t, which are respectively concave and linear (additive) in M. We cannot expect anything so simple for a contact process, but we will see that under appropriate sets of conditions both $p_t(\xi)$ and $m_t(\xi)$, considered for fixed t as set functions of ξ, are monotone, or subadditive, or submodular; that is, strongly subadditive. (The latter property corresponds to concavity; see (5.4').) These properties are used to give some bounds on sets of parameter values λ_k, μ for which $p_\infty(\xi) = \lim_{t\to\infty} p_t(\xi) = 0$ for each $\xi \in \Xi_0$, and also to yield some inequalities for m_t.

If $\{\eta_t\}$ is a process with $\lambda_0 > 0$, we associate with it a certain contact process $\{\xi_t\}$. In case the parameters of $\{\xi_t\}$ are such that

$$\lim_{t\to\infty} \text{Prob}\{\xi_t(x) = 1 \mid \xi_0 = \xi\} = 0$$

for each ξ and x, $\{\eta_t\}$ is ergodic. (Section 8.)

In Section 9 we consider briefly contact processes for which $p_\infty(\xi) > 0$ for all $\xi \neq \emptyset$, and in Section 10 certain inequalities are improved for the case $d = 1$.

ADDED IN PROOF. Griffeath (1974) has recently improved some of the results of this paper. Richardson (1973) has obtained results of quite a different nature about certain contact processes; his technical preliminaries about "paths" are similar to mine about "chains". I have recently seen a technical report of Vasil'ev, Mityushin, Pyatetshiĭ–Shapiro, and Toom (1973) about discrete-time neuron nets. The "minorant" process of Section 3 of that report corresponds to the contact process defined in Section 8 of the present paper and is used for a similar purpose, although the methods of proof differ.

2. Preliminaries. Let S be a finite set and Ξ_S the set of mappings $\xi : Z_d \to S$. Give S the discrete topology and Ξ_S the product topology. The measurable sets of Ξ_S are the Borel sets. The elements of S are called *types*, and $\xi(x)$ is a *coordinate* of ξ. Let C_S be the continuous functions $\Xi_S \to R_1$ and C_{S0} the functions in C_S depending on only finitely many coordinates of ξ. Norms are supremum norms. If $S = \{0, 1\}$ write simply Ξ, C, and C_0. The terminology of Dynkin (1965) will be used.

A *nearest-neighbor interaction* (NNI) is a stationary Ξ_S-valued Markov process whose generator $\mathscr{A}$ satisfies

$$(2.1) \qquad (\mathscr{A}f)(\xi) = \sum_{x\in Z_d} \sum_{s\in S} c(x, \xi, s)(f(\xi_{xs}) - f(\xi)), \qquad f \in C_{S0}.$$

In (2.1), ξ_{xs} is defined by $\xi_{xs}(x) = s$, $\xi_{xs}(y) = \xi(y)$ if $y \neq x$; $c(x, \xi, s) \geqq 0$ is the intensity for a jump $\xi \to \xi_{xs}$, $c(x, \xi, s)$ depends on ξ only through $\{\xi(y), y \sim x\}$, and we take $c(x, \xi, s) = 0$ if $s = \xi(x)$; we assume the translation-invariance property $c(x + y, \xi + y, s) = c(x, \xi, s)$, where $(\xi + y)(z) = \xi(z - y)$. If $S = \{0, 1\}$ it is convenient to write

$$(2.1') \qquad (\mathscr{A}f)(\xi) = \sum_x c(x, \xi)(f(\xi_{(x)}) - f(\xi)), \qquad f \in C_0,$$

where $\xi_{(x)}(x) = 1 - \xi(x)$ and $\xi_{(x)}(y) = \xi(y)$ if $y \neq x$; $c(x, \xi)$ is the intensity for a jump $\xi \to \xi_{(x)}$. There exists a unique Markovian semigroup $\{T_t\}$ on C_S whose generator satisfies (2.1); see for example Holley (1972) or Liggett (1972). Since $\{T_t\}$ has the Feller property and Ξ_S is compact and metrizable, we can and will assume that any NNI to be considered is "standard" (Dynkin (1965) page 104). In particular the sample functions are right-continuous and have left-hand limits. Note that $T_t 1 = \mathbf{1}$.

(a) *Birth-death interactions.* Take $S = \{0, 1\}$ and let $\lambda_0, \lambda_1, \cdots, \lambda_{2d}, \mu_0, \mu_1, \cdots, \mu_{2d} \geqq 0$ be given. In (2.1′) put $c(x, \xi) = \lambda_k$ if $\xi(x) = 0$ and μ_k if $\xi(x) = 1$, where $k = \xi(N_x)$.

(b) *Contact interactions* (*or contact processes*). In (a) take $\lambda_0 = 0$ and all $\mu_k = \mu \geqq 0$. In this case the generator (2.1′) satisfies

$$(2.2) \qquad \begin{aligned} \mathscr{A}f(\xi) = \mu \textstyle\sum_{x \in \xi} (f(\xi \backslash x) - f(\xi)) \\ + \textstyle\sum_{x \sim \xi} \lambda_{\xi(N_x)}(f(\xi \cup x) - f(\xi)), \qquad f \in C_0. \end{aligned}$$

Let $\{\eta_t\}$ be a NNI and let φ map Ξ_S into Ξ. Then $\{\varphi(\eta_t)\}$ is in general not Markovian but is so under the conditions of the following lemma, which we derive from a result in Dynkin (1965). (Put $\xi_t = \varphi(\eta_t)$.)

(2.3) LEMMA. *Let $S = S_0 \cup S_1$, where S_0 and S_1 are disjoint and not empty. Write $s_1 \equiv s_2$ if s_1 and s_2 are in the same set S_i. Define φ: Ξ_S onto Ξ by $\varphi(\eta)(x) = i$ if $\eta(x) \in S_i$, $i = 0, 1$. Suppose for each $x \in Z_d$ and $\eta, \eta' \in \Xi_S$ such that $\varphi(\eta) = \varphi(\eta')$ we have*

$$\sum_{s: s \not\equiv \eta(x)} (c(x, \eta, s) - c(x, \eta', s)) = 0.$$

Then $\{\xi_t\}$ is a NNI. *Its generator has the form* (2.1′) *with*

$$c(x, \xi) = \sum_{s: s \not\equiv \eta(x)} c(x, \eta, s), \qquad \xi \in \Xi,$$

for any η such that $\varphi(\eta) = \xi$.

SKETCH OF PROOF. Let $\mathscr{B}$ be the Borel sets in Ξ and let $\mathscr{H}$ be the set of $\varphi^{-1}(\mathscr{B})$-measurable functions in C_S. From Dynkin (1965), page 325, especially (10.58), it suffices to show that the transition semigroup $\{T_t\}$ corresponding to $\{\eta_t\}$ maps $\mathscr{H}$ into itself.[2] Let $\mathscr{A}$ be the generator of $\{T_t\}$. From Holley (1972), we have, even though $\mathscr{A}$ is unbounded,

$$(2.4) \qquad T_t f = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathscr{A}^k f, \qquad f \in C_{0S},\ 0 \leqq t \leqq \Delta,$$

[2] On the left side of (10.58) in one printing x appears incorrectly instead of γx.

for some $\Delta > 0$ not depending on f, the series converging in norm. Since our assumptions imply that $\mathscr{A}$, given by (2.1), maps $C_{S0} \cap \mathscr{H}$ into itself, a routine argument shows that T_t maps $\mathscr{H}$ into itself.

3. Contact chains. In the rest of this paper $\{\xi_t\} = \{\xi_t(\omega)\}$ is a contact process with parameters $\mu, \lambda_0 = 0, \lambda_1, \cdots, \lambda_{2d}$, governed by probability measures $\{P_\xi, \xi \in \Xi\}$ on the measurable space $(\Omega, \mathscr{M}^0)$; $\{\xi_t\}$ is adapted to the increasing family of σ-fields $\{\mathscr{M}_t\}$, $\mathscr{M}_t \subset \mathscr{M}^0$; $\mathscr{E}_\xi(\cdot) = \int (\cdot)\, dP_\xi$; $P(t, \xi, \Gamma) = P_\xi\{\xi_t \in \Gamma\}$. Let $\xi_t(\omega, x)$ be the x-coordinate of $\xi_t(\omega)$, sometimes written $\xi_t(x)$; similarly $\xi_t(E) = \xi_t(\omega, E) = \sum_{x \in E} \xi_t(x)$. Since $\{\xi_t\}$ can be assumed standard (see Section 2), for each ω and each finite $E \subset Z_d$ the mapping $t \to \xi_t(\omega, E)$ has only finitely many jumps in any finite interval.

(3.1) DEFINITIONS. Recall $N_x = \{y : y \sim x\}$. Let $\mathscr{N}_t$ be the σ-field in Ω generated by $\{\xi_s, 0 \leqq s \leqq t\}$. Let $\tau(\omega, x) = \tau(x) = \inf\{s : \xi_s(x) = 1\}$. Since $\{\xi : \xi(x) = 1\}$ and its complement are compact, Lemma 4.1, page 106 of Dynkin (1965) implies that $\tau(x)$ is a Markov time and $\{\tau(x) > t\} \in \mathscr{N}_t$. Let $\mathscr{A}$ denote the generator of $\{\xi_t\}$, given by (2.2).

(3.2) LEMMA. *For each $\xi \in \Xi$, $P_\xi\{$two coordinates of $\{\xi_t\}$ ever jump simultaneously$\} = 0$.*

PROOF. Fix $\bar{x}, \bar{y} \in Z_d$, $\bar{x} \neq \bar{y}$; let $f_{ij}(\xi) = 1$ if $\xi(\bar{x}) = i$ and $\xi(\bar{y}) = j$, and 0 otherwise, $i, j = 0, 1$. Then $\mathscr{A} f_{ij}(\xi) = 0$ if simultaneously $\xi(\bar{x}) \neq i$ and $\xi(\bar{y}) \neq j$. Now fix i and j and let W_{kn} be the event

$$\{\xi_{k/n}(\bar{x}) = 1 - i, \xi_{(k+1)/n}(\bar{x}) = i, \xi_{k/n}(\bar{y}) = 1 - j, \xi_{(k+1)/n}(\bar{y}) = j\}, \qquad 0 \leqq k < n .$$

Let $W = \{\xi_{t-0}(\bar{x}) = 1 - i, \xi_t(\bar{x}) = i, \xi_{t-0}(\bar{y}) = 1 - j, \xi_t(\bar{y}) = j$ for some $t \in [0, 1]\}$. Then for all sufficiently large n we have $W \subset \bigcup_{k=0}^{n-1} W_{kn}$. Now $P_\xi(W_{0n}) = 0$ unless $\xi(\bar{x}) = 1 - i$ and $\xi(\bar{y}) = 1 - j$, in which case $P_\xi(W_{0n}) = T_{1/n} f_{ij}(\xi) = o(1/n)$ uniformly for those ξ such that $\xi(\bar{x}) = 1 - i$ and $\xi(\bar{y}) = 1 - j$. Thus $P_\xi(W_{0n}) = o(1/n)$ uniformly for all ξ and hence from the Markov property $P_\xi(W_{kn}) = o(1/n)$ uniformly in ξ and k, whence $P_\xi(\bigcup_{k=0}^{n-1} W_{kn}) \to 0$ as $n \to \infty$. The lemma is an easy consequence of this, the argument extending to all finite intervals. □

(3.3) LEMMA. *If $\xi(x) = 0$, then $P_\xi\{\xi_s(N_x) > 0$ everywhere on some open interval containing $\tau(x) \mid \tau(x) < \infty\} = 1$.*

PROOF. Arguing somewhat as in the preceding lemma we have $P_\xi\{\xi_{k/n}(x \cup N_x) = 0, \xi_{(k+1)/n}(x) = 1$ for some $k = 0, 1, \cdots, n - 1\} \to 0$ as $n \to \infty$, uniformly in ξ. It follows that if $0 < \tau(x) < 1$ then $\tau(x)$ is a.s. (P_ξ) the right end-point of an s-interval in which $\xi_s(N_x) > 0$. Using Lemma (3.2), we have the desired result if $0 < \tau(x) < 1$, and similarly for all finite intervals. □

(3.4) LEMMA. *Let $\Lambda = \max \lambda_i$. Then $P_\xi\{\tau(x) \leqq u\} \leqq 1 - e^{-\Lambda u}$, $u \geqq 0$ if $\xi(x) = 0$.*

PROOF. $P_\xi\{\tau(x) > 1\} = \lim_n P_\xi\{\xi_{k/n}(x) = 0, k = 0, 1, \cdots, n\}$. From (2.2), $P_{\xi'}\{\xi_{1/n}(x) = 0\} \geqq 1 - \Lambda/n + o(1/n)$ uniformly in ξ' such that $\xi'(x) = 0\}$, whence $P_\xi\{\tau(x) > 1\} \geqq e^{-\Lambda}$, and similarly for other values of u. □

(3.5) DEFINITIONS. For $n = 1, 2, \cdots$ call $(x_1, \cdots, x_n)$ a *chain* (of length n from x_1 to x_n) if the x_i are distinct and $x_i \sim x_{i+1}$, $1 \leqq i \leqq n - 1$. The infinite sequence $x_1, x_2, \cdots$ is called a chain if each $(x_1, \cdots, x_n)$ is a chain. The chain $(x_1, \cdots, x_n)$ is a *contact chain* for x, relative to some sample path, if $x_1 = x$ and if

(a) $\tau(x_1) = 0$ in case $n = 1$, and
(b) $\infty > \tau(x_1) > \cdots > \tau(x_n) = 0$ in case $n > 1$.

The chain $(x_1, x_2, \cdots)$ is a contact chain (infinite) for x_1 if $\infty > \tau(x_1) > \tau(x_2) > \cdots$. A contact chain for x "reaches x before t" if $\tau(x) \leqq t$.

Let $L_n{}^t(x_1)$ be the event

$$\{\exists \text{ chain } (x_1, \cdots, x_n) : t \geqq \tau(x_1) > \cdots > \tau(x_n) > 0\}.$$

Then $L_n{}^t(x_1) \in \mathscr{N}_t$. It can be shown that the event {an infinite contact chain reaches x_1 before t} is precisely equal to $\bigcap_n L_n{}^t(x_1)$ and hence is also $\mathscr{N}_t$-measurable. The event that a contact chain of length n reaches x_1 before t is also $\mathscr{N}_t$-measurable.

REMARK. If $(x_1, \cdots, x_n)$ is a contact chain for x_1, the probability may be strictly positive, if $d > 1$, that $\xi_{\tau(x_1)}(x_2) = 0$. Hence it would not be quite right to think of the chain as a path of "infection" reaching x_1, but the analogy is helpful.

(3.6) LEMMA. *$P_\xi\{$some contact chain reaches x_1 before $t \mid \tau(x_1) \leqq t\} = 1$, $\xi \in \Xi$, $x_1 \in Z_d$, $t \geqq 0$.*

PROOF. Suppose $\xi(x_1) = 0$. Given $\tau(x_1) \leqq t$, by Lemma 3.3 there is a.s. (P_ξ) a chain $x_1, x_2, \cdots$ such that $t \geqq \tau(x_1) > \tau(x_2) > \cdots$, either continuing indefinitely or terminating with some x_n having $\tau(x_n) = 0$. If $\xi(x_1) = 1$, the result is obvious. □

(3.7) LEMMA. *Let $(x_1, \cdots, x_n)$ be a chain. Defining Λ as in (3.4), we have for every ξ and $t \geqq 0$*

$$P_\xi\{0 < \tau(x_1) < \cdots < \tau(x_n) \leqq t\} \leqq F_n(t) = \frac{\Lambda^n}{(n-1)!} \int_0^t e^{-\Lambda u} u^{n-1}\, du, \qquad n = 1, 2, \cdots.$$

PROOF. For chains of length 1, Lemma 3.7 follows from Lemma 3.4. Now suppose Lemma 3.7 is true for all chains of length $\leqq n - 1$, for some $n \geqq 2$. Let $(x_1, \cdots, x_n)$ be a chain and fix ξ. Assume $\sum \xi(x_i) = 0$ since otherwise the result is obvious. Let $\tau_k = \tau(x_k)$ and $V_k = \{0 < \tau_1 < \cdots < \tau_k < \infty\}$, $1 \leqq k \leqq n$. Let $G_k(t) = P_\xi\{V_k \cap (\tau_k \leqq t)\}$. We must show $G_n(t) \leqq F_n(t)$. Let $1(V) = 1(V)(\omega)$ be the indicator of the event V. To save notation in the inductive step, we take

$n = 3$, the argument being exactly the same in general. Then

$$G_3(t) \leqq P_\xi\{V_2, \xi_{\tau_2}(x_3) = 0, \xi_s(x_3) = 1 \text{ for some } s \in (\tau_2, t]\}$$
$$= P_\xi\{V_2, \xi_{\tau_2}(x_3) = 0, \theta_{\tau_2}\tau_3 \leqq t - \tau_2\},$$

where $(\tau_2, t] = \varnothing$ if $\tau_2 \geqq t$ and θ_{τ_2} is the usual shift operator. Noting that $V_2 \cap \{\xi_{\tau_2}(x_3) = 0\} \in \mathscr{M}_{\tau_2}$ we have

$$G_3(t) \leqq \mathscr{E}_\xi\{1(V_2, \xi_{\tau_2}(x_3) = 0)P_\xi[\theta_{\tau_2}\tau_3 \leqq t - \tau_2 \mid \mathscr{M}_{\tau_2}]\}. \tag{3.8}$$

Letting $H(\xi', u) = P_{\xi'}(\tau_3 \leqq u)$, we have for fixed u

$$P_\xi(\theta_{\tau_2}\tau_3 \leqq u \mid \mathscr{M}_{\tau_2}) = H(\xi_{\tau_2}, u) \qquad \text{a.s.} \quad (P_\xi) \quad \text{on} \quad \{\tau_2 < \infty\}$$

(see Dynkin (1965) page 100), whence, by routine arguments we can put

$$P_\xi(\theta_{\tau_2}\tau_3 \leqq t - \tau_2 \mid \mathscr{M}_{\tau_2}) = H(\xi_{\tau_2}, t - \tau_2) \tag{3.9}$$

in (3.8). Let F be the exponential distribution with mean $1/\Lambda$. From Lemma 3.4, $H(\xi', u) \leqq F(u)$ if $\xi'(x_3) = 0$. Hence (3.8) and (3.9) imply $G_3(t) \leqq \mathscr{E}_\xi\{1(V_2)F(t - \tau_2)\} = (G_2 * F)(t)$. Since the inductive hypothesis implies $G_2 \leqq F_2$, we have $G_3 \leqq F_3$. □

(3.10) DEFINITION. Let $\nu(n)$ be the number of chains of length n ($= (n-1)$-step self-avoiding walks) from any vertex, $n = 1, 2, \cdots$. Obviously $\nu(n) \leqq 2d(2d-1)^{n-2}$ for $n \geqq 2$; for sharper estimates see Hammersley (1961), (1962).

(3.11) LEMMA. *For $n = 1, 2, \cdots$ and $E \subset Z_d$ let $D_n^t(x)$ be the event that a contact chain of length $> n$ (or an infinite contact chain) reaches x before t. Let $D_n^t = \bigcup_x D_n^t(x)$. Then*

$$P_\xi(D_n^t(x)) \leqq \nu(n)F_n(t), \tag{3.12}$$

$$P_\xi(D_n^t) \leqq |\xi|\nu(n+1)F_n(t), \tag{3.13}$$

where F_n is as in (3.7). *Moreover for each ξ,*

$$P_\xi\{\textit{an infinite contact chain reaches some } x \textit{ before } t\} = 0.$$

PROOF. If $\omega \in D_n^t(x)$, then $\xi_0(\omega, x_1) = 0$ and there is a chain $(x_1, \cdots, x_n)$ such that $t \geqq \tau(x_1) > \cdots > \tau(x_n) > 0$. Thus (3.12) follows from Lemma 2.7. Since $\lim_{n\to\infty} \nu(n)F_n(t) = 0$, we see that a.s. (P_ξ) no infinite contact chains reach any x before t. Next suppose $\omega \in D_n^t$. Then, a.s. (P_ξ), there is a finite chain $(x_1, \cdots, x_k)$ with $k > n$ such that $t \geqq \tau(x_1) > \cdots > \tau(x_k) = 0$, where necessarily $x_k \in \xi$. Hence there is a chain of length $n + 1$, namely $(x_{k-n}, x_{k-n+1}, \cdots, x_k)$, such that $x_k \in \xi$ and $t \geqq \tau(x_{k-n}) > \cdots > \tau(x_{k-1}) > 0$. Since there are $\nu(n+1)$ chains of length $n + 1$ ending at each vertex in ξ, we obtain (3.13) by using Lemma 3.7. □

4. The imbedded jump process. Recall that Ξ_0 is the set of *finite* subsets of Z_d, and is thus a countable set.

(4.1) THEOREM. *If* $\xi \in \Xi_0$ *then* $P_\xi\{\xi_t \in \Xi_0$ *for all* $t\} = 1$; i.e., Ξ_0 *is stochastically closed.*

PROOF. For each $t > 0$ we have for each $\xi \in \Xi_0$

$$P_\xi\{\xi_s \notin \Xi_0 \text{ for some } s \leqq t\} \leqq P_\xi\{\tau(x) \leqq t \text{ for infinitely many } x\}.$$

But Lemma 3.6 implies that $\{\tau(x) \leqq t$ for infinitely many $x\}$ is a.s. (P_ξ) a subset of $D_n{}^t$ for each n. Letting $n \to \infty$ and using (3.13), we complete the proof. □

(4.2) REMARK. If $x_1, \cdots, x_k$ are distinct vertices in the initial set ξ, it can be verified directly from the form of the generator in (2.2) that the random variables $\inf\{t: \xi_t(x_i) = 0\}$, $i = 1, \cdots, k$, are independent exponential, mean $1/\mu$. Hence $\Xi \backslash \Xi_0$ is also stochastically closed.

(4.3) THEOREM. *Let* $J_t = \{\{\xi_s\}$ *has infinitely many jumps for* $0 \leqq s \leqq t\}$. *Then* $P_\xi(J_t) = 0$, $\xi \in \Xi_0$, $0 \leqq t < \infty$.

PROOF. For each ω, x, and $t > 0$ the function $s \to \xi_s(\omega, x)$, $0 \leqq s \leqq t$, has only finitely many jumps, whence $J_t \subset \{\tau(x) \leqq t$ for infinitely many $x\}$. The proof is completed as in Theorem 4.1. □

It follows from (4.1) and (4.3) that we can consider $\{\xi_t\}$ confined to Ξ_0 as a strictly stochastic countable state process, all the states being stable (having nonzero holding times), $\xi = \varnothing$ being an absorbing state. The holding time in the state ξ is exponential with mean $(q_\xi)^{-1}$, where

$$q_\xi = \mu|\xi| + \textstyle\sum_{x \sim \xi} \lambda_{\xi(N_x)} . \tag{4.4}$$

To see this, note from (3.6) that

$$\begin{aligned} P_\xi\{\xi_s = \xi, 0 \leqq s \leqq t\} &= P_\xi\{\xi_s(x) = \xi(x), 0 \leqq s \leqq t, x \in \xi^+\} \\ &= P_\xi\{\xi_t(x) = \xi(x), x \in \xi^+\} + o(t), \qquad t \downarrow 0, \end{aligned}$$

where ξ^+ is the union of ξ and its neighbors. Since the indicator of the set $E = \{\xi' : \xi'(x) = \xi(x), x \in \xi^+\}$ depends on only finitely many coordinates, it is in the domain of $\mathscr{A}$ and hence

$$P_\xi\{\xi_t \in E\} = 1 + t\mathscr{A} 1_E(\xi) + o(t) = 1 - tq_\xi + o(t).$$

Similarly the intensity $q_{\xi\eta}$ for a jump $\xi \to \eta$, $\xi \neq \eta$, is 0 except

$$\begin{aligned} q_{\xi\eta} &= \mu & &\text{if } \eta = \xi \backslash x \text{ for some } x \in \xi, \\ &= \lambda_{\xi(N_x)} & &\text{if } \eta = \xi \cup x \text{ for some } x \sim \xi. \end{aligned} \tag{4.5}$$

(4.6) DEFINITION. Let $m_t(\xi) = \mathscr{E}_\xi\, \xi_t(Z_d)$.

We establish the following crude bound.

(4.7) LEMMA. *If* $\xi \in \Xi_0$,

$$m_t(\xi) \leqq |\xi| e^{(2d\Lambda - \mu)t}, \qquad t \geqq 0,$$

where $\Lambda = \max \lambda_i$.

PROOF. Write $m_t(\xi) = m_t'(\xi) + m_t''(\xi)$, where $m_t'(\xi) = \mathscr{E}_\xi\, \xi_t(\xi^+)$ and $m_t''(\xi) = \mathscr{E}_\xi\, \xi_t(Z_d \backslash \xi^+)$. Write $Z_d \backslash \xi^+ = \bigcup_{k=3}^\infty E_k$, where $E_k = \{x\colon$ the shortest chain from x to any vertex of ξ has length $k\}$. Then

$$m_t''(\xi) \leqq \sum_{k=3}^\infty \sum_{x \in E_k} P_\xi\{\tau(x) \leqq t\}\,.$$

From Lemma 3.6, we have $\{\tau(x) \leqq t\} \subset D_{k-1}^t(x)$ a.s. (P_ξ) if $x \in E_k$, and hence $P_\xi\{\tau(x) \leqq t\} \leqq \nu(k-1)F_{k-1}(t)$ from (3.12). It can be seen that $|E_k| \leqq |\xi| \cdot M_d k^{d-1}$, where M_d depends only on d. Hence

$$\begin{aligned} m_t''(\xi) &\leqq \sum_{k=3}^\infty |\xi| \cdot M_d k^{d-1} \nu(k-1) F_{k-1}(t) \\ &\leqq |\xi| \cdot M_d' t^2\,, \qquad 0 \leqq t \leqq 1\,, \end{aligned} \tag{4.8}$$

where M_d' depends only on d. Next, letting $\tau'(x) = \inf\{s\colon \xi_s(x) = 0\}$, note that

$$P_\xi\{\xi_t(x) = 0\} \geqq P_\xi\{\tau'(x) \leqq t;\ \xi_{\tau'(x)+s}(x) = 0,\ 0 \leqq s \leqq t\}\,.$$

From (3.4), (4.2), and the Markov property we see that if $\xi(x) = 1$, the above probability is $\geqq e^{-\Lambda t}(1 - e^{-\mu t})$. Hence

$$\begin{aligned} m_t'(\xi) &= \sum_{x \in \xi} \mathscr{E}_\xi(\xi_t(x)) + \sum_{x \sim \xi} \mathscr{E}_\xi(\xi_t(x)) \\ &\leqq |\xi|[1 - e^{-\Lambda t}(1 - e^{-\mu t})] + 2d|\xi|(1 - e^{-\Lambda t})\,, \end{aligned} \tag{4.9}$$

where we have again used (3.4) to bound the second sum in (4.9). Combining (4.8) and (4.9) we get

$$m_t(\xi) \leqq |\xi|(1 + (2d\Lambda - \mu)t + Mt^2)\,, \qquad 0 \leqq t \leqq 1\,, \tag{4.10}$$

where M does not depend on ξ or t. From (4.10) and the Markov property we have

$$m_{t+h}(\xi) \leqq (1 + (2d\Lambda - \mu)h + Mh^2) m_t(\xi)\,, \qquad 0 \leqq h \leqq 1\,,$$

which leads to Lemma 4.7. □

(4.11) LEMMA. *For all* $\xi \in \Xi$, *we have*

$$m_{t+h}(\xi) \geqq e^{-\mu h} m_t(\xi)\,, \qquad t \geqq 0,\ h \geqq 0\,.$$

PROOF. From (4.2), $m_t(\xi) \geqq e^{-\mu t}|\xi|$. Lemma 4.11 follows from this and the Markov property. □

5. Monotonicity and subadditivity. In this section $\{\xi_t\}$ is still a contact process, with probability measure P_ξ, and $m_t(\xi)$ is as in (4.6).

(5.1) DEFINITIONS. Let $p_t(\xi) = P_\xi\{\xi_t \neq \varnothing\}$, $p_\infty(\xi) = \lim_{t\to\infty} p_t(\xi)$ (exists because $p_t(\xi) \downarrow$ in t). Note $p_t(\xi) = 1$ if $|\xi| = \infty$, from (4.2). Denote $p_t(\xi)$ by $p_t(x)$ if $\xi = \{x\}$. Call $\{\xi_t\}$ *increasing*, *subadditive*,[3] or *submodular* if respectively (5.2), (5.3) or (5.4) hold for arbitrary $\xi, \eta \in \Xi$ and $t \geqq 0$:

$$p_t(\xi) \leqq p_t(\xi \cup \eta)\,, \tag{5.2}$$

$$p_t(\xi \cup \eta) \leqq p_t(\xi) + p_t(\eta)\,, \tag{5.3}$$

$$p_t(\xi \cup \eta) + p_t(\xi \cap \eta) \leqq p_t(\xi) + p_t(\eta)\,. \tag{5.4}$$

[3] Certain "subadditive" processes have been studied by Hammersley and Welsh (1965), where however the ideas and results are different from those of the present paper.

The significance of (5.4) is perhaps clearer from the equivalent property: if $\xi' \subset \xi$ and $\xi \cap \eta = \emptyset$ then

$$p_t(\xi \cup \eta) - p_t(\xi) \leqq p_t(\xi' \cup \eta) - p_t(\xi') . \tag{5.4'}$$

(5.5) REMARK. We will actually prove more than (5.2) by constructing contact processes $\{\xi_t\}$ and $\{\xi_t'\}$ on the same probability field, with the same transition law, in such a way that $\xi_t \subset \xi_t'$, $\xi_0 = \xi$, $\xi_0' = \xi \cup \eta$. Thus other functions such as $m_t(\xi)$ and $P_\xi\{\xi_t(x) = 1\}$ will also be increasing in ξ. A similar remark applies to (5.3) and (5.4).

(5.6) THEOREM. (a) *If $\lambda_k \uparrow$ in k then $\{\xi_t\}$ is increasing.*
(b) *If also $\lambda_k/k \downarrow$ in k, $k \geqq 1$, then $\{\xi_t\}$ is subadditive.*

PROOF. We will exploit a method employed effectively by Vasershtein (1969) and Dobrushin (1971) to prove ergodic theorems, i.e., defining two processes on the same space. It is convenient first to prove (5.6b), so we assume till further notice that $\lambda_k \uparrow$ and $\lambda_k/k \downarrow$. These conditions imply that for any nonnegative integers m', n', and k we have

$$\begin{aligned} m'\lambda_{m'+n'} &\leqq (m'+n')\lambda_{m'} \\ &\leqq (m'+n')\lambda_{m'+k} . \end{aligned} \tag{5.7}$$

Let $\{\eta_t\}$ be a NNI (Section 2) with S the set of eight types $\{0, A, A', B, B', AB, A'B, AB'\}$, and probability measure $\tilde{P}_\eta$ corresponding to $\eta_0 = \eta$. The function $c(x, \eta, s)$ in (2.1) will be given in Table 1. The first column in each row is the initial type $\eta(x)$; in the second column are listed the types s for which $c(x, \eta, s) > 0$, the value of c being given in parentheses after the type. Fix x and η; let m, m', n, n' be the numbers of vertices in N_x with type respectively in the sets $\{A, AB, AB'\}$, $\{A', A'B\}$, $\{AB, A'B, B\}$, $\{B', AB'\}$. Let $M = m + m'$, $N = n + n'$, $\lambda' = \lambda_{m'+n'}$, $\alpha = m'/(m'+n')$, $\beta = n'/(m'+n')$. Take $\alpha = \beta = 0$ if $m' + n' = 0$. From (5.7), all the indicated intensities are $\geqq 0$.

Let $S_1 = \{A, A', AB, A'B, AB'\}$, $S_0 = S\backslash S_1 = \{0, B, B'\}$. Define $\varphi_A : \Xi_S \to \Xi$ by $\varphi_A(\eta)(x) = i$ if $\eta(x) \in S_i$, $i = 0, 1$. Let $\eta_t^A = \varphi_A(\eta_t)$. We apply Lemma 2.3 to show that $\{\eta_t^A\}$ is Markov with the same set of intensities as our basic contact process $\{\xi_t\}$. In each row of Table 1 corresponding to a type in S_0, the sum of

TABLE 1

$\eta(x)$	New Type
0	$A(\lambda_M - \alpha\lambda')$, $A'(\alpha\lambda')$, $B(\lambda_N - \beta\lambda')$, $B'(\beta\lambda')$
A	$0(\mu)$, $A'(\alpha\lambda')$, $AB(\lambda_N - \beta\lambda')$, $AB'(\beta\lambda')$
A'	$0(\mu)$, $A'B(\lambda_N)$
B	$0(\mu)$, $B'(\beta\lambda')$, $AB(\lambda_M - \alpha\lambda')$, $AB'(\alpha\lambda')$
B'	$0(\mu)$, $AB'(\lambda_M)$
AB	$0(\mu)$, $A'B(\lambda')$
$A'B$	$0(\mu)$
AB'	$0(\mu)$

intensities into types in S_1 is $\lambda_M = \lambda_{m+m'}$, where $m + m'$ is the number of vertices in N_x with type in S_1, as can be seen from the definitions of m and m'. Similarly, in each row corresponding to a type in S_1, the sum of the intensities into types in S_0 is μ; hence $\{\eta_t^A\}$ has the same transition law as $\{\xi_t\}$. An observer who can see only the symbol A, alone or in combination, will see η_t^A when he looks at η_t.

Next put $\varphi_B(\eta)(x) = 1$ if $\eta(x) \in \{B, B', AB, A'B, AB'\}$, and 0 otherwise. Let $\varphi'(\eta)(x) = 1$ if $\eta(x) \in \{A', B', A'B, AB'\}$ and 0 otherwise. An argument like that given above shows that $\{\eta_t^B\} = \{\varphi_B(\eta_t)\}$ and $\{\eta_t'\} = \{\varphi'(\eta_t)\}$ are Markov with the same transition law as $\{\xi_t\}$.

Let $A_1, B_1 \subset Z_d$ and consider the process $\{\eta_t\}$, taking $\eta_0(x) = A'$ if $x \in A_1 \backslash B_1$, B' if $x \in B_1 \backslash A_1$, $A'B$ if $x \in A_1 \cap B_1$, and 0 otherwise. Then $\eta_0^A = A_1$, $\eta_0^B = B_1$, $\eta_0' = A_1 \cup B_1$. Moreover $\eta_t' \subset \eta_t^A \cup \eta_t^B$, $t \geqq 0$, since the symbol ($'$) can occur only with an A or a B. Hence

$$\begin{aligned} P_{A_1 \cup B_1}\{\xi_t \neq \emptyset\} = \tilde{P}_{\eta_0}\{\eta_t' \neq \emptyset\} &\leqq \tilde{P}_{\eta_0}\{(\eta_t^A \neq \emptyset) \cup (\eta_t^B \neq \emptyset)\} \\ &\leqq P_{A_1}\{\xi_t \neq \emptyset\} + P_{B_1}\{\xi_t \neq \emptyset\}\,. \end{aligned}$$

This proves (5.6 b).

Now assuming only $\lambda_k \uparrow$, let $\{\eta_t\}$ be a NNI with $S = \{0, A, B\}$. For any $\eta \in \Xi_S$ and $x \in Z_d$ let m be the number of neighbors of x of type A and n the number of type B. The intensities are given in Table 2.

TABLE 2

$\eta(x)$	New Type
0	$A(\lambda_m)$, $B(\lambda_{m+n} - \lambda_m)$
A	$0(\mu)$
B	$A(\lambda_m)$, $0(\mu)$

Let $\eta_t^A(x) = 1$ if $\eta_t(x) = A$ and 0 otherwise. Let $\eta_t^{AB}(x) = 1$ if $\eta_t(x) = A$ or B and 0 otherwise. Using Lemma 2.3 as above, we see that $\{\eta_t^A\}$ and $\{\eta_t^{AB}\}$ are Markov with the same transition law as $\{\xi_t\}$. Let A_1 and B_1 be disjoint subsets of Z_d and take $\eta_0(x) = A$ if $x \in A_1$, B if $x \in B_1$, 0 otherwise; then $\eta_0^A = A_1$ and $\eta_0^{AB} = A_1 \cup B_1$. Since $\eta_t^A \subset \eta_t^{AB}$, we have $p_t(A_1) \leqq p_t(A_1 \cup B_1)$. □

It can be shown by examples that the two conditions $\lambda_k \uparrow$ and $\lambda_k \leqq k\lambda_1$ together do not imply subadditivity, at least of $m_t(\xi)$, for $d = 2$.

(5.8) LEMMA. *Let $\{\xi_t\}$ and $\{\xi_t'\}$ be contact processes with parameters $\mu, \lambda_1, \lambda_2, \cdots$ and $\mu, \lambda_1', \lambda_2', \cdots$. Suppose $\lambda_k \leqq \min_{j:j \geqq k} \lambda_j'$, $k = 1, 2, \cdots$. Then (with the obvious definitions for p' and m') $p_t(\xi) \leqq p_t'(\xi)$ and $m_t(\xi) \leqq m_t'(\xi)$, $t \geqq 0$, $\xi \in \Xi$.*

PROOF. Let $\{\eta_t\}$ be a NNI with $S = \{(00), (01), (11)\}$. Given $\eta \in \Xi_S$ and $x \in Z_d$, let m be the number of neighbors of x of type (11) and m' the number of type (01) or (11). The intensities are in Table 3.

Let $\xi_t(x) = 1$ if $\eta_t(x) = (11)$ and 0 otherwise; $\xi_t'(x) = 1$ if $\eta_t(x) = (01)$ or (11) and 0 otherwise. From Lemma 2.3, $\{\xi_t\}$ and $\{\xi_t'\}$ are contact processes with respective parameters $\mu, \lambda_1, \lambda_2, \cdots$ and $\mu, \lambda_1', \lambda_2', \cdots$; also $\xi_t \subset \xi_t'$. Taking

TABLE 3

$\eta(x)$	New Type
(00)	$(01)(\lambda'_{m'} - \lambda_m)$, $(11)(\lambda_m)$
(01)	$(00)(\mu)$, $(11)(\lambda_m)$
(11)	$(00)(\mu)$

$\eta_0(x) = (11)$ for $x \in \xi$ and $\eta_0(x) = (00)$ otherwise, we have $\xi_0 = \xi_0' = \xi$, and the lemma follows. □

(5.9) LEMMA. *Suppose* $\mu > 0$, $\lambda_k \uparrow$, $\xi \subset \xi'$, $\xi \neq \xi'$, $|\xi| < \infty$. *Then* $p_t(\xi) < p_t(\xi')$, $0 < t < \infty$. *Moreover* $p_\infty(\xi) < p_\infty(\xi')$ *if* $p_\infty(x) > 0$. (Note $p_\infty(x) = p_\infty(\{x\})$ is the same for each x.)

PROOF. Define η_t, $\eta_t{}^A$, and $\eta_t{}^{AB}$ again as in Table 2, denoting the corresponding probability measure by Q_η if $\eta_0 = \eta$. We take $\eta_0(x) = A$ if $x \in \xi$, $\eta_0(x) = B$ if $x \in \xi'\backslash\xi$, $\eta_0(x) = 0$ otherwise, so $\eta_0{}^A = \xi$, $\eta_0{}^{AB} = \xi'$. Then

$$(5.10) \qquad \begin{aligned} p_t(\xi') - p_t(\xi) &= Q_\eta(\eta_t{}^{AB} \neq \varnothing) - Q_\eta(\eta_t{}^A \neq \varnothing) \\ &= Q_\eta\{\eta_t{}^{AB} \neq \varnothing, \eta_t{}^A = \varnothing\}, \qquad t > 0 . \end{aligned}$$

Suppose first that $|\xi'| < \infty$. Since all jumps of $\{\eta_t\}$ are jumps of $\{\eta_t{}^A\}$ or $\{\eta_t{}^{AB}\}$, and from Section 4 the two latter processes are countable-state chains with stable states, it follows that $\{\eta_t\}$ is such a chain. Fix $T > 0$. From the construction of $\{\eta_t\}$ the probability is strictly positive that the first $|\xi|$ transitions of $\{\eta_t\}$ occur before T and are changes of type from A to 0 of the vertices in ξ, whence $Q_\eta\{\eta_T{}^{AB} \neq \varnothing, \eta_T{}^A = \varnothing\} > 0$. This proves the first assertion of the lemma. Now for each $t > 0$, $p_\infty(\xi') - p_\infty(\xi) \geqq Q_\eta\{\eta_s{}^{AB} \neq \varnothing$ for each $s \geqq t$, $\eta_t{}^A = \varnothing\} = Q$, say. Conditioning on η_s, $0 \leqq s \leqq t$, we have, if $p_\infty(x) > 0$,

$$\begin{aligned} Q &= \textstyle\sum_{\zeta \in \Xi, 0<|\zeta|<\infty} Q_\eta\{\eta_t{}^A = \varnothing, \eta_t{}^{AB} = \zeta\} p_\infty(\zeta) \\ &\geqq p_\infty(x)(p_t(\xi') - p_t(\xi)) > 0 . \end{aligned}$$

If $|\xi'| = \infty$, the result is obvious because $p_t(\xi') = 1$. □

6. Submodularity. Now let $\{\lambda_i : i = 0, 1, \cdots, 2d\}$ be concave and non-decreasing in i, with $\lambda_0 = 0$. Concavity means that $\lambda_{i+1} - \lambda_i$ is non-increasing in i, implying

$$(6.1) \qquad \lambda_n - \lambda_0 \geqq \lambda_{n+1} - \lambda_1 \geqq \lambda_{n+2} - \lambda_2 \geqq \cdots, \qquad n = 1, 2, \cdots .$$

The simplest cases are $\lambda_k = k\lambda_1$, $k = 1, 2, \cdots$, and $\lambda_k = \lambda_1$, $k = 1, 2, \cdots$.

(6.2) THEOREM. *If* $\{\lambda_i\}$ *is concave and non-decreasing,* $\lambda_0 = 0$, *then* $p_t(\xi)$ *is submodular.*

Note from (5.4′) that submodularity is a kind of concavity property.

PROOF. Again we use an auxiliary NNI, this time with types in $S = \{0, A, BC, B, C', C\}$. Let m, r, n, k', k denote the respective numbers of neighbors of x of type $A, BC, B, C'. C$ respectively. The intensities are given by the following table.

TABLE 4

$\eta(x)$	New Type
0	$A(\lambda_m)$, $BC(\lambda_{m+r} - \lambda_m)$, $B(\lambda_{m+r+n} - \lambda_{m+r})$, $C'(\lambda_{m+r+n+k'} - \lambda_{m+r+n})$, $C((\lambda_{m+r+k+k'} - \lambda_{m+r}) - (\lambda_{m+r+n+k'} - \lambda_{m+r+n}))$
A	$0(\mu)$
BC	$A(\lambda_m)$, $0(\mu)$
B	$A(\lambda_m)$, $BC(\lambda_{m+r+k+k'} - \lambda_m)$, $0(\mu)$
C'	$A(\lambda_m)$, $BC(\lambda_{m+r+n} - \lambda_m)$, $0(\mu)$
C	$A(\lambda_m)$, $BC(\lambda_{m+r+n} - \lambda_m)$, $C'(\lambda_{m+r+n+k'} - \lambda_{m+r+n})$, $0(\mu)$

Since $\lambda_k \uparrow$ and (6.1) holds, the indicated intensities are nonnegative.

Define four processes, whose values at x are 1 under the following conditions and 0 otherwise:

$$\begin{aligned} \eta_t^A(x) &= 1 \quad \text{if} \quad \eta_t(x) = A \\ \eta_t^{AB}(x) &= 1 \quad \text{if} \quad \eta_t(x) = A\,,\ BC\,,\ \text{or}\ B \\ \eta_t^{ABC'}(x) &= 1 \quad \text{if} \quad \eta_t(x) = A\,,\ BC\,,\ B\,,\ \text{or}\ C' \\ \eta_t^{AC}(x) &= 1 \quad \text{if} \quad \eta_t(x) = A\,,\ BC\,,\ C'\,,\ \text{or}\ C\,. \end{aligned}$$

From Lemma 2.3, these are contact processes. Let A_1, B_1, C_1 be disjoint subsets of Z_d. Let $\eta(x) = A$ on A_1, B on B_1, C' on C_1, and 0 elsewhere, and let Q_η be the probability measure for $\{\eta_t\}$ with $\eta_0 = \eta$. Then

$$\eta_0^A = A_1\,, \qquad \eta_0^{AB} = A_1 \cup B_1\,, \qquad \eta_0^{ABC'} = A_1 \cup B_1 \cup C_1\,, \qquad \eta_0^{AC} = A_1 \cup C_1\,.$$

The general idea is that $\eta_t^{ABC'} \backslash \eta_t^{AB}$ represents the difference between a process starting on $A_1 \cup B_1 \cup C_1$ and one starting on $A_1 \cup B_1$.

We have

$$p_t(A_1 \cup B_1 \cup C_1) - p_t(A_1 \cup B_1) = Q_\eta\{\eta_t^{ABC'} \neq \emptyset, \eta_t^{AB} = \emptyset\} = Q_\eta(U)\,, \quad \text{say},$$

since $\eta_t^{AB} \subset \eta_t^{ABC'}$ from the construction. Similarly,

$$p_t(A_1 \cup C_1) - p_t(A_1) = Q_\eta\{\eta_t^{AC} \neq \emptyset, \eta_t^A = \emptyset\} = Q_\eta(V)\,, \quad \text{say}.$$

However, from the definition of our various processes we have $U \subset V$ whence $Q_\eta(U) \leqq Q_\eta(V)$, and thus

$$p_t(A_1 \cup B_1 \cup C_1) - p_t(A_1 \cup B_1) \leqq p_t(A_1 \cup C_1) - p_t(A_1)\,. \qquad \square$$

7. Computation and bounds for p_∞ and m_t. In principle we can compute $p_\infty(\xi)$ as follows. For $A \subset Z_d$ let

$$q(A, \xi) = P_\xi\{\xi_s \subset A, 0 \leqq s < \infty, \xi_s = \emptyset \ \text{ultimately}\}\,.$$

Let $\xi_{(0)} = \xi_0, \xi_{(1)}, \xi_{(2)}, \cdots$ be the imbedded jump process of $\{\xi_t\}$, assuming $\xi_0 \in \Xi_0$ (see Section 4), with the transition matrix $r(\xi, \eta)$. Now let A be a fixed cube oriented along the axes, and suppose $0 < \mu, \lambda_1, \lambda_2, \cdots$. Let $r_A(\xi, \eta)$ be the submatrix of r with $\xi, \eta \subset A$. Then $q(A, \xi) = \lim_{n\to\infty} r_A^{(n)}(\xi, \phi)$, $\xi \subset A$, and $1 - p_\infty(\xi) = \lim_{A \uparrow Z_d} q(A, \xi)$, $\xi \in \Xi_0$.

For subadditive processes, letting p_∞ denote $p_\infty(\xi)$ when $\xi = \{x\}$, we have, for each $n = 1, 2, \cdots$,

$$\begin{aligned} p_\infty &= \textstyle\sum_{\eta \in \Xi_0} r^{(n)}(x, \eta) p_\infty(\eta) \\ &\leqq \textstyle\sum r^{(n)}(x, \eta) \cdot |\eta| \cdot p_\infty = p_\infty \mathscr{E}_x |\xi_{(n)}| \,. \end{aligned}$$

If we find computationally that $\mathscr{E}_x|\xi_{(n)}| < 1$ for some n, we can conclude $p_\infty = 0$. This procedure apparently requires a large n; for example if $d = 1$, $\mu = \lambda_1 = 1$, $\lambda_2 = 2$, then $\mathscr{E}_x|\xi_{(4)}| = 1.36$ approximately, although actually $p_\infty = 0$ and probably $\mathscr{E}_x|\xi_{(n)}| \to 0$. (See Section 10.)

(7.1) THEOREM.[4] *Let $\{\xi_t\}$ be a contact process with $(2d-1)\lambda_k < k\mu$, $k = 1, 2, \cdots$. Then $p_\infty(\xi) = 0$, $\xi \in \Xi_0$.*

PROOF. Using Lemma 5.8 and a change of time scale, it is sufficient to treat the case $\mu = 1$, $\lambda_k = k\lambda$, where $(2d-1)\lambda < 1$. For this case denote $p_\infty(\xi)$ by $\pi_1 = \pi_1(\lambda)$ or $\pi_2 = \pi_2(\lambda)$ respectively if $\xi = \{x\}$ or ξ is a pair of neighbors. Let r be the transition matrix of the imbedded jump process. Examining the intensities in (4.4) and (4.5), we have for any pair of neighbors ξ

$$\pi_1 = \frac{2d\lambda}{1 + 2d\lambda} \pi_2 \,, \tag{7.2}$$

$$\begin{aligned} \pi_2 &= \frac{2}{2 + (4d-2)\lambda} \pi_1 + \textstyle\sum' r(\xi, \xi \cup x) p_\infty(\xi \cup x) \\ &= \frac{\pi_1}{1 + (2d-1)\lambda} + \textstyle\sum' r(\xi, \xi \cup x)(p_\infty(\xi \cup x) - \pi_2) \\ &\quad + \pi_2 \textstyle\sum' r(\xi, \xi \cup x) \,, \end{aligned} \tag{7.3}$$

where $\sum'$ is taken over $x \sim \xi$. The last term in (7.3) is $\pi_2(2d-1)\lambda/(1 + (2d-1)\lambda)$. Subtracting this from both sides, multiplying by $1 + (2d-1)\lambda$, and subtracting π_1, we get

$$\begin{aligned} \pi_2 - \pi_1 &= (1 + (2d-1)\lambda) \textstyle\sum' r(\xi, \xi \cup x)(p_\infty(\xi \cup x) - \pi_2) \\ &\leqq (1 + (2d-1)\lambda)(\pi_2 - \pi_1) \textstyle\sum' r(\xi, \xi \cup x) \\ &= (\pi_2 - \pi_1)(2d-1)\lambda \,, \end{aligned} \tag{7.4}$$

where Theorem 6.2 was used to give the inequality $p_\infty(\xi \cup x) - \pi_2 \leqq \pi_2 - \pi_1$. If $\pi_1 > 0$, then $\pi_2 > \pi_1$ from (7.2). Dividing (7.4) by $\pi_2 - \pi_1$ shows that $(2d-1)\lambda \geqq 1$ if $\pi_1 > 0$. □

In Lemma (7.5), m_t denotes the value of $m_t(\xi)$ if $\xi = \{x\}$.

(7.5) LEMMA. *Let $\{\xi_t\}$ be a contact process such that $\lambda_k \uparrow$ and suppose $\int_0^\infty m_t\, dt = \infty$. Then $\int_0^\infty (m_t(\xi) - m_t)\, dt = \infty$, $|\xi| \geqq 2$.*

[4] At least if $d = 1$ this result can be improved by pushing the method harder; see Section 10. F. Spitzer and R. Holley have recently obtained related results; in particular Holley has shown Theorem 7.1 by different methods. (Personal communication.)

The proof is rather like that of Lemma 5.9 and is omitted.

(7.6) THEOREM. *Let* $\{\xi_t\}$ *be a contact process having* $\lambda_k/\mu < k/(2d-1)$, $k \geqq 1$. *Then* $\int_0^\infty m_t(\xi)\,dt < \infty$, $\xi \in \Xi_0$.

PROOF. It is sufficient to consider the case $\mu = 1$ and $\lambda_k = k\lambda$, where $(2d-1)\lambda < 1$, as in the proof of Theorem 7.1. Let $m_1(t)$ and $m_2(t)$ denote $m_t(\xi)$ if respectively $\xi = \{x\}$ or ξ is a pair of neighbors. It follows from Lemma 4.7 that m_1 and m_2 are bounded on each finite t-interval. Consideration of the first jump shows that for any pair of neighbors ξ we have

$$m_2(t) = 2e^{-Kt} + K\int_0^t e^{-Ks}\left\{\frac{2}{K}m_1(t-s) + \sum{}' r(\xi, \xi \cup x)m_{t-s}(\xi \cup x)\right\}ds\,, \tag{7.7}$$

where $K = 2 + (4d-2)\lambda$. From submodularity (see the remark (5.5)),

$$\begin{aligned} m_{t-s}(\xi \cup x) &= m_{t-s}(\xi \cup x) - m_{t-s}(\xi) + m_{t-s}(\xi) \\ &\leqq m_2(t-s) - m_1(t-s) + m_2(t-s)\,. \end{aligned} \tag{7.8}$$

From (7.8), the relation $\sum' r(\xi, \xi \cup x) = (4d-2)\lambda/K$, and (7.7), we have

$$\begin{aligned} m_2(t) \leqq 2e^{-Kt} &+ 2\int_0^t e^{-Ks}m_1(t-s)\,ds \\ &+ (4d-2)\lambda\int_0^t e^{-Ks}(2m_2(t-s) - m_1(t-s))\,ds\,. \end{aligned} \tag{7.9}$$

Let $M_i(t) = \int_0^t m_i(s)\,ds$, $i = 1, 2$. Integrating (7.9) from 0 to T, we get

$$\begin{aligned} M_2(T) &\leqq \frac{2}{K}(1 - e^{-KT}) + \frac{2}{K}\int_0^T m_1(s)(1 - e^{-K(T-s)})\,ds \\ &\quad + \frac{(4d-2)\lambda}{K}\int_0^T (2m_2(s) - m_1(s))(1 - e^{-K(T-s)})\,ds \\ &\leqq \frac{2}{K} + \frac{2}{K}M_1(T) + \frac{(4d-2)\lambda}{K}(2M_2(T) - M_1(T))\,. \end{aligned} \tag{7.10}$$

Subtracting $(4d-2)\lambda M_2/K + 2M_1/K$ from both sides and multiplying by $K/2$ we get

$$M_2(T) - M_1(T) \leqq 1 + (2d-1)\lambda(M_2(T) - M_1(T))\,. \tag{7.11}$$

Since $(2d-1)\lambda < 1$, we have

$$M_2(T) - M_1(T) \leqq \frac{1}{1-(2d-1)\lambda}\,, \qquad T \geqq 0\,.$$

Letting $T \to \infty$ and using Lemma 7.5, we see that $\int_0^\infty m_1(t)\,dt < \infty$. The desired result follows from subadditivity. □

(7.12) REMARK. Using Lemma 4.11, we see that under the conditions of Theorem 7.6, we have $\lim_{t\to\infty} m_t(\xi) = 0$, $\xi \in \Xi_0$.

(7.13) THEOREM. *Under the assumptions of Theorem* 7.6, *we have*

$$\lim_{t\to\infty} P_\xi\{\xi_t(x) = 1\} = 0\,, \qquad \xi \in \Xi,\ x \in Z_d\,.$$

PROOF. From the proof of Lemma 5.8, it is sufficient to suppose $\mu = 1$, $\lambda_k = k\lambda$, $(2d-1)\lambda < 1$. For an arbitrary ξ, choose $\xi^1, \xi^2, \cdots \in \Xi_0$ such that $\xi^n \uparrow \xi$. From monotonicity and the Feller property of $\{\xi_t\}$ we have $P_{\xi^n}\{\xi_t(x) = 1\} \uparrow P_\xi\{\xi_t(x) = 1\}$. From subadditivity, taking x as the origin O, we have

$$\begin{aligned} P_{\xi^n}\{\xi_t(O) = 1\} &\leqq \textstyle\sum_{x \in \xi^n} P_x\{\xi_t(O) = 1\} = \sum_{x \in \xi^n} P_O\{\xi_t(-x) = 1\} \\ &\leqq \textstyle\sum_{x \in Z^d} P_O\{\xi_t(-x) = 1\} = m_t(O)\,. \end{aligned}$$

Theorem 7.13 follows from (7.12). □

Note that in Theorem 7.13, ξ is not necessarily in Ξ_0.

8. An ergodic theorem. Let $\{\eta_t\}$ be a NNI with state space $\{0, 1\}$ and intensities β and α_k for the change of type $1 \to 0$ and $0 \to 1$ at x, where $k = \eta(N_x)$. We will find conditions under which $\{\eta_t\}$ has a limiting distribution not depending on η_0, i.e., is *ergodic*. As noted before, the procedure of Dobrushin (1971) was to construct processes $\{\eta_t\}$ and $\{\eta_t'\}$ with the same transition law, on the same probability space. We do this here and also construct a contact process $\{\xi_t\}$, such that $\xi_t(x) = 0$ implies $\eta_t(x) = \eta_t'(x)$. This enables us to sharpen Dobrushin's result somewhat for our particular process.

(8.1) DEFINITIONS. Let $\mu = \beta + \min_{k \geqq 0} \alpha_k$,

$$\lambda_k = \max_{|m-n| \leqq k} |\alpha_m - \alpha_n|\,, \qquad k = 0, 1, \cdots, 2d\,.$$

Note that then

$$\lambda_k \leqq k\lambda_1\,, \qquad k = 0, 1, 2, \cdots\,. \tag{8.2}$$

Consider a NNI process with set of types $S = \{(000), (001), (011), (101), (110), (111)\}$. It is convenient to denote the process by (η_t, η_t', ξ_t), the state space being the set $\Xi^* = \{(\eta, \eta', \xi) \in \Xi^3\colon |\eta(x) - \eta'(x)| \leqq \xi(x)\}$. Given some $(\eta, \eta', \xi) \in \Xi^*$, let $m = \eta(N_x)$, $m' = \eta'(N_x)$, $n = \xi(N_x)$. Then

$$\lambda_n \geqq |\alpha_{m'} - \alpha_m|\,. \tag{8.3}$$

For, let m_{10} be the number of neighbors y of x such that $\eta(y) = 1$ and $\eta'(y) = 0$. At each such y, $\xi(y) = 1$, whence $m - m' \leqq m_{10} \leqq n$. Similarly $m' - m \leqq n$ and (8.3) follows from the definition of λ_n.

TABLE 5

$(\eta(x), \eta'(x), \xi(x))$	New Type
(000)	(001)[$\lambda_n - \lvert\alpha - \alpha'\rvert$], (011)[$\alpha' - \min(\alpha, \alpha')$], (101)[$\alpha - \min(\alpha, \alpha')$], (110)[$\min(\alpha, \alpha')$]
(001)	(000)[μ], (011)[$\alpha' - \min(\alpha, \alpha')$], (101)[$\alpha - \min(\alpha, \alpha')$], (111)[$\min(\alpha, \alpha')$]
(011)	(000)[β], (110)[α^*], (111)[$\alpha - \alpha^*$]
(101)	(000)[β], (110)[α^*], (111)[$\alpha' - \alpha^*$]
(110)	(000)[β], (111)[λ_n]
(111)	(000)[β], (110)[α^*]

In the accompanying table, intensities are in square brackets after the new type. For convenience put

$$\alpha_m = \alpha\,, \qquad \alpha_{m'} = \alpha'\,, \qquad \min_i \alpha_i = \alpha^*\,, \qquad \text{and} \qquad \beta + \min \alpha_i = \mu$$

as above.

From (8.3), all the indicated intensities are $\geqq 0$. From Lemma 2.3, $\{\eta_t\}$, $\{\eta_t'\}$, and $\{\xi_t\}$ are Markovian, the first two having the transition law of the process introduced at the beginning of the section, while $\{\xi_t\}$ is a contact process with the parameters given in (8.1).

(8.4) THEOREM. *Let $\{\eta_t\}$ be the process defined at the beginning of this section. Suppose $\beta > 0$ and*

$$\max_{0 \leqq n \leqq 2d-1} |\alpha_{n+1} - \alpha_n| < \frac{\beta + \min_n \alpha_n}{2d - 1}\,. \tag{8.5}$$

Then as $t \to \infty$ the distribution of $\{\eta_t\}$ converges weakly to a limit not depending on the distribution of η_0.

PROOF. Consider the process $\{\eta_t, \eta_t', \xi_t\}$ defined above. Since $\{\eta_t\}$ is Feller on a compact state space it has at least one invariant distribution π. Let η_0 have this distribution, let η_0' have an arbitrary distribution, and let $\xi_0(x) = 1$ for all x. Then

$$\text{Prob}\,\{\eta_t'(x) = \eta_t(x)\} \geqq \text{Prob}\,\{\xi_t(x) = 0\}\,. \tag{8.6}$$

From (8.1), (8.2), (8.5), and Theorem 7.13, the right side of (8.6) $\to 1$ as $t \to \infty$, which shows that the distribution of $\{\eta_t'\}$ converges weakly to π. $\square$

REMARK. If $\beta = 0$, we again have Theorem 7.1, which will be strengthened in Section 10 for the case $d = 1$.

A criterion for ergodicity for rather general processes is given in Theorem 4 of Dobrushin (1971). His result is apparently stronger than is stated in his theorem, since in the argument on page 83 of the paper one can apparently use f_b and g_{cb} rather than $\bar{f}_b$ and $\bar{g}_{cb}$. The corresponding stronger result would then give $2d$ in the denominator of the right side of (8.5) above, rather than $2d - 1$. Thus (8.5) is an improvement by a factor of 2 in case $d = 1$ and a smaller improvement for higher dimensions. Note that any strengthening of Theorem 7.13 for contact processes would give a corresponding strengthening of Theorem 8.4. These remarks apply for $\beta \geqq 0$.

If $d = 2$, it is known that a process $\{\eta_t\}$ as defined at the beginning of this section, with $\beta > 0$, can be nonergodic for certain $\alpha_k > 0$. This stems from the non-uniqueness of the random fields corresponding to certain sets of parameters of the Ising model. (See, e.g., Spitzer (1971) Chapter 7.) However, no nonergodic example is known for $\beta > 0$ if $d = 1$. If there is none, then Theorem 8.4 is of no use when $\beta > 0$ and $d = 1$.

Another criterion for ergodicity has been given by Holley (1972c).

9. Permanence. We call the contact process $\{\xi_t\}$ *permanent* if

$$\liminf_{t\to\infty} P_\xi\{\xi_t(x) = 1\} > 0\,, \qquad \xi \neq \emptyset\,,\ x \in Z_d\,.$$

This implies $p_\infty(\xi) > 0$, $\xi \neq \emptyset$. We will show that permanent processes exist.

Related results for discrete time have been given by Toom (1968), Vasil'ev (1969), and others. We follow them in treating a 1-dimensional process by means of a 2-dimensional space-time diagram.[5] However, for the discrete-time processes mentioned above, say, $\eta_0, \eta_1, \cdots$, the random variables $\{\eta_{t+1}(x), x \in Z_1\}$ are conditionally independent, given $\{\eta_t(x), x \in Z_1\}$, while such a property is not true in continuous time. This is why some additional proof is required for the case of continuous time.

If S_1 and S_2 are random subsets of Z_d with respective probability laws ν_1 and ν_2, we say that S_2 *dominates* S_1 if there is a joint law for S_1 and S_2 having ν_1 and ν_2 as marginals, such that Prob $\{S_1 \subset S_2\} = 1$. An interesting criterion for domination has been given by Holley (1972b). Here we use only the easily proved remark that if under ν_i all the events $\{x \in S_i\}$, $x \in Z_d$, are independent, $i = 1, 2$, and if for each x we have $\nu_1\{x \in S_1\} \leqq \nu_2\{x \in S_2\}$, then S_2 dominates S_1.

(9.1) THEOREM. *Let $\{\xi_t\}$ be a contact process. Then $\{\xi_t\}$ is permanent if the parameters λ_k/μ, $k = 1, 2, \cdots$, are sufficiently large.*

SKETCH OF PROOF. We omit most of the technical details, but include enough to indicate how the above-indicated lack of independence is treated. Setting aside the trivial case $\mu = 0$, we can suppose $\mu = 1$. We suppose $d = 1$, since the general case follows by embedding Z_1 in Z_d. From Lemma 5.8 and its proof, it is sufficient to consider the case $\lambda_1 = \lambda$, $\lambda_2 = 2\lambda$.

Let $\alpha_i(x)$, $i = 1, 2, \cdots$, $x \in Z_1$ be independent exponential random variables, mean 1. With each x associate a clock function $t \to S_x(t)$, $S_x(0) = 0$, running at the rate $dS_x/dt = 1$ if $\xi_t(x) = 1$ and $dS_x/dt = \lambda(\xi_t(x-1) + \xi_t(x+1))$ if $\xi_t(x) = 0$. The ith jump of $\{\xi_t(x)\}$ occurs at time $t_i(x) = \inf\{s\colon S_x(s) = \alpha_1(x) + \cdots + \alpha_i(x)\}$. (See Harris (1972) for the details of a similar case.)

Fix $\Delta > 0$ and $\xi \in \Xi_0$ such that $\xi(x) = 0$ if x is odd. Pick some even y, and suppose first $\xi(y-2) = \xi(y) = 1$. Letting $\beta' = \beta'(y-1) = \min(\alpha_1(y-2), \alpha_1(y))$ and $\beta'' = \max(\alpha_1(y-2), \alpha_1(y))$, note that $dS_{y-1}/dt = 2\lambda$ for $0 \leqq t < \beta'$ and $dS_{y-1}/dt \geqq \lambda$ for $\beta' \leqq t < \beta''$, provided in each case that $\xi_t(y-1) = 0$. Let $A = A(y-1) = \{\alpha_2(y-1) > \Delta\}$; let $U(y-1)$ be the event

$$\left\{\frac{\alpha_1(y-1)}{2\lambda} \leqq \min\left[\Delta\,, \frac{\beta' + \Delta}{2}\,, \frac{\beta' + \beta''}{2}\right]\right\}. \tag{9.2}$$

Considering the construction of $\{\xi_t\}$ we find

$$U \cap A \subset \{\xi_\Delta(y-1) = 1\}\,. \tag{9.3}$$

[5] G. R. Grimmett and D. R. Stirzaker have treated similar 1-dimensional processes in discrete time by reduction to plane percolation problems. (Oral communication.)

(Look separately at the cases $\Delta \leqq \beta'$, $\beta' < \Delta < \beta''$, and $\Delta \geqq \beta''$.) Suppose now that $\alpha_1(y)$, for each even y, is realized as $\alpha_1(y) = \max(\alpha^*(y), \alpha^{**}(y))$, where α^* and α^{**} are independent with the distribution $G(t) = (1 - e^{-t})^{\frac{1}{2}}$, $t \geqq 0$. Let $\beta^* = \beta^*(y-1) = \min(\alpha^*(y-2), \alpha^{**}(y))$, $\beta^{**} = \max(\alpha^*(y-2), \alpha^{**}(y))$. Then $\beta^* \leqq \beta'$, $\beta^{**} \leqq \beta''$. Define U^* like U but with β^* and β^{**} replacing β' and β''. Let $w^*(y-1)$ be the indicator of $U^* \cap A$. Then $U^* \subset U$ and

$$w^*(y-1) \leqq \xi_\Delta(y-1), \qquad y \text{ even.} \tag{9.4}$$

If $\xi(y-2) = 1$ and $\xi(y) = 0$, let $w^*(y-1)$ be the indicator of the event

$$A \cap \left\{\frac{\alpha_1(y-1)}{\lambda} \leqq \min(\Delta, \alpha^*(y-2))\right\},$$

and define $w^*(y-1)$ similarly if $\xi(y) = 1$ and $\xi(y-2) = 0$ except with $\alpha^{**}(y)$ instead of $\alpha^*(y-2)$. Again (9.4) holds. Finally, if $\xi(y-2) = \xi(y) = 0$, put $w^*(y-1) = 0$.

From the construction we see that the random variables $w^*(y-1)$, y even, are independent. Moreover, Prob $\{w^*(y-1) = 1\}$ is 0 if $\xi(y-2) = \xi(y) = 0$ and is given by (9.5) and (9.6) if $\xi(y-2) + \xi(y)$ is 2 or 1 respectively:

$$\begin{aligned} &e^{-\Delta} \int\int_{0<u<v<\infty} 2\, dG(u)\, dG(v) \\ &\qquad \times \left\{1 - \exp\left[-2\lambda \min\left(\Delta, \frac{u+\Delta}{2}, \frac{u+v}{2}\right)\right]\right\}; \end{aligned} \tag{9.5}$$

$$e^{-\Delta} \int_{u>0} (1 - e^{-\lambda \min(\Delta, u)})\, dG(u)\,. \tag{9.6}$$

We can make (9.5) and (9.6) arbitrarily close to 1 by proper choice of λ and Δ, since each expression approaches $e^{-\Delta}$ as $\lambda \to \infty$.

Let $W_0, W_1, \cdots$ be a Markov process whose state at time n is a subset of the even integers if n is even and of the odd integers if n is odd. We may visualize W_n as a set of points in the nth row of the plane lattice $\mathscr{L} = \{(x, n): n = 0, 1, 2, \cdots; x - n \text{ even}\}$. If $x - n$ is even, then Prob $\{x - 1 \in W_{n+1} \mid W_0, \cdots, W_n\}$ is p if either $x - 2 \in W_n$ or $x \in W_n$; otherwise the probability is 0; moreover, the indicated events are conditionally independent for different values of x. It is known that there is a $p_0 < 1$ such that if $p > p_0$ and $W_0 \neq \emptyset$, then

$$\liminf_{n\to\infty,\, n-x \text{ even}} \text{Prob}\,\{x \in W_n\} > 0\,.$$

(See the references cited above). Pick Δ and λ so that (9.5) and (9.6) are $> p_0$. Take $W_0 = \xi$ as chosen above. Then $\{\xi_\Delta(y-1), y \text{ even}\}$, considered as a random set, dominates $\{w^*(y-1)\}$ because of (9.4). In turn $\{w^*(y-1)\}$ dominates W_1, by virtue of the remarks preceding Theorem 9.1. Hence $\{\xi_\Delta(y-1), y \text{ even}\}$ dominates W_1. It can then be shown that $\{\xi_{2\Delta}(y), y \text{ even}\}$ dominates W_2, and so on. Theorem 9.1 follows. □

The proof could also be carried out using a result of Hammersley (1959) about percolation processes, rather than the process $\{W_n\}$.

10. The 1-dimensional case. We now show how to improve Theorem 7.1 if $d = 1$. Note that if $d = 1$, the conditions of Theorem 5.6 for subadditivity and Theorem 6.2 for submodularity are both $\lambda_1 \leqq \lambda_2 \leqq 2\lambda_1$. Let $b = \lambda_2/\lambda_1$, assume $1 \leqq b \leqq 2$, and suppose $\mu = 1$. Put $\lambda_1 = \lambda$.

Let π_i be the value of $p_\infty(\xi)$ if $\xi(x) = 1$ for some i consecutive values of x and $\xi(x) = 0$ otherwise, $i = 1, 2, 3, 4$. Let $\pi' = p_\infty(\xi)$ if $\xi(0) = \xi(2) = 1$ and $\xi(x) = 0$ otherwise. Let $\pi'' = p_\infty(\xi)$ if $\xi(0) = \xi(1) = \xi(3) = 1$ and $\xi(x) = 0$ otherwise. Arguing as for Theorem 7.1, we have

$$
\begin{aligned}
&\pi_1 = \frac{2\lambda\pi_2}{1 + 2\lambda}, \qquad \pi_2 = \frac{\pi_1}{1+\lambda} + \frac{\lambda\pi_3}{1+\lambda}, \\
&\pi_3 = \frac{2\pi_2}{3+2\lambda} + \frac{\pi'}{3+2\lambda} + \frac{2\lambda\pi_4}{3+2\lambda}, \\
&\pi' = \left(\frac{1}{2 + b\lambda + 2\lambda}\right)(2\lambda\pi'' + 2\pi_1 + b\lambda\pi_3).
\end{aligned}
\tag{10.1}
$$

From submodularity we have

$$\pi_4 \leqq 2\pi_3 - \pi_2, \tag{10.2}$$

$$\pi'' \leqq \pi' + \pi_2 - \pi_1. \tag{10.3}$$

Using (10.1) we can write π_1, π_2, π_3, and π' as linear functions of π_4 and π'', whose coefficients are certain complicated rational functions of b and λ that are positive for $1 \leqq b \leqq 2$ and $\lambda > 0$. Replacing π_1, π_2, π_3, and π' by these functions on the right side of (10.2) and (10.3), we have

$$
\begin{aligned}
\pi_4 &\leqq a_{11}\pi_4 + a_{12}\pi'', \\
\pi'' &\leqq a_{21}\pi_4 + a_{22}\pi'',
\end{aligned}
\tag{10.4}
$$

where the a_{ij} are again positive functions of b and λ. The eigenvalue of (a_{ij}) of largest modulus is then positive and is given by

$$\gamma(b, \lambda) = \frac{a_{11} + a_{22} + ((a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}))^{\frac{1}{2}}}{2}.$$

If $\gamma < 1$ then (10.4) implies $\pi_4 = \pi'' = 0$, implying $p_\infty(\xi) = 0$ for $\xi \in \Xi_0$.

We find computationally[6] that $\gamma(2, 1.18) = .9999$, $\gamma(2, 1.19) = 1.0015$; also $\gamma(1, 1.22) = .9993$, $\gamma(1, 1.23) = 1.0008$. (It is likely that $\gamma = 1$ when λ is the largest root of $(b+1)\lambda^2 - (b-1)\lambda - 3 = 0$, although this has not been checked.)

To summarize: suppose $\mu = 1$, $\lambda_1 = \lambda$, $\lambda_2 = b\lambda$, $1 \leqq b \leqq 2$. If $\gamma(b, \lambda) < 1$, then $p_\infty(\xi) = 0$ for each $\xi \in \Xi_0$. This is true in particular if $b = 2$ and $\lambda = 1.18$ or if $b = 1$ and $\lambda = 1.22$. Other cases can be treated using Lemma 5.8.

One could push the method farther (it is not clear how far) by writing more

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[6] I am indebted to S. J. Harris for programming the computations.

equalities such as (10.1) before closing up the system with inequalities such as (10.2) and (10.3).

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REFERENCES

[1] DOBRUSHIN, R. L. (1971). Markov processes with a large number of locally interacting components (Russian). *Problems of Transmission of Information* **7**, No. 2, 70–87 and No. 3, 57–66.
[2] GRIFFEATH, DAVID (1974). Ergodic theorems for graph interactions. To appear.
[3] DYNKIN, E. B. (1965) *Markov Processes* **1**. Springer-Verlag, Berlin.
[4] HAMMERSLEY, J. M. (1959). Bornes supérieures de la probabilité critique dans un processus de filtration. *Le Calcul des Probabilités et ses Applications*. Centre National de la Recherche Scientifique, Paris, 17–37.
[5] HAMMERSLEY, J. M. (1961). On the rate of convergence to the connective constant of the hypercubical lattice. *Quart. J. Math.* **12** 250–256.
[6] HAMMERSLEY, J. M. (1962). Further results on the rate of convergence etc. *Quart. J. Math.* **13** 108–110.
[7] HAMMERSLEY, J. M. and WELSH, D. J. A. (1965). First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory, in *Bernoulli, Bayes, Laplace*. Springer-Verlag, Berlin.
[8] HARRIS, T. E. (1972). Nearest-neighbor Markov interaction processes on multidimensional lattices. *Adv. in Math.* **9** 66–89.
[9] HOLLEY, RICHARD (1972 a). Markovian interaction processes with finite range interactions. *Ann. Math. Statist.* **43** 1961–1967.
[10] HOLLEY, RICHARD (1972 b). Recent results on the stochastic Ising model. To appear.
[11] HOLLEY, RICHARD (1972 c). An ergodic theorem for attractive interactions. *Z. Wahrscheinlichkeitstheorie und Ver. Gebiete* **24** 325–334.
[12] LIGGETT, T. M. (1972). Existence theorems for infinite particle systems. *Trans. Amer. Math. Soc.* **165** 471–481.
[13] RICHARDSON, DANIEL (1973). Random growth in a tessellation. *Proc. Camb. Philos. Soc.* **74** 515–528.
[14] SPITZER, F. (1971). *Random Fields and Interacting Particle Systems*. Mathematical Association of America.
[15] TOOM, A. L. (1968). A family of uniform nets of formal neurons (Russian). *Dokl. Akad. Nauk SSSR.* **183** 49–53.
[16] VASERSHTEIN, L. N. (1969). Markov processes on countable product spaces, describing large systems of automata (Russian). *Problems of the Transmission of Information* **5**, No. 3, 64–72.
[17] VASIL'EV, N. B. (1969). On the limiting behavior of a random medium (Russian). *Problems of the Transmission of Information* **5**, No. 4, 68–74.
[18] VASIL'EV, N. B. (1970). Correlation equations for stationary measures of a Markov chain (Russian). *Theor. Probability Appl.* **15** 536–540.
[19] VASIL'EV, N. B., MITYUSHIN, L. G., PYATETSKIĬ-SHAPIRO, I. I., and TOOM, A. L. (1973). Operators of Stavskaya (Russian), Preprint No. 12, *Inst. Priklad. Mat. Akad. Nauk SSSR.* (work dated 1972.)

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